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Contributions to the ideal theory of semigroups

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Let S be a semigroup. A subsemigroup A of S is said to be an *(m, n)-ideal* of S if the inclusion $A^m SA^n \subseteq A$ holds, where m, n are non-negative integers, A^0 is the empty symbol. The author [4] proved that the product of two (1, 1)-ideals of S is again a (1, 1)-ideal. Thus the collection of all (1, 1)-ideals of a semigroup S is a semigroup with respect to the ordinary set product. This semigroup will be denoted by $\mathbf{B}(S)$. Also, the collection of all left [right] ideals of S is a multiplicative semigroup. This semigroup will be denoted by $\mathbf{L}(S)[\mathbf{R}(S)]$. It is easy to see, that $\mathbf{L}(S)$ is a right ideal and $\mathbf{R}(S)$ is a left ideal of $\mathbf{B}(S)$. Their intersection, the multiplicative semigroup of all two-sided ideals of S is a quasi-ideal of $\mathbf{B}(S)$.

In this short note certain classes of semigroups will be characterized by properties of the semigroups $\mathbf{B}(S)$ and $\mathbf{L}(S)$. For the undefined notions and notations we refer to [1], [2], and [12].

We begin with two lemmas.

Lemma 1. *A semigroup S is regular if and only if $BSB=B$ holds for every bi-ideal B of S.*

This is an easy consequence of a result by J. Luh [10].

Lemma 2. *A semigroup S is a semilattice of groups if and only if the intersection of any two* (1, 1)*-ideals of S is equal to their product.*

For this criterion, see the author [5] or [6].

Our first main result is contained in the following

Theorem 1. *For a semigroup S the following conditions are equivalent:*

(1) *S is a semilattice of groups.*

(2) $\mathbf{B}(S)$ *is a distributive lattice with respect to the set product and the set-theoretical union.*

(3) $\mathbf{B}(S)$ *is a regular monoid with respect to the set product.*

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Proof. (1)$\Rightarrow$(2): If S is a semigroup which is a semilattice of groups, then every bi-ideal of S is a two-sided ideal of S. Hence this implication is straightforward by Lemma 2.

(2)$\Rightarrow$(1): by Theorem 1 of [6].

(1)$\Rightarrow$(3) is obvious.

(3)$\Rightarrow$(1): Suppose that S is a semigroup whose bi-ideal semigroup $\mathbf{B}(S)$ is a regular monoid with respect to set product. If A is the identity element of $\mathbf{B}(S)$, we have $S=ASA\subseteqq A$. Hence $A=S$. Therefore $BS=SB=B$ holds for any bi-ideal B of S, whence B is a two-sided ideal of S. On the other hand, the regularity of $\mathbf{B}(S)$ together with Lemma 1 implies that S is regular. Thus S is a regular duo semigroup which is a semilattice of groups.

Corollary 1. *If S is a semilattice, then $\mathbf{B}(S)$ is a distributive lattice. In particular, if S is a diagonal semilattice (i.e., every non-zero element of S is an atom), then $\mathbf{B}(S)$ is a Boolean algebra.*

Corollary 2. *The bi-ideal semigroup $\mathbf{B}(S)$ of a semigroup S is a Boolean algebra if and only if S is a diagonal semilattice of groups.*

The following criterion is due to the author [7].

Lemma 3. *A semigroup S is a semilattice of left groups if and only if $B\cap L=BL$ holds for every bi-ideal B and every left ideal L of S.*

By making use of Lemma 3, further characterizations can be given for semigroups that are semilattices of left groups in term of the bi-ideal semigroup $\mathbf{B}(S)$.

Theorem 2. *For a semigroup S the following conditions are equivalent:*

(1) *S is a semilattice of left groups.*

(2) *$\mathbf{B}(S)$ is a band and S is a right identity of it.*

(3) *$\mathbf{B}(S)$ is a regular semigroup and S is a right identity of it.*

Proof. (1)$\Rightarrow$(2): If S is a semigroup which is a semilattice of left groups, then, by Lemma 3, the relation $L\cap R=RL$ holds for every left ideal L and every right ideal R of S, thus S is regular. Moreover Lemma 3 implies $BS=B$ for every bi-ideal B of S, whence S is a right identity of $\mathbf{B}(S)$. Then Lemma 1 implies that every bi-ideal of S is globally idempotent, i.e., $\mathbf{B}(S)$ is a band.

(2)$\Rightarrow$(3) is clear.

(3)$\Rightarrow$(1): If (3) holds, then it follows that S is a regular left duo semigroup which is a semilattice of left groups.

T. Saitô [11] has proved the following criterion.

Lemma 4. *A semigroup S is a semilattice of left simple semigroups if and only if the intersection of any two left ideals of S is equal to their product.*

Now we are ready to prove the following result.

Theorem 3. *For a semigroup S the following conditions are equivalent:*

(1) *S is a semilattice of left simple semigroups.*

(2) $\mathbf{L}(S)$ *is a distributive lattice with respect to the set product and set-theoretical union.*

(3) $\mathbf{L}(S)$ *is a multiplicative semilattice.*

Proof. (1)$\Rightarrow$(2): If S is a semilattice of left simple semigroups, then, by Lemma 4, every left ideal of S is a two-sided ideal. Hence the implication follows by Lemma 4.

(2)$\Rightarrow$(3) is obvious. Finally, (3)$\Rightarrow$(1) by [11].

Next an ideal-theoretical characterization will be given for homogroups. A semigroup S is called a *homogroup* if it has a subgroup which is at the same time a two-sided ideal of S (for an equivalent definition see [13]). For instance, a semigroup with zero element is a homogroup.

Theorem 4. *A semigroup S is a homogroup if and only if the bi-ideal semigroup* $\mathbf{B}(S)$ *has a zero element.*

Proof. Let S be a homogroup with the group-ideal G. Let B be a bi-ideal of S. Then the product BG is a right ideal of S, and $BG \subseteqq G$. Hence it follows that $BG=G$, because a group has no proper right ideals. Similarly, we get $GB=G$ and G is the zero element of $\mathbf{B}(S)$.

Conversely, if S is a semigroup whose bi-ideal semigroup has a zero element Z, then we have $SZ=ZS=Z$. Hence Z is a two-sided ideal of S. For any element z of Z the product Zz is a left ideal of S. Thus we have $Z=Z(Zz)=Zz$, since the set product is associative for non-empty subsets of S. Similarly we get $zZ=Z$ for any element z of Z. Therefore Z is a subgroup of S, and S is a homogroup, indeed.

Remark. It is easy to see that Theorem 4 remains true with $\mathbf{P}(S)$ instead of $\mathbf{B}(S)$, where $\mathbf{P}(S)$ is the power semigroup of S, i.e., the multiplicative semigroup of all non-empty subsets of S.

Finally, we are interested in semigroups whose bi-ideal semigroup is a monoid.

Theorem 5. *For a semigroup S the bi-ideal semigroup* $\mathbf{B}(S)$ *is a monoid if and only if* (i) *every bi-ideal of S is a two-sided ideal of S, and* (ii) *every two-sided ideal of S is complete (i.e. $IS=SI=I$).*

Proof. First, let S be a semigroup having properties (i), (ii). Then the bi-ideal semigroup $\mathbf{B}(S)$ is the multiplicative monoid of all two-sided ideals of S with the identity S.

Secondly, if the (1, 1)-ideals of a semigroup S form a monoid with identity A, then we have $S=ASA\subseteqq A$, whence it follows that $A=S$. Thus $BS=SB=B$ for every bi-ideal B of S, that is, every (1, 1)-ideal B is a complete (two-sided) ideal of S. Theorem 5 is completely proved.

For the characterizations of completely regular semigroups in terms of (m, n)-ideals, see the author [8] and [9].

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