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On C_0-operators with property (P)

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1. H. BERCOVICI [1] has considered the class $\mathscr{P}$ of Hilbert space operators T of class C_0 having the following property:

(P) any injection $X\in\{T\}'$ is a quasi-affinity.

He has shown that $T\in\mathscr{P}$ if and only if $\bigwedge_{n=1}^{\infty} m_n[T]=1$, where $m_n[T]$ $(n=1, 2, ...)$ are the inner functions in the Jordan model of T. (Cf. Theorem 4.1 of [1].)

He has proved, furthermore, that every operator $T\in\mathscr{P}$ has the following stronger property also:

(P*) for any $X\in\{T\}'$ we have $\gamma_T(\ker X)=\gamma_T(\ker X^*)$.

(Cf. Theorem 7.9 of [1].) Here $\gamma_T(\ker X)$ and $\gamma_T(\ker X^*)$ are generalized inner functions; they play the roles of determinants of the operators $T|\ker X$ and $T_{\ker X^*}$. (Cf. sections 6 and 7 of [1].)

Let ϱ be the following relation on the class $\mathscr{P}$: $T_1\varrho T_2$ if there exist $T\in\mathscr{P}$ and $X\in\{T\}'$ such that T_1 and T_2 are quasisimilar to $T|\ker X$ and $T_{\ker X^*}$, that is, $T_1\sim T|\ker X$, and $T_2\sim T_{\ker X^*}$. Then the previous statement can be written in the following form. If $T_1, T_2\in\mathscr{P}$ and $T_1\varrho T_2$, then $\gamma_{T_1}=\gamma_{T_2}$ (because γ_T is a quasi-similarity invariant).

Bercovici has also proved a partial converse of this statement. Namely, he has proved that if $T_1, T_2\in\mathscr{P}$ are weak contractions and $\gamma_{T_1}=\gamma_{T_2}$, then $T_1\varrho T_2$. On the other hand he has shown that if $T_1, T_2\in\mathscr{P}$ are such that $\gamma_{T_1}=\gamma_{T_2}$, then there exists $S\in\mathscr{P}$ such that $T_1\varrho S$ and $S\varrho T_2$. The main purpose of this note is to prove the complete converse of the statement mentioned above, namely,

Theorem. *If $T_1, T_2\in\mathscr{P}$ are such that $\gamma_{T_1}=\gamma_{T_2}$, then $T_1\varrho T_2$.*

Thus the operators of class $\mathscr{P}$ have, in general, no stronger property than (P*). In particular, in general it is not true that an operator $T\in\mathscr{P}$ has the property:

(Q) $T|\ker X$ and $T_{\ker X^*}$ are quasisimilar for any $X\in\{T\}'$.

(Cf. [2].)

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Furthermore, from the Theorem we can easily infer that ϱ is an equivalence relation on $\mathscr{P}$.

2. In the sections 6 and 7 of [1] Bercovici introduced the notions of "generalized inner function" and "C_0-dimension of a subspace" in the following way. Any inner function $m\in H_i^\infty$ has a factorization $m=cbs$, where c is a complex constant of modulus one, b is a Blaschke product and s is a singular inner function deriving from a finite Borel measure μ on $[0, 2\pi]$, singular with respect to Lebesgue measure. (Cf. [3], Ch. III.) Let us denote by $\sigma(z)$ the multiplicity of the zero z $(|z|<1)$ in the Blaschke product b. Then $\gamma(m)$ will denote the pair $\gamma(m)=(\sigma, \mu)$. The class $\tilde{\Gamma}$ of "generalized inner functions" will be the set of pairs $\gamma=(\sigma, \mu)$, where σ is a natural number valued function defined on $D=\{z\colon |z|<1\}$ such that $\sum_{\sigma(z)\neq 0}(1-|z|)<\infty$, and μ is a (not necessarily finite) Borel measure on $[0, 2\pi]$, which is absolute continuous with respect to a finite Borel measure ν singular with respect to Lebesgue measure. We define addition and lattice operations in $\tilde{\Gamma}$ by components.

If $T\in\mathscr{P}$, then it can be proved that $\gamma_T:=\sum_{j=0}^{\infty}\gamma(m_j)\in\tilde{\Gamma}$, where the $m_j=m_j[T]$ are the inner functions in the Jordan model of T. (Cf. Theorem 4.1 and Proposition 6.6 of [1].) If T is an operator of class C_0 and $\mathfrak{M}\in\operatorname{Lat}_{\frac{1}{2}}(T)$ is such that $T_{\mathfrak{M}}\in\mathscr{P}$, then $\gamma_T(\mathfrak{M})$ is defined as $\gamma_T(\mathfrak{M})=\gamma_{T_{\mathfrak{M}}}$.

For two operators T and T' we denote by $\mathscr{I}(T', T)$ the set of intertwining operators $\mathscr{I}(T', T)=\{X|T'X=XT\}$. If $T'=T$, then $\mathscr{I}(T, T)=\{T\}'$ is the commutant of T.

The next Lemmas will be frequently used in the sequel.

Lemma 1. *Let $\{m_i\}_{i=0}^{\infty}$ be a sequence of pairwise relatively prime inner functions having a least common multiple m. Then the operator $T=\bigoplus_{i=0}^{\infty}S(m_i)$ is quasi-similar to $S(m)$.*

Proof. Cf. Theorem 2.7 of [4].

Lemma 2. *Let m_1, m_2 be inner functions.*

(i) *If m_2 divides m_1 $(m_1\geqq m_2)$ and $Xu=P_{\mathfrak{H}(m_2)}u$ for all $u\in\mathfrak{H}(m_1)$, then $X\in\mathscr{I}(S(m_2), S(m_1))$ is surjective and $S(m_1)|\ker X$ is unitarily equivalent to $S\left(\frac{m_1}{m_2}\right)$ $\left(S(m_1)|\ker X\cong S\left(\frac{m_1}{m_2}\right)\right)$.*

(ii) *If $m_1\leqq m_2$ and $Xu=\frac{m_2}{m_1}u$ for all $u\in\mathfrak{H}(m_1)$, then $X\in\mathscr{I}(S(m_2), S(m_1))$ is injective and $S(m_2)_{\ker X^*}\cong S\left(\frac{m_2}{m_1}\right)$.*

Proof. We can easily verify this statement by a short computation.

Lemma 3. (Proposition 4.6 of [1]) *Let T be an operator of class C_0 acting on $\mathfrak{H}$ and let $\mathfrak{H}_j \in \mathrm{Lat}(T)$ be such that $\mathfrak{H}_j \subset \mathfrak{H}_{j+1}$ $(j=1, 2, \dots)$, and $\mathfrak{H} = \bigvee_{j=1}^{\infty} \mathfrak{H}_j$. Then $T \in \mathcal{P}$ if and only if $T_{\mathfrak{K}_j} \in \mathcal{P}$, $\mathfrak{K}_j = \mathfrak{H}_{j+1} \ominus \mathfrak{H}_j$ $(j=0, 1, 2, \dots;\ \mathfrak{H}_0 = \{0\})$ and $\bigwedge_{j=1}^{\infty} m_0[T_{\mathfrak{H}_j^{\perp}}] = 1$.* (If S is an operator of class C_0, then $m_0[S]$ denotes its minimal function.)

3. Firstly we shall prove the statement of the Theorem in different special cases in the Propositions 1 and 2, from which the general situation can be derived. We remark that it can be always supposed that T_1 and T_2 are Jordan operators. In the proofs of Propositions 1 and 2 we shall need the next Lemma.

Let us denote by $\sum$ the set of injections $\sigma\colon N \to N \cup (-N) = \hat{N}$ satisfying the conditions:

(i) if $1 \leqq i < j$ and $\sigma(i)\sigma(j) \geqq 0$, then $|\sigma(i)| < |\sigma(j)|$;

(ii) if $r \in \sigma(N)$, then for all $s \in \hat{N}$ such that $s \cdot r \geqq 0$ and $|s| < |r|$ we have $s \in \sigma(N)$.

(Here and in the sequel N is the set of natural numbers $1, 2, \dots$.) Let $\mathcal{G}$ be the set of sequences: $a = \{a_n\}_{n=1}^{\infty}$ of real numbers such that $a_1 \geqq a_2 \geqq \dots \geqq 0$ and $a_n \to 0$ as $n \to \infty$. If $a, b \in \mathcal{G}$, then let $F_{(a,b)}$ denote the mapping $\hat{N} \to R$ defined by

$$F_{(a,b)}(i) = \begin{cases} a_i, & \text{if } i \in N, \\ -b_i, & \text{if } i \in (-N). \end{cases}$$

Lemma 4. *Let $a, b \in \mathcal{G}$ satisfy the condition: if $b_n = 0$ for some $n \in N$, then there exists $m \in N$ such that $a_m = 0$. If $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} b_n$, then there exists a $\sigma \in \sum$ such that for all $n \in N$ we have*

$$0 \leqq \sum_{i=1}^{n} F_{(a,b)}(\sigma(i)) \leqq 2 \max(a_1, b_1)$$

furthermore $\sum_{i=1}^{n} F_{(a,b)}(\sigma(i))$ tends to 0, if n tends to ∞.

Proof. Let $\sigma(1) = 1$. If we have already defined σ for $i = 1, 2, \dots, j$, and $\max\{\sigma(i) | i = 1, \dots, j\} = r_j$, $\min\{\{\sigma(i) | i = 1, \dots, j\} \cup \{0\}\} = -s_j$, then

$$\sigma(j+1) := \begin{cases} -(s_j+1) & \text{if } \sum_{i=1}^{j} F_{(a,b)}(\sigma(i)) \geqq b_{s_j+1}, \\ r_j + 1 & \text{otherwise.} \end{cases}$$

It can be easily seen that this $\sigma \in \sum$ will be suitable.

Proposition 1. *If $T_1, T_2 \in \mathcal{P}$ are such that $\gamma_{T_1} = \gamma_{T_2} = \gamma$ and γ has the form $\gamma = (\sigma, 0)$, then $T_1 \varrho T_2$.*

Proof. Let T_1 and T_2 be the Jordan operators $T_1=\bigoplus_{n=1}^{\infty} S(u_n)$ and $T_2=\bigoplus_{n=1}^{\infty} S(v_n)$. From the assumption it follows that u_1 and v_1 are Blaschke products having the same zeros (disregarding multiplicities): $\lambda_1, \lambda_2, \dots$. For all n u_n and v_n have factorizations $u_n=\prod_{l=1}^{\infty} u_{n,l}$, $v_n=\prod_{l=1}^{\infty} v_{n,l}$, where $u_{n,l}$ and $v_{n,l}$ are Blaschke factors containing only λ_l as a zero. (If λ_l is not a zero of u_n (v_n), then $u_{n,l}:=1$ $(v_{n,l}:=1)$.)

Let us denote by $a_n^{(l)}$ and $b_n^{(l)}$ the multiplicities of λ_l as zero of $v_{n,l}$ and of $u_{n,l}$, respectively. Then $a_l=\{a_n^{(l)}\}_{n=1}^{\infty}$, $b_l=\{b_n^{(l)}\}_{n=1}^{\infty}\in\mathcal{G}$, and by virtue of $\gamma_{T_1}=\gamma_{T_2}$ we have $\sum_{n=1}^{\infty} a_n^{(l)}=\sum_{n=1}^{\infty} b_n^{(l)}$ for all $l\in N$.

By Lemma 4 there exists a $\sigma_l\in\sum$ such that

$$0\leqq \sum_{i=1}^{j} F_{(a_l,b_l)}(\sigma_l(i))\leqq 2\max(a_1^{(l)}, b_1^{(l)})$$

for all $j\in N$. Let $c_j^{(l)}$ be defined by $c_j^{(l)}=\sum_{i=1}^{j} F_{(a_l,b_l)}(\sigma_l(i))$, and let

$$w_j^{(l)}(z):=\begin{cases}\left(\dfrac{\bar\lambda_l}{|\lambda_l|}\dfrac{\lambda_l-z}{1-\bar\lambda_l z}\right)^{c_j^{(l)}} & \text{if } \lambda_l\neq 0,\\ z^{c_j^{(l)}} \quad \text{if } \lambda_l=0; & j\in N,\ z\in D.\end{cases}$$

It is clear that $w_j^{(l)}=1$, if j is large enough. So the operator T_l defined by $T_l=\bigoplus_{j=1}^{\infty} S(w_j^{(l)})$ has finite multiplicity. On the other hand by the construction it follows that $m_0[T_l]\leqq (u_{1,l}\vee v_{1,l})^2$.

Let X_l be the contraction defined by $X_l(\bigoplus_{j=1}^{\infty} f_j)=\bigoplus_{j=1}^{\infty} g_j$, where $\bigoplus_{j=1}^{\infty} f_j$, $\bigoplus_{j=1}^{\infty} g_j\in\bigoplus_{j=1}^{\infty}\mathfrak{H}(w_j^{(l)})$ and $g_1=0$,

$$g_j=\begin{cases}P_{\mathfrak{H}(w_j^{(l)})}f_{j-1} & \text{if } w_{j-1}^{(l)}\geqq w_j^{(l)},\\ \dfrac{w_j^{(l)}}{w_{j-1}^{(l)}}f_{j-1} & \text{if } w_{j-1}^{(l)}\leqq w_j^{(l)} \quad \text{for } j\geqq 2.\end{cases}$$

By Lemma 2 we infer that $X_l\in\{T_l\}'$ and $T_l|\ker X_l\cong\bigoplus_{n=1}^{\infty} S(u_{n,l})$, $(T_l)_{\ker X_l^*}\cong\bigoplus_{n=1}^{\infty} S(v_{n,l})$.

Since $\bigwedge_{j=1}^{\infty} m_0[\bigoplus_{l=j}^{\infty} T_l]\leqq\bigwedge_{j=1}^{\infty}(\prod_{l=j}^{\infty}(u_{1,l}\vee v_{1,l})^2)=1$, by Lemma 3 we see that $T=\bigoplus_{l=1}^{\infty} T_l\in\mathcal{P}$. Then $X=\bigoplus_{l=1}^{\infty} X_l\in\{T\}'$ and using Lemma 1 we get

$$T|\ker X=\bigoplus_{l=1}^{\infty} T_l|\ker X_l\cong\bigoplus_{l=1}^{\infty}\left(\bigoplus_{n=1}^{\infty} S(u_{n,l})\right)\cong\bigoplus_{n=1}^{\infty}\left(\bigoplus_{l=1}^{\infty} S(u_{n,l})\right)\sim\bigoplus_{n=1}^{\infty} S(u_n)=T_1$$

and similarly $T_{\ker X^*}\sim T_2$. Therefore, $T_1\varrho T_2$ and Proposition 1 is proved.

Proposition 2. *If* $T_1, T_2 \in \mathcal{P}$ *are such that* $\gamma_{T_1} = \gamma_{T_2} = \gamma$ *and* γ *has the form* $\gamma = (0, \mu)$, *then* $T_1 \varrho T_2$.

Proof.

(i) Let T_1 and T_2 be the Jordan operators $T_1 = \bigoplus_{n=1}^{\infty} S(u_n)$ and $T_2 = \bigoplus_{n=1}^{\infty} S(v_n)$. From the assumption it follows that there exist a finite Borel measure ν in $[0, 2\pi]$, singular with respect to Lebesgue measure, and non-increasing sequences $\{f_n\}_{n=1}^{\infty}$, $\{g_n\}_{n=1}^{\infty}$ of non-negative Borel functions from $L^1(\nu)$ which are tending to 0 and such that

$$\operatorname{Exp}[f_n] = u_n \quad \text{and} \quad \operatorname{Exp}[g_n] = v_n \quad \text{for all } n.$$

Here and in the sequel we use the notations

$$\operatorname{Exp}[f, E](z) = \exp\left[-\int_E \frac{e^{it}+z}{e^{it}-z} f(t)\, d\nu(t)\right] \quad (z \in D), \text{ and } \operatorname{Exp}[f] = \operatorname{Exp}[f, [0, 2\pi]],$$

for any non-negative Borel function $f \in L^1(\nu)$, and measurable set $E \subset [0, 2\pi]$.

Therefore we see that $f(t) = \{f_n(t)\}_{n=1}^{\infty}$, $g(t) = \{g_n(t)\}_{n=1}^{\infty} \in \mathcal{G}$ for all t in $[0, 2\pi]$. Furthermore we can assume that

$$\sum_{n=1}^{\infty} f_n(t) = \sum_{n=1}^{\infty} g_n(t) \quad \text{for all } t \text{ in } [0, 2\pi].$$

(ii) Let E be the measurable set of points t in $[0, 2\pi]$ such that $a = g(t)$ and $b = f(t)$ satisfy the assumptions of Lemma 4. If $t \in E$ let $\sigma_t \in \sum$ be the function constructed in the proof of Lemma 4 taking $a = g(t)$ and $b = f(t)$. For all $j \in N$ let $h_j \in L^1(\nu)$ be the measurable function defined by

$$h_j(t) = \begin{cases} \sum_{i=1}^{j} F_{(g(t), f(t))}(\sigma_t(i)) & \text{if } t \in E, \\ 0 & \text{otherwise.} \end{cases}$$

By Lemma 4 we infer that

$$0 \leqq h_j(t) \leqq 2 \max (f_1(t), g_1(t))$$

for all $j \in N$, $t \in [0, 2\pi]$, and

$$\lim_{j \to \infty} h_j(t) = 0 \quad \text{for all} \quad t \in [0, 2\pi].$$

Introducing the inner functions $\{w_j\}_{j=1}^{\infty}$ by $w_j = \operatorname{Exp}[h_j]$, we consider the operator $\bigoplus_{j=1}^{\infty} S(w_j)$.

(iii) We shall show that $\bigoplus_{j=1}^{\infty} S(w_j) \in \mathcal{P}$. By Lemma 3 it is enough to prove that $m = \bigwedge_{k=1}^{\infty} m_0[\bigoplus_{j=k}^{\infty} S(w_j)] = 1$.

Let ε be an arbitrary positive number. There exists a positive δ such that if H is a Borel set and $\nu(H)<\delta$, then $\int_H 2\max\{f_1(t), g_1(t)\}d\nu(t)<\varepsilon$. By Egorov's theorem we infer that there exists a Borel set H_ε such that $\nu(H_\varepsilon)<\delta$ and the sequence $\{h_j\}_{j=1}^{\infty}$ converges uniformly to zero on the complement $CH_\varepsilon=[0,2\pi]\setminus H_\varepsilon$. So there exists a k_0 such that for all $j>k_0$ and $t\in CH_\varepsilon$ we have $h_j(t)<\varepsilon$. Therefore if $j>k_0$, then for all $t\in[0,2\pi]$ we have $h_j(t)\leqq\tilde{h}_\varepsilon(t)$, where $\tilde{h}_\varepsilon$ is the function defined by

$$\tilde{h}_\varepsilon(t)=\begin{cases}\varepsilon & \text{if } t\in CH_\varepsilon,\\ 2\max\{f_1(t), g_1(t)\} & \text{if } t\in H_\varepsilon.\end{cases}$$

We infer that the inner function m satisfies the inequality

$$m\leqq \operatorname{Exp}[\tilde{h}_\varepsilon].$$

Therefore we have

$$|m(0)|\geqq|\operatorname{Exp}[\tilde{h}_\varepsilon](0)|=\exp\Big[-\int_0^{2\pi}\tilde{h}_\varepsilon(t)\,d\nu(t)\Big]=$$

$$=\exp\Big[-\int_{H_\varepsilon}\tilde{h}_\varepsilon(t)\,d\nu(t)-\int_{CH_\varepsilon}\tilde{h}_\varepsilon(t)\,d\nu(t)\Big]\geqq\exp[-\varepsilon-\varepsilon\cdot\nu([0,2\pi])].$$

Since ε can be chosen arbitrary small, so $|m(0)|=1$. That is, $m=1$.

(iv) Let $E_{j,i}$ denote the measurable subset of E defined by

$$E_{j,i}=\{t\in E:\ \sigma_t(j+1)=i\}$$

for all $j\in N$ and $i\in\hat{N}$. Then $\{E_{j,i}\}_{j\in N, i\in\hat{N}}$ will be a system of subsets of E such that the systems $\{E_{j,i}\}_{i\in\hat{N}}$ and $\{E_{j,i}\}_{j\in N}$ consist of pairwise disjoint sets for all fixed $j\in N$ and $i\in\hat{N}$, respectively; furthermore $\bigcup_{i\in\hat{N}}E_{j,i}=E$ for all $j\in N$, $(\bigcup_{j\in N}E_{j,i})\supset\{t\in E|g_i(t)>0\}$ if $i\in N$ and $(\bigcup_{j\in N}E_{j,i})\supset\{t\in E|f_i(t)>0\}$ if $i\in(-N)$.

For all $j\in N$ let S_j be the operator defined by $S_j=S_{j,1}\oplus S_{j,2}$, where $S_{j,1}=\bigoplus_{i\in\hat{N}}S(\operatorname{Exp}[h_j, E_{j,i}])$ and $S_{j,2}=\bigoplus_{i\in\hat{N}}S(\operatorname{Exp}[h_{j+1}, E_{j,i}])$.

By Lemma 1 we infer that $S_{j,1}$ and $S_{j,2}$ are quasisimilar to $S(w_j)$ and $S(w_{j+1})$, respectively, for all $j\in N$. Therefore the operator $S=\bigoplus_{j=1}^{\infty}S_j$ is quasisimilar to the operator $(\bigoplus_{j=1}^{\infty}S(w_j))\oplus(\bigoplus_{j=2}^{\infty}S(w_j))$, which belongs to $\mathscr{P}$ by section (iii) and Proposition 4.4 of [1]. By Corollary 4.3 of [1] we see that $S\in\mathscr{P}$.

Since $S_{j,2}$ is quasisimilar to $S_{j+1,1}$, there exists a quasiaffinity $Y_j\in\mathscr{I}(S_{j+1,1}, S_{j,2})$ $(j\in N)$. We may assume that Y_j is a contraction.

For all $j\in N$, $i\in \hat{N}$ let

$$Z_{j,i}\in \mathcal{I}(S(\mathrm{Exp}\,[h_{j+1}, E_{j,i}]), S(\mathrm{Exp}\,[h_j, E_{j,i}]))$$

be the operator defined by

$$Z_{j,i}\,m = \begin{cases} \dfrac{\mathrm{Exp}\,[h_{j+1}, E_{j,i}]}{\mathrm{Exp}\,[h_j, E_{j,i}]}\,m & \text{if } i\in N, \\ P_{\mathfrak{H}(\mathrm{Exp}\,[h_{j+1}, E_{j,i}])}\,m & \text{if } i\in(-N), \end{cases}$$

where $m\in \mathfrak{H}(\mathrm{Exp}\,[h_j, E_{j,i}])$.

Then for all $j\in N$ we infer that $Z_j=\bigoplus_{i\in\hat{N}} Z_{j,i}\in\mathcal{I}(S_{j,2}, S_{j,1})$.

Let $X\in\{S\}'$ be the operator defined by

$$X|\bigoplus_{i\in\hat{N}} \mathfrak{H}(\mathrm{Exp}\,[h_j, E_{j,i}]) = Z_j \quad\text{and}\quad X|\bigoplus_{i\in\hat{N}} \mathfrak{H}(\mathrm{Exp}\,[h_{j+1}, E_{j,i}]) = Y_j$$

for all $j\in N$.

Then by Lemmas 1 and 2 we infer

$$S|\ker X \cong \bigoplus_{j=1}^{\infty}\left(\bigoplus_{i=1}^{\infty} S(\mathrm{Exp}\,[f_i, E_{j,i}])\right) \cong \bigoplus_{i=1}^{\infty}\left(\bigoplus_{j=1}^{\infty} S(\mathrm{Exp}\,[f_i, E_{j,i}])\right) \sim$$

$$\sim \bigoplus_{i=1}^{\infty} S(\mathrm{Exp}\,[f_i, E]), \quad \text{and similarly,}$$

$$S_{\ker X^*} \sim \bigoplus_{i=1}^{\infty} S(\mathrm{Exp}\,[g_i, E]).$$

(v) It is clear that for all $t\in CE=[0, 2\pi]\setminus E$ we have that $a=f(t)$ and $b=g(t)$ satisfy the assumptions of Lemma 4. Replacing E, $f_n(t)$, $g_n(t)$, $dv(t)$ by $(CE)^{\tilde{}}=\{t\in[0, 2\pi]|2\pi-t\in CE\}$, $g_n(2\pi-t)$, $f_n(2\pi-t)$ and $dv(2\pi-t)$, respectively, we repeat the reasoning of the sections (ii), (iii) and (iv). Also taking adjoints we get that there exist operators $R\in\mathcal{P}$ and $Y\in\{R\}'$ such that

$$R|\ker Y \sim \bigoplus_{i=1}^{\infty} S(\mathrm{Exp}\,[f_i, CE]) \quad\text{and}\quad R_{\ker Y^*} \sim \bigoplus_{i=1}^{\infty} S(\mathrm{Exp}\,[g_i, CE]).$$

Therefore, the operator $T=S\oplus R$ will belong to $\mathcal{P}$, $Z=X\oplus Y\in\{T\}'$, and by Lemma 1

$$T|\ker Z \sim \bigoplus_{i=1}^{\infty} S(\mathrm{Exp}\,[f_i]) = T_1, \quad T_{\ker Z^*} \sim \bigoplus_{i=1}^{\infty} S(\mathrm{Exp}\,[g_i]) = T_2.$$

That is, $T_1\,\varrho\,T_2$ and the Proposition 2 is proved.

Proof of the Theorem. Let T_1 and T_2 be the Jordan operators $T_1=\bigoplus_{n=1}^{\infty} S(u_n)$ and $T_2=\bigoplus_{n=1}^{\infty} S(v_n)$. The inner functions u_n, v_n have canonical factorizations $u_n=u_{n,1}\cdot u_{n,2}$, $v_n=v_{n,1}\cdot v_{n,2}$, where $u_{n,1}$, $v_{n,1}$ are Blaschke products, $u_{n,2}$, $v_{n,2}$ are singular inner functions for all $n\in N$. Introducing the operators $T_{1,i}=\bigoplus_{n=1}^{\infty} S(u_{n,i})$ and $T_{2,i}=\bigoplus_{n=1}^{\infty} S(v_{n,i})$ $(i=1,2)$ we infer by Propositions 1 and 2 that $T_{1,1}\varrho T_{2,1}$ and $T_{1,2}\varrho T_{2,2}$. Taking direct sums and using Lemma 1 we see that $T_1\varrho T_2$. The proof is done.

4. By this Theorem and Theorem 7.9 of [1] we infer:

Corollary 1. *For $T_1, T_2\in\mathcal{P}$ we have $T_1\varrho T_2$ if and only if $\gamma_{T_1}=\gamma_{T_2}$.*

We list some immediate consequences of this Corollary.

Corollary 2. *ϱ is an equivalence relation on $\mathcal{P}$.*

Corollary 3. *Let us suppose that $T_i\in\mathcal{P}$, $\mathfrak{H}_i\in\operatorname{Lat}(T_i)$ and $\gamma_{T_i}(\mathfrak{H}_i)=(\sigma_i,\mu_i)$, where μ_i is σ-finite $(i=1,2)$. If $T_1\varrho T_2$ and $(T_1|\mathfrak{H}_1)\varrho(T_2|\mathfrak{H}_2)$, then $(T_1)_{\mathfrak{H}_1^\perp}\varrho(T_2)_{\mathfrak{H}_2^\perp}$.*

Proof. This follows from Corollary 7.10 and Lemma 6.5 of [1], and from the above Corollary 1.

Corollary 4. *Let T, S be operators of class $\mathcal{P}$ acting on the spaces $\mathfrak{H}$ and $\mathfrak{K}$, respectively, and let $\mathfrak{H}_j\in\operatorname{Lat}(T)$, $\mathfrak{K}_j\in\operatorname{Lat}(S)$ be such that $\mathfrak{H}_j\subset\mathfrak{H}_{j+1}$, $\mathfrak{K}_j\subset\mathfrak{K}_{j+1}$ $(j=1,2,\ldots)$ and $\bigvee_{j=1}^{\infty}\mathfrak{H}_j=\mathfrak{H}$, $\bigvee_{j=1}^{\infty}\mathfrak{K}_j=\mathfrak{K}$. If $(T|\mathfrak{H}_j)\varrho(S|\mathfrak{K}_j)$ for all $j=1,2,\ldots$, then $T\varrho S$.*

Proof. This follows from Corollary 1 and Lemma 7.4 of [1].

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