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Selecting independent lines from a family of lines in a space

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0. Introduction. A family of flats in a projective space is called *independent*, if no member of the family intersects the flat spanned by the other members. It is an interesting combinatorial problem to select a maximum number of independent flats from a given family of flats. In the special case when all the flats are faces of a simplex, this question is equivalent to the so-called *matching problem* for hypergraphs: given a collection of sets, find the maximum number of disjoint ones among them. This problem is known to belong to the class of (in a sense) hardest combinatorial problems, the so-called NP-complete problems (see [6]). Hence there is no hope to solve it in a satisfactory way.

However, the special case of the matching problem when all the given sets are pairs, is well-solved [2, 4]. This suggests that probably the problem of selecting a maximum set of independent lines from a family of lines is solvable.

"Solution" here may mean two different things:

(a) find a minimax formula for the number in question;

(b) find an algorithm to determine this number such that the running time of the algorithm is polynomial in the number of data.

We shall present a solution in the sense of (a) (Theorem 2). It remains open if these methods can be extended (or other methods found) to yield a solution in the sense (b), but we hope the answer is affirmative. The problem we discuss can be considered as the so-called "matroid parity problem" for representable matroids (see LAWLER [3], Ch. 9). We shall discuss the difficulties of generalizing our methods, along with other connections to matroid theory in section 5.

1. Some special cases and equivalents. The famous *f-factor problem*, solved by TUTTE [5], is the following. Let G be a graph and f an integer-valued function on its vertex set $V(G)$. Does there exist a subgraph G' such that the degree of x in G' is $f(x)$, for every $x \in V(G)$?

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This problem can be reduced to the line-selection problem as follows. Let, for each $x \in V(G)$, A_x be a flat of rank* $f(x)$ in a projective space, such that the flats $\{A_x: x \in V(G)\}$ are independent. For each edge $e=(x, y)$, select two points $p_{e,x} \in A_x$ and $p_{e,y} \in A_y$ such that the points $\{p_{e,x}$: e is a line adjacent to $x\}$ are in general position on A_x (i.e. no $f(x)$ of them are contained in a proper subflat of A_x). Let $\hat{e}$ denote the line connecting $p_{e,x}$ to $p_{e,y}$. Then it is easy to verify that

a subgraph H has degree $\leqq f(x)$ at each point x iff the lines $\{\hat{e}: e \in E(H)\}$ are independent.

So G has an f-factor iff the family $\{\hat{e}: e \in E(G)\}$ contains $\frac{1}{2} \sum_x f(x)$ independent lines. Our results therefore yield a necessary and sufficient condition for the existence of an f-factor in a graph. Although our condition has features similar to Tutte's, to derive Tutte's theorem from it is somewhat lengthy.

We may place the points $p_{e,x}$ on A_x in such a way that they form an arbitrary matroid embeddable in the projective space we consider. This yields then a solution to the "matchoid problem" of Edmonds in the special case when the matroids prescribed at the vertices are representable.

Finally, we mention an equivalent version of our problem. Let $\mathcal{H}$ be a collection of lines which spans a rank r projective space P. Let $\nu(\mathcal{H})$ be the maximum number of independent lines in $\mathcal{H}$ and $\mu(\mathcal{H})$ the minimum number of lines in $\mathcal{H}$ which still span P. Then $\nu(\mathcal{H})+\mu(\mathcal{H})=r$. This identity is a generalization of Gallai's identity in graph theory, and can be proved along the same lines. So we also have a minimax formula for the minimum number of lines in a family which span the same flat as the whole family. The transformation of Theorem 2 to this version is left to the reader.

2. Preliminaries. Let P be a projective geometry over a (possibly skew) field. We shall denote by $\overline{X}$ the *span* of the set $X \subseteq P$, i.e. the smallest flat (subspace) containing X. Each flat A in P has a rank $r(A)$, which is one larger than its dimension. So $\emptyset$ has rank 0, points have rank 1, lines have rank 2. We extend the notation of rank over arbitrary subsets of P by $r(X)=r(\overline{X})$ and even over a collection $\mathcal{H}$ of subsets of P by $r(\mathcal{H})=r(\cup \mathcal{H})$. Similarly, if $\mathcal{H}$ is a collection of subsets of P we set $\overline{\mathcal{H}} = \overline{\cup \mathcal{H}}$.

The rank satisfies the important identity

$$r(A \cup B)+r(A \cap B) = r(A)+r(B),$$

where A, B are flats.

* The rank $r(A)$ of a projective space A is its dimension plus 1.

We shall make use of the following very simple lemma:

Lemma 1. *Let* $A_1, \dots, A_k$, D *be flats in a projective geometry and* $A_i \subseteqq D$. *Assume that*

$$\sum_{i=1}^{k} \{r(D)-r(A_i)\} < r(D).$$

Then $\bigcap_{i=1}^{k} A_i \neq \emptyset$.

Proof. We show by induction on k that

$$r(A_1 \cap \dots \cap A_k) \geqq r(D) - \sum_{i=1}^{k} \{r(D)-r(A_i)\}.$$

This is trivially true if $k=1$. Let $k \geqq 2$. Then

$$r(A_1 \cap \dots \cap A_k) = r(A_1 \cap \dots \cap A_{k-1}) + r(A_k) - r((A_1 \cap \dots \cap A_{k-1}) \cup A_k) \geqq$$
$$\geqq r(A_1 \cap \dots \cap A_{k-1}) + r(A_k) - r(D).$$

Applying the induction hypothesis the assertion follows.

Q.E.D.

Recall that a set of lines in a projective geometry is called *independent* if no member of the set meets the span of the rest. It is immediately seen that each subset of an independent set is independent.

Lemma 2. *Let* $\mathscr{F}$ *be a set of lines in a projective space. Then*

$$r(\mathscr{F}) \leqq 2|\mathscr{F}|. \tag{1}$$

Equality holds iff $\mathscr{F}$ *is independent.*

Proof. Let $e \in \mathscr{F}$. Then

$$r(\mathscr{F}) = r(e) + r(\mathscr{F}-e) - r(e \cap \overline{\mathscr{F}-e}) = r(\mathscr{F}-e) + 2 - r(e \cap \overline{\mathscr{F}-e}).$$

Hence the inequality (1) follows by induction. If $\mathscr{F}$ is independent, then clearly so is $\mathscr{F}-e$ and then equality in (1) follows by induction. On the other hand, if equality holds in (1) then the computation above implies that $r(e \cap \overline{\mathscr{F}-e})=0$, i.e. $e \cap \overline{\mathscr{F}-e} = \emptyset$. Since this holds for every $e \in \mathscr{F}$, it follows that $\mathscr{F}$ is independent.

Q.E.D.

A set $\mathscr{C}$ of lines is called a *circuit*, if $r(\mathscr{C}) = 2|\mathscr{C}|-1$ but every proper subset of $\mathscr{C}$ is independent. Thus a circuit is a minimal dependent set of lines; but not every minimal dependent set of lines is a circuit, as shown by 3 lines in general position in the space.

Note that if $\mathscr{C}$ is a circuit and $e\in\mathscr{C}$ then

$$2\,|\mathscr{C}|-1 = r(\mathscr{C}) = r(\mathscr{C}-e)+2-r(e\cap\overline{\mathscr{C}-e}) = 2|\mathscr{C}|-r(e\cap\overline{\mathscr{C}-e}),$$

whence it is seen that e meets $\overline{\mathscr{C}-e}$ in exactly one point.

Lemma 3. *Let $\mathscr{K}$ be a set of lines such that $r(\mathscr{K})=2|\mathscr{K}|-1$. Then $\mathscr{K}$ contains exactly one circuit.*

Proof. Let $\mathscr{C}$ be a minimal subset of $\mathscr{K}$ with $r(\mathscr{C})=2|\mathscr{C}|-1$. We claim that all proper subsets of $\mathscr{C}$ are independent. In fact,

$$r(\mathscr{C}-e) = r(\mathscr{C})-2+r(e\cap\overline{\mathscr{C}-e}) = 2|\mathscr{C}|-3+r(e\cap\overline{\mathscr{C}-e}) \geqq 2|\mathscr{C}-e|-1.$$

Equality here would contradict the minimality property of $\mathscr{C}$. Hence $\mathscr{C}-e$ is independent for every e. This implies that $\mathscr{C}$ is a circuit.

Assume now indirectly that there is another circuit $\mathscr{C}'$. Let e.g. $f\in\mathscr{C}-\mathscr{C}'$. We have

$$r(\mathscr{K}-f) = r(\mathscr{K})-2+r(f\cap\overline{\mathscr{K}-f}) = 2|\mathscr{K}|-3+r(f\cap\overline{\mathscr{K}-f}).$$

But $f\cap\overline{\mathscr{C}-f}\neq\emptyset$ and so $f\cap\overline{\mathscr{K}-f}\neq\emptyset$. Hence $\mathscr{K}-f$ is independent and so it cannot contain any circuit.

Q.E.D.

Let $\mathscr{H}$ be an arbitrary set of lines in a projective geometry. Let $\nu(\mathscr{H})$ denote the maximum number of independent lines in $\mathscr{H}$. A set of $\nu(\mathscr{H})$ independent lines will be called a *basis* of $\mathscr{H}$.

Let $\mathscr{B}$ be a basis of $\mathscr{H}$ and e a line not contained in $\overline{\mathscr{B}}$. Obviously, e must intersect $\overline{\mathscr{B}}$. Lemma 3 implies that $\mathscr{B}+e$ contains a unique circuit, which will be called the *fundamental circuit* of e relative to $\mathscr{B}$. Trivially, the fundamental circuit of e contains e. If e intersects a line $f\in\mathscr{B}$ then the fundamental circuit of e, relative to $\mathscr{B}$, is the set $\{e,f\}$.

If $\mathscr{B}$ is a basis, e a line not contained in $\overline{\mathscr{B}}$, and f a line of the fundamental circuit of e relative to $\mathscr{B}$, then $\mathscr{B}-f+e$ is another basis. We say that $\mathscr{B}-f+e$ arises from $\mathscr{B}$ by *elementary augmentation.* Trivially, the inverse of an elementary augmentation is an elementary augmentation as well.

3. Primitive sets of lines. In this section we discuss a special type of arrangement of lines. These sets will be the most difficult cases in the proof of the main result. A set $\mathscr{H}$ of lines in a projective space is called *primitive*, if the intersection of spans of all bases is void.

Lemma 4. *Let $\mathscr{H}$ be a primitive set of lines and $\mathscr{B}_1$, $\mathscr{B}_2$ two bases of $\mathscr{H}$. Then $\mathscr{B}_1$ can be transformed into $\mathscr{B}_2$ by elementary augmentations.*

Proof. Let $\mathscr{B}_1'$, $\mathscr{B}_2'$ be two bases such that $\mathscr{B}_i'$ arises from $\mathscr{B}_i$ by elementary augmentations and $|\mathscr{B}_1' \cap \mathscr{B}_2'|$ is maximal. If $\mathscr{B}_1' = \mathscr{B}_2'$ we are done by the remark after the definition of elementary augmentation.

We claim that $\overline{\mathscr{B}_1'} = \overline{\mathscr{B}_2'}$. In fact, if $\overline{\mathscr{B}_1'} \neq \overline{\mathscr{B}_2'}$ then there exists a line $e \in \mathscr{B}_1'$ such that $e \not\subseteq \overline{\mathscr{B}_2'}$. Let $\mathscr{C}$ be the fundamental circuit of e relative to $\mathscr{B}_2'$. Since $\mathscr{B}_1'$ is independent, there exists a line $f \in \mathscr{C} - \mathscr{B}_1'$. $\mathscr{B}_2'' = \mathscr{B}_2' + e - f$ is a basis which arises from $\mathscr{B}_2$ by elementary augmentations and which has $|\mathscr{B}_1' \cap \mathscr{B}_2''| > |\mathscr{B}_1' \cap \mathscr{B}_2'|$, a contradiction.

So we know that $\overline{\mathscr{B}_1'} = \overline{\mathscr{B}_2'}$. We want to show that $\mathscr{B}_1' = \mathscr{B}_2'$. Assume indirectly that there exists a line $e \in \mathscr{B}_1' - \mathscr{B}_2'$. Consider a basis $\mathscr{B}_0$ which does not span e. Such a basis exists since $\mathscr{H}$ is primitive. Choose $\mathscr{B}_1'$, $\mathscr{B}_2'$ and $\mathscr{B}_0$ so that $|\mathscr{B}_1' \cap \mathscr{B}_0|$ is maximal. Obviously, $\overline{\mathscr{B}_0} \neq \overline{\mathscr{B}_1'}$, and hence, there exists a line $g \in \mathscr{B}_0$ such that $g \not\subseteq \overline{\mathscr{B}_1'}$. Let $\mathscr{C}_1$, $\mathscr{C}_2$ denote the fundamental circuits of g relative to $\mathscr{B}_1'$ and $\mathscr{B}_2'$. We distinguish two cases.

Case 1. $\mathscr{C}_1 \neq \mathscr{C}_2$. Then $\mathscr{C}_1 \not\subseteq \mathscr{B}_2' + g$, since otherwise, $\mathscr{B}_2' + g$ would contain two distinct circuits, contradicting Lemma 3. Similarly $\mathscr{C}_2 \not\subseteq \mathscr{B}_1' + g$. So we can select lines $f_1 \in \mathscr{C}_1 - \mathscr{B}_2' - g$ and $f_2 \in \mathscr{C}_2 - \mathscr{B}_1' - g$. Now $\mathscr{B}_i'' = \mathscr{B}_i' - f_i + g$ is a basis arising from $\mathscr{B}_i$ by elementary augmentations, and $|\mathscr{B}_1'' \cap \mathscr{B}_2''| > |\mathscr{B}_1' \cap \mathscr{B}_2'|$, a contradiction.

Case 2. $\mathscr{C}_1 = \mathscr{C}_2$. Then $e \notin \mathscr{C}_1 = \mathscr{C}_2$. Let $f \in \mathscr{C}_1 - \mathscr{B}_0$, and put $\mathscr{B}_i'' = \mathscr{B}_i' + g - f$. Now $|\mathscr{B}_1'' \cap \mathscr{B}_2''| = |\mathscr{B}_1' \cap \mathscr{B}_2'|$, $|\mathscr{B}_1'' \cap \mathscr{B}_0| > |\mathscr{B}_1' \cap \mathscr{B}_0|$, and since $e \in \mathscr{B}_1''$, $e \notin \mathscr{B}_2''$, this is a contradiction.

Lemma 5. *Let $\mathscr{H}$ be a primitive set of lines and $\mathscr{K} \subseteq \mathscr{H}$ such that $r(\mathscr{K}) \geqq 2v+2$. Then the flats spanned by the circuits in $\mathscr{K}$ have no element in common.*

Proof. Suppose indirectly that a point p is contained in the span of each circuit in $\mathscr{K}$. Since $\mathscr{H}$ is primitive, there exists a basis $\mathscr{B} \subseteq \mathscr{H}$ such that the span of $\mathscr{B}$ does not contain p. Choose such a $\mathscr{B}$ with $|\mathscr{B} \cap \mathscr{K}|$ maximal. Since $r(\mathscr{B}+p) = 2v+1 < r(\mathscr{K})$, there is a line $e \in \mathscr{K}$ such that $e \not\subseteq \overline{\mathscr{B}+p}$. Then $p \notin \overline{\mathscr{B}+e}$. Let $\mathscr{C}$ be the fundamental circuit of e relative to $\mathscr{B}$. Then $p \notin \overline{\mathscr{B}}$ and so by the definition of p, $\mathscr{C} \not\subseteq \mathscr{K}$. Let $f \in \mathscr{C} - \mathscr{K}$, then $\mathscr{B}' = \mathscr{B} - f + e$ is a basis such that $p \notin \overline{\mathscr{B}'}$ and $|\mathscr{B}' \cap \mathscr{K}| > |\mathscr{B} \cap \mathscr{K}|$, which is a contradiction.

Lemma 6. *Let $\mathscr{H}$ be a primitive set of lines, $\mathscr{B}$ a basis, $e, f \in \mathscr{H}$ such that $r(\mathscr{B}+e+f) = 2v+2$. Let $\mathscr{C}_1$ and $\mathscr{C}_2$ be the fundamental circuits of e and f, respectively, relative to $\mathscr{B}$. Then $\mathscr{C}_1 \cap \mathscr{C}_2 = \emptyset$.*

Proof. Suppose indirectly that $\mathscr{C}_1 \cap \mathscr{C}_2 \neq \emptyset$. Let $\mathscr{C}_1, \ldots, \mathscr{C}_t$ denote the circuits in $\mathscr{K} = \mathscr{B}+e+f$ and let D_i be the flat spanned by $\mathscr{C}_i$. For each $u \in \mathscr{K}$, $r(\mathscr{K}-u) \leqq$

$\leqq 2v+1$, since otherwise $\mathscr{K}-u$ would be an independent set of $v+1$ lines. Let

$$\mathscr{K}_0 = \{u \in \mathscr{K}\colon\ r(\mathscr{K}-u) = 2v\}.$$

If $u \in \mathscr{K}-\mathscr{K}_0$ then $\mathscr{K}-u$ contains a unique circuit by Lemma 3. Let

$$\mathscr{K}_i = \{u \in \mathscr{K}-\mathscr{K}_0\colon \mathscr{C}_i \subseteqq \mathscr{K}-u\}.$$

Thus $\{\mathscr{K}_0, \mathscr{K}_1, \dots, \mathscr{K}_t\}$ is a partition of $\mathscr{K}$, and

$$\mathscr{C}_i = \mathscr{K}-\mathscr{K}_0-\mathscr{K}_i.$$

$\mathscr{C}_1 \cap \mathscr{C}_2 \neq \emptyset$ implies now by Lemma 5 that $t \geqq 3$. Furthermore, we have

$$r(D_i) = 2|\mathscr{K}-\mathscr{K}_0-\mathscr{K}_i|-1, \quad r\left(\bigcup_{i=1}^{t} D_i\right) = 2|\mathscr{K}-\mathscr{K}_0|-2.$$

(Here $r(\bigcup_{i=1}^{t} D_i) \geqq 2|\mathscr{K}-\mathscr{K}_0|-2$ is trivial; in the case of strict inequality Lemma 3 would imply that $\bigcup D_i$ contains only one circuit, which is not the case.) Now apply Lemma 1:

$$\sum_{i=1}^{t}\left\{r\left(\bigcup_{j=1}^{t} D_j\right)-r(D_i)\right\} = \sum_{i=1}^{t}\{2|\mathscr{K}_i|-1\} = 2|\mathscr{K}-\mathscr{K}_0|-t < r\left(\bigcup_{i=1}^{t} D_i\right),$$

and so $\bigcap_{i=1}^{t} D_i \neq \emptyset$. But this contradicts Lemma 5.

Call two lines e, f of a primitive set $\mathscr{H}$ *coherent*, if $r(\mathscr{B}+e+f) \leqq 2v+1$ for every basis $\mathscr{B}$.

Lemma 7. *Coherence of lines is an equivalence relation.*

Proof. Symmetry and reflexivity of coherence are evident. Suppose e and f, moreover f and g, are coherent. We show that e and g are coherent. Suppose indirectly that there exists a basis $\mathscr{B}_1$ such that $r(\mathscr{B}_1+e+g)=2v+2$. Since $\mathscr{H}$ is primitive, there exists another basis $\mathscr{B}_2$ such that $f \not\subseteqq \overline{\mathscr{B}_2}$, and so $r(\mathscr{B}_2+f)=2v+1$. Choose $\mathscr{B}_2$ such that $|\mathscr{B}_1 \cap \mathscr{B}_2|$ is maximum. Since $r(\mathscr{B}_1+e+g) > r(\mathscr{B}_2+f)$, there exists a line $h \in \mathscr{B}_1+e+g$ such that $h \not\subseteqq \overline{\mathscr{B}_2+f}$, i.e. such that

$$r(\mathscr{B}_2+f+h) = 2v+2. \tag{3}$$

So h is not coherent with f and so $h \neq e, g$. Thus $h \in \mathscr{B}_1$. Let $\mathscr{C}$ denote the fundamental circuit of h relative to $\mathscr{B}_2$. Since $\mathscr{C} \not\subseteqq \mathscr{B}_1$, we may choose a line $l \in \mathscr{C}-\mathscr{B}_1$. Then $\mathscr{B}_2'=\mathscr{B}_2+h-l$ is a basis such that $f \not\subseteqq \overline{\mathscr{B}_2'}$ (since $f \not\subseteqq \overline{\mathscr{B}_2+h}$ by (3) and $\mathscr{B}_2' \subset \mathscr{B}_2+h$), and moreover, $|\mathscr{B}_2' \cap \mathscr{B}_1| > |\mathscr{B}_2 \cap \mathscr{B}_1|$. This contradicts the choice of $\mathscr{B}_2$.

Q.E.D.

Lemma 8. *Let $\mathscr{B}$ be a basis and e a line not in the span of $\mathscr{B}$. Let $\mathscr{C}$ denote the fundamental circuit of e relative to $\mathscr{B}$. Then all lines in $\mathscr{C}$ are coherent.*

Proof. Suppose indirectly that there is a line $f\in\mathscr{C}$ such that e, f are not coherent. Let $\mathscr{B}'$ be a basis such that $r(\mathscr{B}'+e+f)=2v+2$. From among all counterexamples choose one in which $|\mathscr{B}'\cap\mathscr{B}|$ is maximal. Since $r(\mathscr{B}'+e+f)>r(\mathscr{B}+f)$ there is a line $g\in\mathscr{B}'+e+f$ not contained in the span of $\mathscr{B}+f$. Obviously, $g\neq e,f$, so $g\in\mathscr{B}'$. Let $\mathscr{C}_2$ denote the fundamental circuit of g relative to $\mathscr{B}$. Since $r(\mathscr{B}+f+g)=$ $=2v+2$, if follows by Lemma 6 that $\mathscr{C}\cap\mathscr{C}_2=\emptyset$. Hence if we replace any element of $\mathscr{C}_2\cap\mathscr{B}$ by g in $\mathscr{B}$, we obtain another basis $\mathscr{B}^*$ which has the property that the fundamental circuit of f relative to $\mathscr{B}^*$ is $\mathscr{C}$, but $|\mathscr{B}^*\cap\mathscr{B}'|>|\mathscr{B}\cap\mathscr{B}'|$, a contradiction.

Q.E.D.

Lemma 9. *If $\mathscr{H}$ is primitive, $e, f\in\mathscr{H}$ and e and f intersect, then e and f are coherent.*

Proof. Suppose not, then there exists a basis $\mathscr{B}$ such that $r(\mathscr{B}+e+f)=2v+2$. By an elementary augmentation we get a basis $\mathscr{B}'$ such that $e\in\mathscr{B}'$ but $f\not\subseteq\mathscr{B}'$. But by $e\cap f\neq\emptyset$ the fundamental circuit of f relative to $\mathscr{B}'$ is $\{e, f\}$, which contradicts Lemma 8.

Q.E.D.

Lemma 10. *Let $\mathscr{H}$ be a primitive set of lines, $\mathscr{B}$ a basis of $\mathscr{H}$, e a line not in the span of $\mathscr{B}$ and $\mathscr{A}$ the set of lines in $\mathscr{B}$ coherent to e. Then every line coherent to e is contained in $\overline{\mathscr{A}+e}$.*

Proof. Let f be a line coherent to e. Let $p\in e-\overline{\mathscr{B}}$ and $q\in f$, and denote by g the line pq. Set $\mathscr{H}'=\mathscr{H}+g$.

Claim 1. $v(\mathscr{H}')=v$. For suppose indirectly that $\mathscr{H}'$ contains an independent set $\mathscr{F}$ of $v+1$ lines. Obviously, $g\in\mathscr{F}$ and $\mathscr{F}-g$ is a basis of $\mathscr{H}$. But

$$r(\mathscr{F}-g+e+f)\geqq r(\mathscr{F})=2v+2,$$

which contradicts the assumption that e and f are coherent.

Claim 2. $\mathscr{H}'$ is primitive. This follows immediately from the fact that all bases of $\mathscr{H}$ are bases of $\mathscr{H}'$.

Claim 3. If two lines of $\mathscr{H}$ are coherent in $\mathscr{H}'$ then they are coherent in $\mathscr{H}$, for the same reason.

Claim 4. e, f and g are coherent in $\mathscr{H}'$. This follows by Lemma 9.

Now by Lemma 8, all lines in the fundamental circuit of g relative to $\mathscr{B}$ are coherent to g in $\mathscr{H}'$. By Claim 4, they are coherent to e in $\mathscr{H}'$ and so by Claim 3, they are coherent to e in $\mathscr{H}$. Thus $g\cap\overline{\mathscr{A}}\neq\emptyset$. Since $p\notin\overline{\mathscr{A}}$ but $p\in e$, it follows that g has at least two points in $\overline{\mathscr{A}+e}$. But then $g\subseteq\overline{\mathscr{A}+e}$, and consequently $q\in\overline{\mathscr{A}+e}$. q being an arbitrary point of f, it follows that $f\subseteq\overline{\mathscr{A}+e}$.

Q.E.D.

Let $\mathscr{H}_1, ..., \mathscr{H}_k$ denote the equivalence classes of the relation of coherence. Consider a basis $\mathscr{B}$ and set $v_i=|\mathscr{B}\cap \mathscr{H}_i|$. Observe that the numbers v_i are independent of the choice of $\mathscr{B}$: in fact, they remain the same when an elementary augmentation is carried out by Lemma 8, and every other basis can be obtained from $\mathscr{B}$ by elementary augmentations by Lemma 4.

Our result on primitive set of lines can be summarized as follows:

Theorem 1. *Let $\mathscr{H}$ be a primitive set of lines. Then there exist flats $A_1, ..., A_k$ with the following properties:*

(i) *$A_1, ..., A_k$ are disjoint.*

(ii) *Every line in $\mathscr{H}$ is contained in exactly one of $A_1, ..., A_k$.*

(iii) $r(A_i)=2v_i+1$.

(iv) *Every basis contains precisely v_i lines in A_i.*

(v) $v(\mathscr{H})=\sum_{i=1}^{k} v_i$.

Proof. Denote by A_i the flat spanned by $\mathscr{H}_i$. First we show that $r(A_i)=2v_i+1$. Let $e\in\mathscr{H}_i$ and $\mathscr{B}$ any basis not spanning e. Let $\mathscr{A}_i=\mathscr{B}\cap\mathscr{H}_i$. By Lemma 8, the fundamental circuit of e relative to $\mathscr{B}$ is contained in $\mathscr{A}_i+e$. Hence $r(\mathscr{A}_i+e)=2v_i+1$. On the other hand, Lemma 10 implies that all lines of $\mathscr{H}_i$ are contained in $\overline{\mathscr{A}_i+e}$. Hence $A_i=\overline{\mathscr{A}_i+e}$ and $r(A_i)=2v_i+1$.

Thus (iii) and (iv) are proved. (v) follows immediately. If we show (i) then (ii) will be trivially true.

So let $1\leqq i<j\leqq k$; we show that $A_i\cap A_j=\emptyset$. Let $e_i\in\mathscr{H}_i$, $e_j\in\mathscr{H}_j$, and let $\mathscr{B}$ be a basis such that $r(\mathscr{B}+e_i+e_j)=2v+2$ (such a basis exists by the definition of the sets $\mathscr{H}_i$). Let $\mathscr{A}_t=\mathscr{B}\cap\mathscr{H}_t$. By the argument above, e_t meets $\overline{\mathscr{A}_t}$ $(t=i,j)$ and $A_t=\overline{\mathscr{A}_t+e_t}$. But

$$r(A_i\cup A_j) = r(\mathscr{A}_i\cup \mathscr{A}_j\cup \{e_i, e_j\}) \geqq r(\mathscr{B}+e_i+e_j)-2|\mathscr{B}-\mathscr{A}_i-\mathscr{A}_j| =$$

$$= 2|\mathscr{B}|+2-2|\mathscr{B}-\mathscr{A}_i-\mathscr{A}_j| = |\mathscr{A}_i|+|\mathscr{A}_j|+2 = r(A_i)+r(A_j).$$

Hence A_i and A_j are disjoint.

Q.E.D.

4. The main result.

Theorem 2. *Let $\mathscr{H}$ be a set of lines in a projective geometry. Then the maximum number $v(\mathscr{H})$ of independent lines in $\mathscr{H}$ is the minimum of the expression*

$$r(A)+\sum_{i=1}^{k}\left[\frac{r(A_i)-r(A)}{2}\right],$$

where $A, A_1, \ldots, A_k$ are flats such that $A\subseteqq A_i$ $(i=1, \ldots, k)$ and for every $e\in\mathscr{H}$ either $e\cap A\neq\emptyset$ or there is an i such that $e\subseteqq A_i$.

Proof. I. First we show that if $\mathscr{F}$ is a set of independent lines, $A, A_1, \ldots, A_k$ are subspaces such that $A\subseteqq A_i$ and each line of $\mathscr{F}$ either meets A or is contained in one of the A_i's then

$$|\mathscr{F}|\leqq r(A)+\sum_{i=1}^{k}\left[\frac{r(A_i)-r(A)}{2}\right].$$

Let $\mathscr{F}_i$ and $\mathscr{F}_0$ denote the set of lines of $\mathscr{F}$ contained in A_i and meeting A, respectively. Let A_i' be the subspace spanned by $\mathscr{F}_i'=\mathscr{F}_i-\mathscr{F}_{i-1}-\ldots-\mathscr{F}_0$. Then $r(A_i')=$ $=2|\mathscr{F}_i'|$. Moreover, the subspaces A_i' are clearly independent and, therefore, so are the subspaces $A_i'\cap A$, $i=0, \ldots, k$. Hence

$$r(A)\geqq\sum_{i=0}^{k} r(A_i'\cap A).$$

Here $r(A_i'\cap A)=r(A_i')+r(A)-r(A_i'\cup A)\geqq r(A_i')+r(A)-r(A_i)$, whence

$$|\mathscr{F}_i'|=\frac{1}{2}r(A_i')\leqq\frac{r(A_i)-r(A)}{2}+\frac{r(A_i'\cap A)}{2}\leqq\frac{r(A_i)-r(A)}{2}+r(A_i'\cap A),$$

and using integrality,

$$|\mathscr{F}_i'|\leqq\left[\frac{r(A_i)-r(A)}{2}\right]+r(A_i'\cap A).$$

Moreover, obviously $|\mathscr{F}_0'|\leqq r(A\cap A_0')$. Hence

$$|\mathscr{F}|=\sum_{i=0}^{k}|\mathscr{F}_i'|\leqq r(A_0'\cap A)+\sum_{i=1}^{k}\left\{\left[\frac{r(A_i)-r(A)}{2}\right]+r(A_i'\cap A)\right\}\leqq$$

$$\leqq r(A)+\sum_{i=1}^{k}\left[\frac{r(A_i)-r(A)}{2}\right].$$

II. We want to construct subspaces $A, A_1, \ldots, A_k$ satisfying the conditions in the theorem. We use induction on $v(\mathscr{H})=v$. If $\mathscr{H}$ is primitive then the result is immediate by Theorem 1. So we may suppose that $\mathscr{H}$ is not primitive, i.e. there exists a point p contained in the span of each basis. Delete the lines containing p from $\mathscr{H}$ to obtain the system $\mathscr{H}_1$. Project the lines of $\mathscr{H}_1$ from p onto a hyperplane T. Let e' be the projection of e on T, and $\mathscr{H}_1'=\{e' : e\in\mathscr{H}_1\}$.

The system $\mathscr{H}_1'$ contains no v independent lines. In fact, if v lines from $\mathscr{H}_1$ are not independent, then neither are the corresponding lines in $\mathscr{H}_1'$; if v lines from $\mathscr{H}_1$ form a basis then p is contained in their span and hence the rank of their span decreases by the projection from p.

So by the induction hypothesis, there exist flats $D, D_1, \dots, D_k$ in T such that $D\subseteqq D_i$ $(i=1, \dots, k)$; for each $e\in\mathscr{H}_1'$ the line e either meets D or is contained in some D_i; and

$$r(D)+\sum_{i=1}^{k}\left[\frac{r(D_i)-r(D)}{2}\right]=v-1.$$

Consider now the subspaces $A=D+p$, $A_i=D_i+p$. Obviously, $A\subseteqq A_i$. Furthermore, the lines in $\mathscr{H}-\mathscr{H}_1$ meet A, and so do all lines e for which e' meets D. If $e'\subseteqq D_i$ then $e\subseteqq A_i$. Finally, $r(A)=r(D)+1$, $r(A_i)=r(D_i)+1$ and hence

$$r(A)+\sum_{i=1}^{k}\left[\frac{r(A_i)-r(A)}{2}\right]=r(D)+1+\sum_{i=1}^{k}\left[\frac{r(D_i)-r(D)}{2}\right]=v.$$

Q.E.D.

5. Connections with matroid theory. The first question which comes up is whether or not Theorem 2 remains valid in an arbitrary matroid. First of all, the definition of independence of lines has to be done more carefully; let us accept the natural solution that a set $\mathscr{F}$ of lines is independent if $r(\mathscr{F})=2|\mathscr{F}|$. In this case the problem is equivalent to the so-called matroid parity problem (see LAWLER [3], Chapter 9).

A counterexample to the analogue of Theorem 2 is any affine space, where $\mathscr{H}$ consists of all lines parallel to a given one. Of course, if we extend our affine space to a projective space then we could choose $k=0$, A the common ideal point of our lines. But in general, there seems to be no hope to extend the original matroid so as to achieve the validity of Theorem 2. The possiblity of "simulating" the flat A inside the matroid seems to be a difficult, and probably not only technical, question.

It is clear that independence of lines does not define, in general, a matroid. See e.g. two disjoint lines and a third one meeting both. There is a class of systems of lines, however, for which the situation is different. Let us call a set $\mathscr{H}$ of lines *flexible*, if $r(e\cap\overline{\mathscr{H}-e})\leqq 1$ for each line $e\in\mathscr{H}$. For each $e\in\mathscr{H}$, let $p(e)$ be the intersection of e with $\overline{\mathscr{H}-e}$, if this exists, and an arbitrary point of e otherwise.

The next proposition shows that independence of lines in a flexible set defines a matroid:

Proposition 1. *Let $\mathscr{H}$ be a flexible set of lines. Then $\mathscr{F}\subseteq\mathscr{H}$ is independent iff the set $\mathscr{F}'=\{p(e)\colon e\in\mathscr{F}\}$ of points is independent.*

Proof. It is trivial that if $\mathscr{F}$ is independent then so is $\mathscr{F}'$. Assume now that $\mathscr{F}'$ is independent. Then we prove by induction on $|\mathscr{G}|$ that if $\mathscr{G}\subseteq\mathscr{F}$ then

$$r(\mathscr{F}'\cup\mathscr{G}) = |\mathscr{F}'|+|\mathscr{G}|. \tag{4}$$

For $\mathscr{G}=\mathscr{F}$ this will mean that $\mathscr{F}$ is independent.

(4) is trivially true for $\mathscr{G}=\emptyset$. Let $\mathscr{G}\neq\emptyset$ and $e\in\mathscr{G}$. Then

$$r(\mathscr{F}'\cup\mathscr{G}) = r\big(\mathscr{F}'\cup(\mathscr{G}-e)\big)+1,$$

since $\mathscr{H}$ being flexible, e intersects $\overline{\mathscr{F}'\cup(\mathscr{G}-e)}$ in precisely one point. This proves (4) by induction.

Q.E.D.

Finally, let us point out one more matroid which is induced by a set of lines. This is a certain analogue of the *matching matroid* of graphs by Edmonds and Fulkerson [1]. Let $\mathscr{H}$ be a set of lines. Call a subset $\mathscr{G}\subseteq\mathscr{H}$ *dispersive*, if there exists a basis $\mathscr{B}$ of $\mathscr{H}$ such that $r(\mathscr{B}\cup\mathscr{G})=2v(\mathscr{H})+|\mathscr{G}|$.

Proposition 2. *Dispersive sets form the independent sets of a matroid.*

This proposition generalizes Lemma 8, and can be proved along the same lines. Details are omitted.

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