Acta Sci. Math., 42 (1980), 67—70

Unbounded operators with spectral decomposition properties

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The general spectral decomposition problem for bounded linear operators on a complex Banach space X has been formulated and studied in [4]. In this paper we extend the problem to the unbounded case and show that the single valued extension property remains valid for a class of closed linear operators on X.

While the theory of unbounded decomposable operators considered in [2, 3] relies heavily upon the concept of spectral capacity [1], here we make the theory independent of such an external constraint.

A short glossary of notations now follows. For a subset S of the complex plane C, $\bar{S}$ denotes the closure, S^c the complement, conv S the convex hull and $d(\lambda, S)$ the distance from a point λ to S. $\mathcal{G}$ denotes the collection of all open sets in C. For a linear operator T on X we use the following notations: the domain D_T, the spectrum $\sigma(T)$, the resolvent set $\varrho(T)$ and the resolvent operator $R(\cdot\,; T)$. A subspace (closed linear manifold) Y of X is invariant under T if $T(Y\cap D_T)\subset Y$. Inv (T) denotes the family of invariant subspaces under T. For $Y\in \operatorname{Inv}(T)$, we write $T|Y$ for the restriction of T to Y and we abbreviate $\lambda I-T$ by $\lambda-T$, where $\lambda\in C$ and I stands for the identity operator.

Let T: $D_T(\subset X)\to X$ be a closed linear operator.

1. Definition. A *spectral decomposition* of X by T is a finite system $\{(G_i, Y_i)\}\subset \mathcal{G}\times \operatorname{Inv}(T)$ with the following properties:

(i) $$\sigma(T)\subset \bigcup_i G_i;$$

(ii) $$X=\sum_i Y_i;$$

(iii) $$\sigma(T|Y_i)\subset G_i \quad \text{for all } i.$$

Received September 14, and in revised form, December 14, 1978.

Research supported by the Temple University Summer Research Fellowship 1978.

2. Definition. T is said to have the *spectral decomposition property* (abbrev. SDP) if for every finite open cover $\{G_i\}$ of $\sigma(T)$, there is a system $\{Y_i\}\subset \mathrm{Inv}\,(T)$ with the following properties:

(I) $Y_i \subset D_T$ if G_i is relatively compact;

(II) $\{(G_i, Y_i)\}$ is a spectral decomposition of X by T.

Our objective is to show that T with the SDP possesses the single valued extension property. For this we need a lemma.

3. Lemma. *Given T, let $f\colon D\to D_T$ be holomorphic on an open connected set $D\subset C$ and satisfy conditions:*

$$f(\lambda)\not\equiv 0 \quad \textit{and} \quad (\lambda-T)f(\lambda)=0 \quad \text{on } D.$$

If $Y\in \mathrm{Inv}\,(T)$ is such that $\{f(\lambda)\colon \lambda\in G\}\subset Y$ for some $G\in\mathcal{G}$ then $D\subset\sigma(T|Y)$.

Proof. Define

$$H=\{\lambda\in D\colon f(\lambda), f'(\lambda), f''(\lambda), \ldots, \in Y\}.$$

H has the following properties:

(a) $H\neq\emptyset$; (b) H is open; (c) H is closed in D; (d) $H\subset\sigma(T|Y)$.

(a): Let $\lambda_0\in G$. For $r>0$ sufficiently small, $\Gamma=\{\lambda\in C\colon |\lambda-\lambda_0|=r\}\subset G$ and then by hypothesis, $\{f(\lambda)\colon \lambda\in\Gamma\}\subset Y$. By Cauchy's formula

$$f^{(n)}(\lambda_0)=\frac{n!}{2\pi i}\int_\Gamma \frac{f(\lambda)\,d\lambda}{(\lambda-\lambda_0)^{n+1}}\in Y, \quad n=0,1,2,\ldots$$

(b): Let $\lambda_0\in H$. Then $f(\lambda_0), f'(\lambda_0), f''(\lambda_0), \ldots\in Y$. Since $f, f', f'', \ldots$ are analytic, they admit Taylor series expansions in an open neighborhood $V(\lambda_0)$ of λ_0 and hence $f^{(n)}(\lambda)\in Y$ on $V(\lambda_0)$ for $n=0,1,2,\ldots$. Thus $V(\lambda_0)\subset H$.

(c):
$$H=\left[\bigcap_{n=0}^{\infty}(f^{(n)})^{-1}(Y)\right]\cap D.$$

(d): Let $\lambda\in H$. The vectors $f^{(n)}(\lambda)$ are not all zero because otherwise $f=0$. Let

$$m=\min\{n\colon f^{(n)}(\lambda)\neq 0\}.$$

If $m=0$ then $Tf(\lambda)=\lambda f(\lambda)$ and

$$Tf^{(m)}(\lambda)=\lambda f^{(m)}(\lambda) \quad \text{for} \quad m>0. \tag{1}$$

(1) holds because f is T-analytic (cf. [5, Lemma 2.1]) on D. In either case, $f^{(m)}(\lambda)$ is an eigenvector of $T|Y$ with respect to the eigenvalue λ.

By properties (a), (b), (c) $H=D$ and then property (d) concludes the proof.

4. Theorem. *Every T with the* SDP *has the single valued extension property.*

Proof. *Let* $f\colon D\to D_T$ be locally holomorphic on an open $D\subset C$ and satisfy identity

$$(\lambda-T)f(\lambda)=0 \quad \text{on } D.$$

We shall adapt the proof of [4, Theorem 8] to the unbounded case. We may assume that D is connected and contained in $\sigma(T)$, for $D\cap\varrho(T)\neq\emptyset$ implies that $f=0$ on some open set and hence on all of D, by analytic continuation. Fix $\lambda_0\in D$ and choose real numbers r_1 and r_2 such that $0<r_2<r_1<d(\lambda_0, D^c)$. Let

$$G_1=\{\lambda\colon |\lambda-\lambda_0|<r_1\},\quad G_2=\{\lambda\colon |\lambda-\lambda_0|>r_2\}.$$

Then G_1, G_2 cover $\sigma(T)$, $\bar{G}_1$ is both convex and compact, $D-\bar{G}_1\neq\emptyset$ and

$$D\not\subset G_2. \tag{2}$$

By the SDP of T, there are $Y_1, Y_2\in\operatorname{Inv}(T)$ verifying the following conditions:

$$X=Y_1+Y_2 \quad \text{with} \quad Y_1\subset D_T;$$

$$\sigma(T|Y_i)\subset G_i,\quad i=1,2. \tag{3}$$

There is an open $V\subset D-\bar{G}_1$ and there are functions $f_i\colon V\to Y_i$ $(i=1,2)$ such that

$$f(\mu)=f_1(\mu)+f_2(\mu) \quad \text{on } V. \tag{4}$$

Since the ranges of both f and f_1 are contained in D_T, so is the range of f_2. There is a function $g\colon V\to Y_1\cap Y_2$ defined by

$$g(\mu)=(\mu-T)f_1(\mu)=(T-\mu)f_2(\mu)\in Y_1\cap Y_2,\quad \mu\in V.$$

Since $Y_1\cap Y_2$ is invariant under $T|Y_1$ and G_1 is convex, we have

$$\sigma(T|Y_1\cap Y_2)\subset \operatorname{conv}\sigma(T|Y_1)\subset G_1.$$

Consequently, $V\subset G_1^c\subset\varrho(T|Y_1\cap Y_2)$. The function $h\colon V\to Y_1\cap Y_2$, defined by

$$h(\mu)=R(\mu;T|Y_1\cap Y_2)\,g(\mu)\in Y_1\cap Y_2,\quad \mu\in V$$

has property

$$(\mu-T)\,[h(\mu)-f_1(\mu)]=0.$$

Since both $h(V)\subset Y_1$, $f_1(V)\subset Y_1$ and $V\subset\varrho(T|Y_1)$, we have

$$f_1(\mu)=h(\mu)\in Y_1\cap Y_2 \quad \text{on } V.$$

Then (4) implies that $f(\mu)\in Y_2$ on V and hence $f(\mu)\in Y_2$ on all of D, by analytic continuation. Now if f is not identically zero on D then Lemma 3 implies that $D\subset\sigma(T|Y_2)$. This, under hypothesis (2), contradicts the second inclusion of (3). □

Acknowledgement. The author expresses his gratitude to the referee for his critical reading of the paper and for his valuable suggestions for improvements.

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