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On the divergence of multiple orthogonal series

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1. Preliminaries. Let $I^d = \times_{j=1}^{d} [0, 1]$ be the unit cube in the d-dimensional Euclidean space, where $d \geqq 1$ is a fixed integer. The points $(x_1, \ldots, x_d), (y_1, \ldots, y_d), \ldots$ of I^d are denoted by the corresponding bold letters $\mathbf{x}, \mathbf{y}, \ldots$. Let Z_+^d be the set of d-tuples $\mathbf{k}=(k_1, \ldots, k_d)$ with positive integral coordinates, the tuple $(1, \ldots, 1)$ is denoted by $\mathbf{1}$. Z_+^d is partially ordered by agreeing that $\mathbf{k} \leqq \mathbf{m}$ iff $k_j \leqq m_j$ for each j. Finally, we write

$$\mathbf{k}_* = \min_{1 \leqq j \leqq d} k_j \quad \text{and} \quad \mathbf{k}^* = \max_{1 \leqq j \leqq d} k_j.$$

Let $\{\varphi_\mathbf{k}(\mathbf{x})\colon \mathbf{k} \in Z_+^d\}$ be a *d-dimensional orthonormal system* on I^d, i.e. for every $\mathbf{k}$ and $\mathbf{m}$ in Z_+^d let

$$\int_{I^d} \varphi_\mathbf{k}(\mathbf{x}) \varphi_\mathbf{m}(\mathbf{x})\, d\mathbf{x} = \delta_{\mathbf{km}} \qquad (d\mathbf{x} = dx_1 \ldots dx_d).$$

In particular, if for each $j=1, 2, \ldots, d$ the system $\{\varphi_k^{(j)}(x)\}_{k=1}^\infty$ is orthonormal on $I=[0, 1]$, then the functions

$$\varphi_{k_1, \ldots, k_d}(x_1, \ldots, x_d) = \prod_{j=1}^{d} \varphi_{k_j}^{(j)}(x_j)$$

are orthonormal on I^d.

We shall consider the d-multiple orthogonal series

$$\sum_{\mathbf{k} \geqq \mathbf{1}} a_\mathbf{k} \varphi_\mathbf{k}(\mathbf{x}) = \sum_{k_1=1}^{\infty} \cdots \sum_{k_d=1}^{\infty} a_{k_1, \ldots, k_d} \varphi_{k_1, \ldots, k_d}(x_1, \ldots, x_d), \tag{1}$$

where $\{a_\mathbf{k}\colon \mathbf{k} \in Z_+^d\}$ is a given system of numbers (coefficients). For any $\mathbf{m} \in Z_+^d$ set

$$S_\mathbf{m}(\mathbf{x}) = \sum_{\mathbf{1} \leqq \mathbf{k} \leqq \mathbf{m}} a_\mathbf{k} \varphi_\mathbf{k}(\mathbf{x}) =$$

$$= \sum_{k_1=1}^{m_1} \cdots \sum_{k_d=1}^{m_d} a_{k_1, \ldots, k_d} \varphi_{k_1, \ldots, k_d}(x_1, \ldots, x_d),$$

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which is a *rectangular partial sum* of (1). In case $m_1 = \ldots = m_d$, $S_{\mathbf{m}}(\mathbf{x})$ is called a *square partial sum* of (1). The *spherical partial sums* of (1) are defined as

$$S_r(\mathbf{x}) = \sum_{k_1^2+\ldots+k_d^2 \leqq r} a_{\mathbf{k}} \varphi_{\mathbf{k}}(\mathbf{x}) \qquad (r = d, d+1, \ldots).$$

The following Theorem A has been published by a few authors, while Theorems B and C were proved by the first author in [3] and [4].

Theorem A. *If*

$$\sum_{\mathbf{k} \geqq 1} a_{\mathbf{k}}^2 \prod_{j=1}^{d} (\log 2k_j)^2 < \infty,$$

then the rectangular partial sums $S_{\mathbf{m}}(\mathbf{x})$ *of* (1) *converge a.e. on* I^d *as* $\mathbf{m}_* \to \infty$.

Here and in the sequel log is of base 2.

Theorem B. *If*

$$\sum_{\mathbf{k} \geqq 1} a_{\mathbf{k}}^2 (\log \mathbf{k}^*)^2 < \infty,$$

then both the square partial sums $S_{n,\ldots,n}(\mathbf{x})$ *and the spherical partial sums* $S_n(\mathbf{x})$ *of* (1) *converge a.e. on* I^d *as* $n \to \infty$.

The part concerning the spherical partial sums was stated in [3] in a slightly different form, but the two statements are equivalent, because

$$(\mathbf{k}^*)^2 \leqq k_1^2 + \ldots + k_d^2 \leqq d(\mathbf{k}^*)^2.$$

Theorem C. *If*

$$\sum_{\mathbf{k} \geqq 1} a_{\mathbf{k}}^2 < \infty,$$

then

$$S_{\mathbf{m}}(\mathbf{x}) = o_{\mathbf{x}}\left(\prod_{j=1}^{d} \log 2m_j\right) \quad \textit{a.e. on } I^d \textit{ as } \quad \mathbf{m}^* \to \infty.$$

2. Results. In this paper we are going to show that these theorems are exact in the sense that $\log n$ cannot be replaced by any sequence $\varrho(n)$ tending to ∞ slower than $\log n$ as $n \to \infty$. More precisely, let $\{\varrho(n)\}_{n=1}^{\infty}$ be a non-decreasing sequence of positive numbers for which

$$\varrho(n) = o(\log n) \quad (n \to \infty). \tag{2}$$

Theorem 1. *For every* $d \geqq 1$ *and* $\{\varrho(n)\}$ *satisfying* (2), *there exist an orthonormal system* $\{\varphi_{\mathbf{k}}(\mathbf{x})\colon \mathbf{k} \in Z_+^d\}$ *and a system* $\{a_{\mathbf{k}}\colon \mathbf{k} \in Z_+^d\}$ *of coefficients such that*

$$\sum_{\mathbf{k} \geqq 1} a_{\mathbf{k}}^2 (\log \mathbf{k}^*)^{2d-2} \varrho^2(\mathbf{k}^*) < \infty \tag{3}$$

and

(4) $$\limsup_{\mathbf{m}_*\to\infty} |S_{\mathbf{m}}(\mathbf{x})| = \infty \quad a.e.\ on\ I^d.$$

By virtue of Theorem B in case $d \geqq 2$ both the square partial sums and the spherical partial sums of the series $\sum_{\mathbf{k}\geqq 1} a_{\mathbf{k}}\varphi_{\mathbf{k}}(\mathbf{x})$ occurring in Theorem 1 converge a.e.

Theorem 2. *For every $d \geqq 1$ and $\{\varrho(n)\}$ satisfying* (2), *there exist an orthonormal system $\{\varphi_{\mathbf{k}}(\mathbf{x})\colon \mathbf{k}\in Z_+^d\}$ and a system $\{b_{\mathbf{k}}\colon \mathbf{k}\in Z_+^d\}$ of coefficients such that*

$$\sum_{\mathbf{k}\geqq 1} b_{\mathbf{k}}^2 < \infty \quad and \quad \limsup_{\mathbf{m}_*\to\infty} \frac{|S_{\mathbf{m}}(\mathbf{x})|}{(\log \mathbf{m}^*)^{d-1}\varrho(\mathbf{m}^*)} = \infty \quad a.e.\ on\ I^d.$$

Theorems 1 and 2 for $d=1$ are well-known (see, e.g. [1, pp. 99—100]).

Theorem 3. *For every $d \geqq 2$ and $\{\varrho(n)\}$ satisfying* (2), *there exist an orthonormal system $\{\varphi_{\mathbf{k}}(\mathbf{x})\colon \mathbf{k}\in Z_+^d\}$ and a system $\{c_{\mathbf{k}}\colon \mathbf{k}\in Z_+^d\}$ of coefficients such that*

$$\sum_{\mathbf{k}\geqq 1} c_{\mathbf{k}}^2 (\log \mathbf{k}^*)^2 < \infty$$

and

$$\limsup_{\mathbf{m}_*\to\infty} \frac{|S_{\mathbf{m}}(\mathbf{x})|}{(\log \mathbf{m}^*)^{d-2}\varrho(\mathbf{m}^*)} = \infty \quad a.e.\ on\ I^d.$$

Again by Theorem B, both the square partial sums and the spherical partial sums of the series $\sum_{\mathbf{k}\geqq 1} c_{\mathbf{k}}\varphi_{\mathbf{k}}(\mathbf{x})$ converge a.e.

Our last theorem states the a.e. divergence of the rectangular partial sums of series (1) for a whole class of coefficient systems. A system $\{a_{\mathbf{k}}\colon \mathbf{k}\in Z_+^d\}$ of coefficients is said to be *non-increasing in absolute value* if for every $\mathbf{k}, \mathbf{m}\in Z_+^d$,

$$\mathbf{k} \leqq \mathbf{m} \Rightarrow |a_{\mathbf{k}}| \geqq |a_{\mathbf{m}}|.$$

It is clear that this is equivalent to the following: for every $\mathbf{k}\in Z_+^d$ and j, $1 \leqq j \leqq d$, we have

$$|a_{k_1,\dots,k_{j-1},k_j,k_{j+1},\dots,k_d}| \geqq |a_{k_1,\dots,k_{j-1},k_j+1,k_{j+1},\dots,k_d}|.$$

Theorem 4. *For every system $\{a_{\mathbf{k}}\colon \mathbf{k}\in Z_+^d\}$ of coefficients, which is non-increasing in absolute value and satisfies the relation*

(5) $$\sum_{\mathbf{k}\geqq 1} a_{\mathbf{k}}^2 \prod_{j=1}^{d} (\log 2k_j)^2 = \infty,$$

there exists an orthonormal system $\{\varphi_{\mathbf{k}}(\mathbf{x})\colon \mathbf{k}\in Z_+^d\}$ such that

(6) $$\limsup_{\mathbf{m}^*\to\infty} |S_{\mathbf{m}}(\mathbf{x})| = \infty \quad a.e.\ on\ I^d.$$

If, in addition, for every $\mathbf{m}\in Z_+^d$ *we have*

(7) $$\sum_{\mathbf{k}\geqq\mathbf{m}} a_{\mathbf{k}}^2 \prod_{j=1}^{d} (\log 2k_j)^2 = \infty,$$

then $\mathbf{m}^*\to\infty$ *can be replaced by* $\mathbf{m}_*\to\infty$ *in* (6).

The two parts of Theorem 4 coincide for $d=1$. In this case Theorem 4 was proved by the second author in [5].

3. Notations and an auxiliary result. For the sake of simplicity in notations, we present the proofs only for the case $d=2$. We write (x, y) instead of $\mathbf{x}=(x_1, x_2)$ and (k, l) instead of $\mathbf{k}=(k_1, k_2)$.

We agree that $\langle a, b\rangle$ means either the open interval (a, b), or one of the half-closed intervals $[a, b)$ and $(a, b]$, or the closed interval $[a, b]$. Given a function $f(x, y)$ defined on I^2 and a rectangle $R=\langle a, b\rangle\times\langle c, d\rangle\subseteqq I^2$, let us put

$$f(R; x, y) = \begin{cases} f\left(\dfrac{x-a}{b-a}, \dfrac{y-c}{d-c}\right) & \text{if} \quad (x, y)\in R, \\ 0 & \text{otherwise.} \end{cases}$$

Given a set $H\subseteqq I^2$, let $H(R)$ denote the set into which H is carried over by the linear transformation $\bar{x}=(b-a)x+a$ and $\bar{y}=(d-c)y+c$.

A set $H\subseteqq I$ (or $\subseteqq I^2$) is said to be *simple* if H can be represented as the union of finitely many disjoint intervals (or rectangles).

The proofs of all our theorems are based on the following basic result of MENŠOV [2].

Lemma. *For every positive integer n there exist a system* $\{\psi_k^{(n)}(x)\}_{k=1}^n$ *of step functions, orthonormal on the interval* $I=[0, 1]$, *and a simple set* $E^{(n)}$ *of I such that*

(8) $$\text{mes}\, E^{(n)} \geqq C_1,$$

and for every $x\in E^{(n)}$ *there exists an integer* $\varkappa(x)$, $1\leqq\varkappa(x)\leqq n$, *such that* $\psi_1^{(n)}(x)\geqq \geqq 0, \dots, \psi_{\varkappa(x)}^{(n)}(x)\geqq 0$ *and*

(9) $$\sum_{k=1}^{\varkappa(x)} \psi_k^{(n)}(x) \geqq C_2\sqrt{n}\log 2n.$$

Here C_1 and C_2 denote positive constants. Further, if $E\subseteqq I$ (or $\subseteqq I^2$), then mes E denotes the Lebesgue measure of the set E on the line (or on the plane).

4. Proof of Theorem 4. *Part* 1. By (5) and the non-increasing property of $\{a_{kl}^2\}_{k,l=1}^\infty$ we have

$$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} 2^{p+q}(p+1)^2(q+1)^2 a_{2^{p+1}-1,2^{q+1}-1}^2 = \infty.$$

With the notation

$$A_r = \sum_{\substack{0\leqq p,q\leqq r\\ \max(p,q)=r}} 2^{p+q}(p+1)^2(q+1)^2 a^2_{2^{p+1}-1,2^{q+1}-1} \quad (r=0,1,\ldots)$$

this can be rewritten into the form $\sum_{r=0}^{\infty} A_r=\infty$. We can find a sequence $\{s_r\}_{r=0}^{\infty}$ of positive numbers with the following properties:

$$\lim_{r\to\infty} s_r = 0, \quad s_r^2 A_r \leqq 1 \quad (r=0,1,\ldots)$$

and

$$\sum_{r=0}^{\infty} s_r^2 A_r = \infty. \tag{10}$$

Without loss of generality we may assume that $a_{kl}\geqq 0$ for every $k, l=1, 2, \ldots$.

Our goal is to construct a system $\{\varphi_{kl}(x, y)\}_{k,l=1}^{\infty}$ of step functions and a system $\{H_{pq}\}_{p,q=0}^{\infty}$ of simple sets of I^2 such that these functions be orthonormal on I^2, these sets be stochastically independent with

$$\text{mes}\, H_{pq} \geqq C_1^2 2^{p+q}(p+1)^2(q+1)^2 s_r^2 a^2_{2^{p+1}-1,2^{q+1}-1} \qquad (p, q = 0, 1, \ldots), \tag{11}$$

and for every $(x, y)\in H_{pq}$

$$\max_{2^p\leqq m<2^{p+1}} \max_{2^q\leqq n<2^{q+1}} \left|\sum_{k=2^p}^{m}\sum_{l=2^q}^{n} a_{kl}\varphi_{kl}(x, y)\right| \geqq \frac{C_2^2}{s_r}, \tag{12}$$

where $r=\max(p, q)$.

The construction will be done by induction on r. If $r=0$, then let $\varphi_{11}(x, y)= =1/s_0 a_{11}$ on a rectangle H_{00} of area $s_0^2 a_{11}^2$ and let $\varphi_{11}(x, y)=0$ otherwise. Then (11) and (12) are satisfied for $p=q=0$ provided $C_1, C_2\leqq 1$, which is the case.

Now let r_0 be a positive integer and assume that the step functions $\varphi_{kl}(x, y)$ are defined for $k, l=1, 2, \ldots, 2^{r_0}-1$ and the simple sets H_{pq} are defined for $p, q= =0, 1, \ldots, r_0-1$ such that these functions are orthonormal on I^2, these sets are stochastically independent, and the relations (11) and (12) are satisfied for $p, q= =0, 1, \ldots, r_0-1$. We are going to define the step functions $\varphi_{kl}(x, y)$ of the r_0th block successively in the following arrangement: for

$k = 2^{r_0}, 2^{r_0}+1, \ldots, 2^{r_0+1}-1$ and $l=1$;

$k = 2^{r_0}, 2^{r_0}+1, \ldots, 2^{r_0+1}-1$ and $l=2, 3$;

....

$k = 2^{r_0}, 2^{r_0}+1, \ldots, 2^{r_0+1}-1$ and $l = 2^{r_0}, 2^{r_0}+1, \ldots, 2^{r_0+1}-1$;

$k = 2^{r_0-1}, 2^{r_0-1}+1, \ldots, 2^{r_0}-1$ and $l = 2^{r_0}, 2^{r_0}+1, \ldots, 2^{r_0+1}-1$;

....

$k = 2, 3$ and $l = 2^{r_0}, 2^{r_0}+1, \ldots, 2^{r_0+1}-1$;

$k = 1$ and $l = 2^{r_0}, 2^{r_0}+1, \ldots, 2^{r_0+1}-1$;

and the simple sets $H_{r_0,0}, H_{r_0,1}, \ldots, H_{r_0,r_0}, H_{r_0-1,r_0}, \ldots, H_{1,r_0}, H_{0,r_0}$ in such a way that the functions $\varphi_{kl}(x, y)$ $(k, l=1, 2, \ldots, 2^{r_0+1}-1)$ be orthonormal on I^2, the sets H_{pq} $(p, q=0, 1, \ldots, r_0)$ be stochastically independent, and the relations (11) and (12) be satisfied for $p, q=0, 1, \ldots, r_0$.

For the sake of definiteness, let us assume that the sets $H_{r_0,0}, H_{r_0,1}, \ldots, H_{r_0,q_0-1}$ $(1\leqq q_0\leqq r_0)$ and the functions $\varphi_{kl}(x, y)$ for $k=2^{r_0}, 2^{r_0}+1, \ldots, 2^{r_0+1}-1$ and $l=1, 2, \ldots \ldots, 2^{q_0}-1$ have been appropriately defined. Let us apply Menšov's lemma firstly with $n=2^{r_0}$, secondly with $n=2^{q_0}$, and set

$$\bar\varphi_{2^{r_0}+k-1,\, 2^{q_0}+l-1}(x, y) = \psi_k^{(2^{r_0})}(x)\,\psi_l^{(2^{q_0})}(y) \quad (k = 1, 2, \ldots, 2^{r_0};\ l = 1, 2, \ldots, 2^{q_0}).$$

Then by (9) for every $(x, y)\in F=E^{(2^{r_0})}\times E^{(2^{q_0})}$ we have

$$\max_{1\leqq m\leqq 2^{r_0}}\ \max_{1\leqq n\leqq 2^{q_0}} \left|\sum_{k=1}^{m}\sum_{l=1}^{n} a_{2^{r_0}+k-1,\, 2^{q_0}+l-1}\,\bar\varphi_{2^{r_0}+k-1,\, 2^{q_0}+l-1}(x, y)\right| \geqq$$

$$\geqq C_2^2 \sqrt{2^{r_0+q_0}}(r_0+1)(q_0+1)a_{2^{r_0+1}-1,\, 2^{q_0+1}-1}.$$

Let Q be an arbitrary rectangle in I^2 with

$$\operatorname{mes} Q = 2^{r_0+q_0}(r_0+1)^2(q_0+1)^2 s_{r_0}^2 a^2_{2^{r_0+1}-1,\, 2^{q_0+1}-1}$$

(the quantity on the right is not greater than 1 because of the choice of s_{r_0}), and let us "contract" the functions $\bar\varphi$ from I^2 to Q:

$$\bar{\bar\varphi}_{2^{r_0}+k-1,\, 2^{q_0}+l-1}(x, y) = \frac{\bar\varphi_{2^{r_0}+k-1,\, 2^{q_0}+l-1}(Q;\ x, y)}{\sqrt{2^{r_0+q_0}}(r_0+1)(q_0+1)s_{r_0}a_{2^{r_0+1}-1,\, 2^{q_0+1}-1}}$$

$$(k = 1, 2, \ldots, 2^{r_0};\ l = 1, 2, \ldots, 2^{q_0}).$$

It is not hard to check that these step functions are also orthonormal on I^2, by (8)

(13) $$\operatorname{mes} F(Q) = \operatorname{mes} F \operatorname{mes} Q = (\operatorname{mes} E^{(2^{r_0})})^2 \operatorname{mes} Q \geqq$$

$$\geqq C_1^2 2^{r_0+q_0}(r_0+1)^2(q_0+1)^2 s_{r_0}^2 a^2_{2^{r_0+1}-1,\, 2^{q_0+1}-1},$$

and for every $(x, y)\in F(Q)$

(14) $$\max_{1\leqq m\leqq 2^{r_0}}\ \max_{1\leqq n\leqq 2^{q_0}} \left|\sum_{k=1}^{m}\sum_{l=1}^{n} a_{2^{r_0}+k-1,\, 2^{q_0}+l-1}\,\bar{\bar\varphi}_{2^{r_0}+k-1,\, 2^{q_0}+l-1}(x, y)\right| \geqq \frac{C_2^2}{s_{r_0}}.$$

Since the functions $\varphi_{kl}(x, y)$ and the sets H_{pq} defined so far are step functions and simple sets, respectively, we can divide I^2 into a finite number of disjoint rectangles $R_1, R_2, \ldots, R_\sigma$ such that each function $\varphi_{kl}(x, y)$ $(k, l=1, 2, \ldots, 2^{r_0}-1;$ $k=2^{r_0}, 2^{r_0}+1, \ldots, 2^{r_0+1}-1$ and $l=1, 2, \ldots, 2^{q_0}-1)$ is constant on each R_s

$(s=1, 2, \ldots, \sigma)$ and each set H_{pq} $(p, q=0, 1, \ldots, r_0-1; p=r_0$ and $q=0, 1, \ldots, q_0-1)$ is the union of certain R_s. Let R_s' and R_s'' denote the two halves of R_s, i.e., if $R_s= =\langle a, b\rangle\times\langle c, d\rangle$, then let $R_s'=\langle a, (a+b)/2]\times\langle c, d\rangle$ and $R_s''=((a+b)/2, b\rangle\times\langle c, d\rangle$. We set

$$\varphi_{2^{r_0}+k-1, 2^{q_0}+l-1}(x, y) = \sum_{s=1}^{\sigma}\bar{\bar{\varphi}}_{2^{r_0}+k-1, 2^{q_0}+l-1}(R_s'; x, y)-$$

$$-\sum_{s=1}^{\sigma}\bar{\bar{\varphi}}_{2^{r_0}+k-1, 2^{q_0}+l-1}(R_s''; x, y) \quad (k=1, 2, \ldots, 2^{r_0};\ l=1, 2, \ldots, 2^{q_0})$$

and

$$H_{r_0, q_0} = \left(\bigcup_{s=1}^{\sigma} G(R_s')\right)\cup\left(\bigcup_{s=1}^{\sigma} G(R_s'')\right),$$

where $G=F(Q)$.

It is easy to verify that the step functions $\varphi_{kl}(x, y)$ $(k, l=1, 2, \ldots, 2^{r_0}-1;$ $k=2^{r_0}, 2^{r_0}+1, \ldots, 2^{r_0+1}-1$ and $l=1, 2, \ldots, 2^{q_0+1}-1)$ form an orthonormal system on I^2, the simple sets H_{pq} $(p, q=0, 1, \ldots, r_0-1; p=r_0$ and $q=0, 1, \ldots, q_0)$ are stochastically independent, by (13)

$$\operatorname{mes} H_{r_0, q_0} = \operatorname{mes} F(Q) \geqq C_1^2 2^{r_0+q_0}(r_0+1)^2(q_0+1)^2 s_{r_0}^2\, a^2_{2^{r_0+1}-1, 2^{q_0+1}-1},$$

and by (14) for every $(x, y)\in H_{r_0, q_0}$

$$\max_{1\leqq m\leqq 2^{r_0}}\ \max_{1\leqq n\leqq 2^{q_0}}\left|\sum_{k=1}^{m}\sum_{l=1}^{n} a_{2^{r_0}+k-1, 2^{q_0}+l-1}\varphi_{2^{r_0}+k-1, 2^{q_0}+l-1}(x, y)\right| \geqq \frac{C_2^2}{s_{r_0}}.$$

The above induction scheme shows that the orthonormal system $\{\varphi_{kl}(x, y)\}_{k, l=1}^{\infty}$ and the system $\{H_{pq}\}_{p, q=1}^{\infty}$ of stochastically independent sets can be defined so that the conditions (11) and (12) hold true.

Putting (10) and (11) together we see that

$$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\operatorname{mes} H_{pq} = \infty. \tag{15}$$

Thus the second Borel—Cantelli lemma implies that almost every $(x, y)\in I^2$ belongs to an infinite number of sets H_{pq}. Taking into account (12) this means that for almost every (x, y) there exist four sequences $\{m_i\}$, $\{M_i\}$, $\{n_i\}$ and $\{N_i\}$ of positive integers such that $m_i\leqq M_i$ and $n_i\leqq N_i$ $(i=1, 2, \ldots)$, $\max(m_i, n_i)\to\infty$ as $i\to\infty$, and

$$\lim_{i\to\infty}\left|\sum_{k=m_i}^{M_i}\sum_{l=n_i}^{N_i} a_{kl}\varphi_{kl}(x, y)\right| = \infty.$$

Since the double sum in absolute value is equal to

$$S_{M_i, N_i}(x, y)-S_{M_i, n_i-1}(x, y)-S_{m_i-1, N_i}(x, y)+S_{m_i-1, n_i-1}(x, y),$$

(6) follows.

Part 2. Now suppose that (7) is also satisfied, i.e. for every m and n we have

$$\sum_{k=m}^{\infty}\sum_{l=n}^{\infty} a_{kl}^2 (\log 2k)^2 (\log 2l)^2 = \infty.$$

Then, using the non-increasing property of $\{a_{kl}^2\}$, for every r we have

$$\sum_{p=r}^{\infty}\sum_{q=r}^{\infty} 2^{p+q}(p+1)^2(q+1)^2 a_{2^{p+1}-1,2^{q+1}-1}^2 = \infty.$$

This makes it possible to define a sequence $0=r_0<r_1<r_2<\dots$ of integers such that

$$A_i' = \sum_{p=r_i+1}^{r_{i+1}}\sum_{q=r_i+1}^{r_{i+1}} 2^{p+q}(p+1)^2(q+1)^2 a_{2^{p+1}-1,2^{q+1}-1}^2 \geqq 1 \quad (i=0,1,\dots).$$

Finally, let $\{s_i'\}_{i=0}^{\infty}$ be a sequence of positive numbers with the properties

$$\lim_{i\to\infty} s_i' = 0, \quad (s_i')^2 A_i' \leqq 1 \quad (i=0,1,\dots)$$

and

$$\sum_{i=0}^{\infty} (s_i')^2 A_i' = \infty. \tag{16}$$

After this modification we have only to repeat the construction of Part 1. Relations (11) and (16) imply that

$$\sum_{i=0}^{\infty}\sum_{p=r_i+1}^{r_{i+1}}\sum_{q=r_i+1}^{r_{i+1}} \operatorname{mes} H_{pq} = \infty,$$

which is stronger than (15). The second Borel—Cantelli lemma yields that almost every $(x,y)\in I^2$ now belongs to an infinite number of sets H_{pq} with $r_i<p, q\leqq r_{i+1}$, $i=0,1,\dots$. This already ensures that $\mathbf{m}^*\to\infty$ can be replaced by $\mathbf{m}_*\to\infty$ in (6).

The proof of Theorem 4 is complete.

5. Proofs of Theorems 1—3 run along the same lines as that of Theorem 4 with the exception that now there is no need of a "contraction" of the functions $\bar{\varphi}$. In particular, at present

$$\varphi_{2^r+k-1,2^r+l-1}(x,y) = \sum_{s=1}^{\sigma} \bar{\varphi}_{2^r+k-1,2^r+l-1}(R_s'; x,y) - \sum_{s=1}^{\sigma} \bar{\varphi}_{2^r+k-1,2^r+l-1}(R_s''; x,y),$$

where

$$\bar{\varphi}_{2^r+k-1,2^r+l-1}(x,y) = \psi_k^{(2^r)}(x)\psi_l^{(2^r)}(y) \quad (k,l=1,2,\dots,2^r;\ r=0,1,\dots),$$

while the other $\varphi_{kl}(x,y)$ are indifferent from our point of view (of course, they have to be normal and orthogonal to each other). Further,

$$H_{rr} = \left(\bigcup_{s=1}^{\sigma} F(R_s')\right) \cup \left(\bigcup_{s=1}^{\sigma} F(R_s'')\right),$$

where $F=E^{(2r)}\times E^{(2r)}$. By (8)

$$\text{mes}\, H_{rr}=\text{mes}\, F=(\text{mes}\, E^{(2r)})^2\geqq C_1^2. \tag{17}$$

Let $\bar{\varrho}(n)=(\varrho(n)\log n)^{1/2}$. Then by (2)

$$\varrho(n)=o(\bar{\varrho}(n)) \quad\text{and}\quad \bar{\varrho}(n)=o(\log n) \quad (n\to\infty).$$

Hence there exists a sequence $\{n_j=2^{r_j}\}_{j=1}^{\infty}$ of integers such that $n_j\geqq 2n_{j-1}$ $(n_0=1)$,

$$\frac{\varrho(n)}{\bar{\varrho}(n)}\leqq\frac{1}{j} \quad\text{and}\quad \frac{\bar{\varrho}(n)}{\log n}\leqq\frac{1}{j} \quad\text{if}\quad n\geqq n_j \quad (j=1,2,\ldots). \tag{18}$$

The definition of the coefficients is the following: set for $k, l=1, 2, \ldots, n_j$; $j=1, 2, \ldots$

$$a_{n_j+k-1,n_j+l-1}=\frac{1}{n_j\bar{\varrho}(2n_j)\log 2n_j} \quad \text{(in Theorem 1)},$$

$$b_{n_j+k-1,n_j+l-1}=\frac{\bar{\varrho}(2n_j)}{n_j\log 2n_j} \quad \text{(in Theorem 2)},$$

$$c_{n_j+k-1,n_j+l-1}=\frac{\bar{\varrho}(2n_j)}{n_j(\log 2n_j)^2} \quad \text{(in Theorem 3)};$$

and $a_{kl}=b_{kl}=c_{kl}=0$ for $k, l=1, 2, \ldots, n_1-1$;

$$k=n_j,\ldots,2n_j-1 \quad\text{and}\quad l=1,2,\ldots,n_j-1;$$
$$k=2n_j,\ldots,n_{j+1}-1 \quad\text{and}\quad l=1,2,\ldots,n_{j+1}-1;$$
$$k=1,2,\ldots,n_j-1 \quad\text{and}\quad l=n_j,\ldots,2n_j-1;$$
$$k=1,2,\ldots,2n_j-1 \quad\text{and}\quad l=2n_j,\ldots,n_{j+1}-1; \quad j=1,2,\ldots.$$

After this preparation it is quite easy to conclude the proofs. For example, let us carry out the proof of Theorem 1.

If $(x, y)\in H_{r_j,r_j}$ (recall $n_j=2^{r_j}$), then by (9) and (18)

$$\max_{n_j\leqq m,n<n_{j+1}}\left|\sum_{k=n_j+1}^{m}\sum_{l=n_j+1}^{n}a_{kl}\varphi_{kl}(x,y)\right|\geqq \tag{19}$$

$$\geqq\frac{C_2^2 n_j(\log 2n_j)^2}{n_j\bar{\varrho}(2n_j)\log 2n_j}\geqq C_2^2 j \quad (j=1,2,\ldots).$$

By virtue of the second Borel—Cantelli lemma (17) implies that

$$\text{mes}\,(\limsup_{j\to\infty} H_{r_j,r_j})=1.$$

Thus (19) provides (4).

Besides, by (18)

$$\sum_{n_j \leqq k, l < 2n_j} a_{kl}^2 (\log \max (k, l))^2 \varrho^2 (\max (k, l)) \leqq$$

$$\leqq n_j^2 \frac{(\log 2n_j)^2 \varrho^2 (2n_j)}{n_j^2 \bar{\varrho}^2 (2n_j) (\log 2n_j)^2} \leqq \frac{1}{j^2} \quad (j = 1, 2, \dots).$$

Since the remaining $a_{kl} = 0$, hence (3) follows immediately. This finishes the proof of Theorem 1.

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