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New generalizations of Banach's contraction principle

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Many research papers have appeared on different generalizations of Banach's contraction principle. A. MEIR and E. KEELER [2] studied mappings $f\colon X\to X$ of a metric space (X, ϱ) having the property that for every $\varepsilon>0$ there exists a $\delta>0$ such that $\varepsilon\leqq\varrho(x,y)<\varepsilon+\delta$ implies $\varrho(f(x),f(y))<\varepsilon$. In the present paper we consider the following generalization of a restriction of this definition. For $x, y\in X$ let $d_f(x,y)=\operatorname{diam}\{x, y, f(x), f(y), f^2(x), f^2(y), \dots\}$. Here "diam" abbreviates diameter.

The mapping $f\colon X\to X$ is called a generalized Meir—Keeler contraction if $d_f(x,y)<\infty$ for $x, y\in X$ and if for every $\varepsilon>0$ there exist ε', ε'' such that $0<\varepsilon'<\varepsilon<\varepsilon''$ and $d_f(x,y)<\varepsilon''$ implies $\varrho(f(x),f(y))<\varepsilon'$.

Lj. B. ĆIRIĆ [1] studied mappings $f\colon X\to X$ for which $d_f(x,y)<\infty$ and there exists a constant α, $0\leqq\alpha<1$, such that

$$\varrho(f(x),f(y))\leqq\alpha\max\{\varrho(x,y),\ \varrho(x,f(x)),\ \varrho(y,f(y)),\ \varrho(x,f(y)),\ \varrho(y,f(x))\}$$

for $x, y\in X$. In the present paper we consider the following class of mappings wider than that considered by Ćirić.

The mapping $f\colon X\to X$ is called a generalized Banach contraction if $d_f(x,y)<\infty$ for $x, y\in X$ and if there exists a constant α, $0\leqq\alpha<1$ such that $\varrho(f(x),f(y))\leqq\alpha d_f(x,y)$ for all $x, y\in X$.

It is obvious that every generalized Banach contraction is a generalized Meir—Keeler contraction. The function $f(x)=\sin x$ on $X=[0,\pi/2]$ is a generalized Meir—Keeler contraction which is not a generalized Banach contraction. This may be seen in the following way. Firstly, if $\sin x$ were a generalized Banach contraction on $[0,\pi/2]$, then we would have $|\sin x|\leqq\alpha|x|$ for all $x\in[0,\pi/2]$ with some α, $0\leqq\alpha<1$. But this is impossible, since $\lim\limits_{x\to 0}\dfrac{\sin x}{x}=1$. Secondly, for any given ε, $0<\varepsilon\leqq 1$ let ε' be a number such that $\sin\varepsilon<\varepsilon'<\varepsilon$. Denote $\varepsilon''=\arcsin\varepsilon'$. If $x, y\in\left[0,\dfrac{\pi}{2}\right]$, $y\leqq x$ and $|x-y|\leqq\varepsilon''$, then $|\sin x-\sin y|=\int\limits_y^x\cos t\,dt=\int\limits_0^{x-y}\cos(x+t)\,dt\leqq$

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$\leqq \int_0^{x-y} \cos t\,dt = \sin(x-y) \leqq \sin(\arcsin \varepsilon') = \varepsilon'$. Consequently, $\sin x$ is a generalized Meir—Keeler contraction on $\left[0, \frac{\pi}{2}\right]$.

Now we give an example of a generalized Banach contraction which is not of Ćirić type. In fact, let $X=\{1,2,3,4\}$ and $\varrho(1,2)=3.9$, $\varrho(1,3)=3.7$, $\varrho(1,4)=4.0$, $\varrho(2,3)=3.9$, $\varrho(2,4)=3.9$, $\varrho(3,4)=3.0$. Furthermore, let f be defined on X by the equalities $f(1)=2$, $f(2)=3$, $f(3)=4$, $f(4)=4$. Then $\varrho(f(1),f(2))= =\max\{\varrho(1,2), \varrho(1,f(1)), \varrho(2,f(2)), \varrho(1,f(2)), \varrho(2,f(1))\}$. However, it is easy to verify that in this case $\varrho(x,y) \leqq 0{,}99 d_f(x,y)$ for all $x, y \in X$.

The objective of the present paper is to prove the following theorems.

Theorem 1. *Let f: $X \to X$ be a generalized Meir—Keeler mapping. Then there exists at most one fixed point of f, and $\{f^n(x)\}_{n=1}^{\infty}$ is a Cauchy sequence for every $x \in X$. If X is complete, then for every $x \in X$, $f^n(x)$ converges to the unique fixed point of f as $n \to \infty$.*

Theorem 2. *Let f: $X \to X$ be a generalized Banach contraction with constant α, let $x_0 \in X$ be fixed, and let $\delta_n = \operatorname{diam}\{f^n(x_0), f^{n+1}(x_0), \ldots\}$. Then*

$$\delta_n \leqq \frac{\alpha^n}{1-\alpha}\,\varrho(x_0, f(x_0)) \quad (n = 0, 1, \ldots),$$

$$\delta_n \leqq \frac{\alpha}{1-\alpha}\,\varrho(f^{n-1}(x_0), f^n(x_0)) \quad (n = 1, 2, \ldots).$$

If X is complete, then

$$\varrho(z, f^n(x_0)) \leqq \frac{\alpha^n}{1-\alpha}\,\varrho(x_0, f(x_0)) \quad (n = 0, 1, \ldots).$$

$$\varrho(z, f^n(x_0)) \leqq \frac{\alpha}{1-\alpha}\,\varrho(f^{n-1}(x_0), f^n(x_0)) \quad (n = 1, 2, \ldots),$$

where z denotes the unique fixed point of f.

The proofs will be based on the following

Lemma. *Let f: $X \to X$ be a generalized Meir—Keeler mapping, and let $x_0 \in X$, $\delta_n = \operatorname{diam}\{f^n(x_0), f^{n+1}(x_0), \ldots\}$. Then $\delta_n = \sup\limits_{k>n} \varrho(f^n(x_0), f^k(x_0))$ $(n=0,1,\ldots)$.*

Proof. It is sufficient to consider the case $n=0$. If $\delta_0=0$, then the statement of the lemma is obvious. If $\delta_0>0$, then choose δ_0', δ_0'' in such a way that we have $0<\delta_0'<\delta_0<\delta_0''$ and $\varrho(f(x), f(y))<\delta_0'$ if $d_f(x,y)<\delta_0''$. Now let $k, l \geqq 1$. Since $\delta_{\min\{k-1,l-1\}} \leqq \delta_0 < \delta_0''$, we have $\varrho(f^k(x_0), f^l(x_0)) < \delta_0' < \delta_0$. This immediately implies the assertion of our lemma.

Proof of Theorem 1. Let z' and z'' be fixed points of f. If $\varrho(z', z'')=\varepsilon>0$, then choose ε', ε'' such that $0<\varepsilon'<\varepsilon<\varepsilon''$ and $\varrho(f(x), f(y))<\varepsilon'$ if $d_f(x,y)<\varepsilon''$.

Since z', z'' are fixed points, $d_f(z', z'')=\varrho(z', z'')=\varepsilon<\varepsilon''$. Consequently, $\varrho(z', z'')= =\varrho(f(z'), f(z''))<\varepsilon'<\varepsilon$, a contradiction. Therefore we must have $\varrho(z', z'')=0$, i.e., that $z'=z''$.

Now let $x_0\in X$ be fixed and use the notations of Lemma. We have to prove that $\delta_n\to 0$. It follows from the definition of δ_n that $\delta_0\geqq\delta_1\geqq\ldots\geqq 0$. Consequently, $\delta_n\to\varepsilon$ for some $\varepsilon\geqq 0$. Assume that $\varepsilon>0$, and choose ε', ε'' so that we have $0<\varepsilon'< <\varepsilon<\varepsilon''$ and $\varrho(f(x), f(y))<\varepsilon'$ if $d_f(x, y)<\varepsilon''$. Let n_0 be so large that $\delta_{n_0}<\varepsilon''$. If $k, l>n_0$, then $\varrho(f^k(x_0), f(x_0))<\varepsilon'<\varepsilon$, since $d_f(f^{k-1}(x_0), f^{l-1}(x_0))=\delta_{\min\{k-1;\, l-1\}}\leqq \leqq\delta_{n_0}<\varepsilon''$. Therefore, $\delta_{n_0+1}\leqq\varepsilon'$, a contradiction since $\delta_n\downarrow\varepsilon$. Hence $\varepsilon=0$, and $\{f^n(x_0)\}_{n=1}^{\infty}$ is a Cauchy sequence.

Now let X be complete. Then $f^n(x_0)$ converges to an element z of X. We have to prove that z is invariant under f. Let $\delta_n^*=d_f(f^n(z), f^n(z))$. We must prove that $\delta_0^*=0$. Assume the contrary, i.e., that $\delta_0^*>0$, and choose $\delta_0^{*\prime}$, $\delta_0^{*\prime\prime}$ so that $0<\delta_0^{*\prime}< <\delta_0^*<\delta_n^{*\prime\prime}$ and $\varrho(f(x), f(y))<\delta_0^{*\prime}$ if $d_f(x, y)<\delta_0^{*\prime\prime}$. If $k\geqq 1$, then for all large enough n, $d_f(f^{k-1}(z), f^{n-1}(x_0))<\delta_0^{*\prime\prime}$ since $f^n(x_0)\to z$ as $n\to\infty$. For such n we have $\varrho(f^k(z), f^n(x_0))<\delta_0^{*\prime}$. If we let n tend to ∞, then we obtain that $\varrho(f^k(z), z)\leqq \leqq\delta_0^{*\prime}$. Consequently, according to Lemma, $\delta_0^*\leqq\delta_0^{*\prime}$, a contradiction. Therefore z is a fixed point of f.

Proof of Theorem 2. The inequalities involving δ_n imply the other two inequalities. The second inequality concerning δ_n is an immediate consequence of the first. To prove the first, we observe that $\delta_n\leqq\alpha^n\delta_0$. This is so since for $k>1\geqq n$ we have $\varrho(f^k(x_0), f^l(x_0))\leqq\alpha\delta_{n-1}$. Consequently, $\delta_n\leqq\alpha\delta_{n-1}$. We obtain from this by recursion that $\delta_n\leqq\alpha^n\delta_0$. Now let $k=1, 2, \ldots$. Then

$$\varrho(x_0, f^k(x_0)) \leqq \varrho(x_0, f(x_0))+\varrho(f(x_0), f^k(x_0)) \leqq$$
$$\leqq \varrho(x_0, f(x_0))+\delta_1 \leqq \varrho(x_0, f(x_0))+\alpha\delta_0$$

on the basis of what we have just observed. According to Lemma we therefore have $\delta_0\leqq\varrho(x_0, f(x_0))+\alpha\delta_0$, i.e., $\delta_0\leqq\dfrac{1}{1-\alpha}\,\varrho(x_0, f(x_0))$. This is the inequality to be proved for $n=0$. For $n=1, 2, \ldots$ we obtain from this and from what we have observed at the beginning of our proof that $\delta_n\leqq\alpha^n\delta_0\leqq\dfrac{\alpha^n}{1-\alpha}\,\varrho(x_0, f(x_0))$.

References

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