Acta Sci. Math., 42 (1980), 143—152

Characterization of Lebesgue-type decomposition of positive operators

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1. Introduction

Our concerns in this paper are (bounded linear) *positive*, i.e. non-negative definite, operators on a Hilbert space. Order relation among operators always refers to this notion of positivity; that is, $B \geqq A$ means $B-A$ is positive. For convenience, a positive operator B is said to *dominate* another A if $\alpha B \geqq A$ for some $\alpha \geqq 0$.

Given a positive operator A, we say a positive operator C to be *A-absolutely continuous* if there exists a sequence $\{C_n\}$ of positive operators such that $C_n \uparrow C$ and $C_n \leqq \alpha_n A$ for some $\alpha_n \geqq 0$ $(n=1, 2, \ldots)$. Here $C_n \uparrow C$ means that $C_1 \leqq C_2 \leqq \ldots$ and C_n converges strongly to C. In [2] ANDO showed that for any positive operator B there is the maximum of all A-absolutely continuous positive operators C such that $C \leqq B$, and established an algorithm for obtaining the maximum, denoted by $[A]B$, in terms of parallel addition;

$$[A]B = \lim_{n\to\infty} (nA):B.$$

Here the *parallel sum* $A:B$ of two positive operators A, B was introduced by ANDERSON and TRAPP [1] in study of electrical network connection; $A:B$ is defined, in operator matrix notation, as the maximum of all positive operators C such that

$$\begin{pmatrix} A & A \\ A & A+B \end{pmatrix} \geqq \begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix}.$$

Meanwhile, PEKAREV and SMULJAN [6] introduced the notion of *complement* of a positive operator B with respect to a positive operator A. When B dominates A^2 the complement, denoted by B_A, exists and is defined as the minimum of all positive operators C such that

$$\begin{pmatrix} B & A \\ A & C \end{pmatrix} \geqq 0.$$

Received February 9, 1979.

They developed detailed analysis of the map $B \mapsto B_A$ as well as the map $B \mapsto (B_A)_A$ in connection with the reverse operation of parallel addition, that is, parallel subtraction.

Our first aim in this paper is to present algorithms for obtaining $[A]B$ in terms of complement operation.

There is still an important binary operation for positive operators A, B. It is the *geometric mean* $A \# B$ introduced by PUSZ and WORONOWICZ [7]; $A \# B$ is defined as the maximum of all positive operators C such that

$$\begin{pmatrix} A & C \\ C & B \end{pmatrix} \geqq 0.$$

Our second aim is to show that $[A]B$ coincides with each of $[A:B]B$, $[(A:B)^2]B$, $[A \# B]B$ and $[(A \# B)^2]B$. Coincidence of $[A:B]B$ and $[(A:B)^2]B$ as well as that of $[A \# B]B$ and $[(A \# B)^2]B$ is not a trivial thing. As a consequence the identities $A \# B = A \# [A]B$ and $A:B = A:[A]B$ will be established.

In the next section fundamental properties and lemmas of parallel sum, complement and geometric mean are established in the form convenient for our aim. The main results will be presented in the final section.

2. Parallel sum, complement and geometric mean

In this section all operators are positive unless otherwise mentioned.

The *parallel sum* $A:B$ of two operators A, B is defined as the maximum of all operators C satisfying

(1) $$\begin{pmatrix} A & A \\ A & A+B \end{pmatrix} \geqq \begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix}.$$

Explicit representation for $A:B$ is given by

$$((A:B)f, f) = \inf\{(Ah, h) + (Bg, g);\ f = h + g\}.$$

If A and B are invertible

(2) $$A:B = (A^{-1} + B^{-1})^{-1}.$$

The following properties of parallel addition can be derived easily from definition (e.g. [1], [5]).

(3) $$A:B = B:A \quad \text{and} \quad (A:B):C = A:(B:C).$$

(4) $$(CAC):(CBC) \geqq C(A:B)C.$$

A consequence of (4) is

(5) $$(CAC):(CBC) = C(A:B)C \quad \text{for invertible } C.$$

(6) $$A_n \downarrow A \quad \text{and} \quad B_n \downarrow B \quad \text{implies} \quad A_n:B_n \downarrow A:B.$$

As mentioned in the Introduction, given positive operators A, B the maximum of all positive A-absolutely continuous C with $C \leqq B$ exists and is determined by

(7) $$[A]B = \lim_{n\to\infty} (nA):B.$$

Moreover, it is known (see [2]) that $[A]B=B$ is equivalent to the condition

(8) the linear manifold $\{h;\ B^{\frac{1}{2}}h \in \operatorname{ran}(A^{\frac{1}{2}})\}$ is dense in the whole space.

The following properties of the operation $[A]$ can be derived easily from definition (e.g. [2]).

(9) $$[A]B \geqq [C]B \quad \text{if } A \text{ dominates } C.$$

(10) $$[A](B+C) \geqq [A]B+[A]C.$$

A consequence of (10) is

(11) $$[A]B \geqq [A]C \quad \text{if} \quad B \geqq C.$$

More delicate properties are summarized in the following lemma.

Lemma 1. *For any positive operator A the operation $[A]$ has the following properties;*

(i) $[A](B+\alpha A)=[A]B+\alpha A$ *for* $\alpha>0$ *and* $B \geqq 0$.

(ii) *If positive operators B and A^p dominate each other for some $p>0$, then* $[A]B=B$.

Proof. (i) follows from the identity

$$(\beta A):(B+\alpha A) = \left(\frac{\beta}{\alpha+\beta}\right)^2 (((\alpha+\beta)A):B)+\frac{\alpha\beta}{\alpha+\beta} A \quad \text{for} \quad \alpha, \beta > 0,$$

which is easily checked for invertible A, B and then for general A, B by (6) through approximation of A by $A+\varepsilon I$ and B by $B+\varepsilon I$.

(ii) Suppose B and A^p dominate each other. If $p \geqq 1$, A dominates A^p, hence B. This implies $[A]B=B$. Suppose $0<p<1$. Then there is X such that $B=A^{p/2}XA^{p/2}$, $\ker(X)=\ker(A^p)$ (see [4]) and the restriction $X|_{\operatorname{ran}(A)^-}$ is an invertible operator on $\operatorname{ran}(A)^-$, the closure of the range of A. Now by (7) and (4)

$$B \geqq [A]B \geqq nA:B \geqq A^{p/2}(nA^{1-p}:X)A^{p/2}.$$

Since $X|_{\mathrm{ran}(A)^-}$ is invertible and $\ker(X)=\ker(A)=\ker(A^{1-p})$, by virtue of condition (8) $[A^{1-p}]X$ must coincide with X itself. Therefore by (7)

$$B \geqq [A]B \geqq A^{p/2}([A^{1-p}]X)A^{p/2} = A^{p/2}XA^{p/2} = B.$$

This completes the proof.

Let A, B be positive operators. It is known (e.g. [4], [7]) that there is a positive operator C for which

$$\begin{pmatrix} B & A \\ A & C \end{pmatrix} \geqq 0$$

if and only if B dominates A^2. In this case, there is the minimum of all such C. According to Pekarev—Smuljan [6] this minimum is called the *complement* of B with respect to A and is denoted by B_A. More explicit representation of B_A is given by

(12) $$B_A = Z^*Z,$$

where Z is the uniquely determined (bounded linear) operator such that $A=B^{\frac{1}{2}}Z$ and $\ker(Z^*)\supset\ker(B)$ (e.g. [4], [7]). If B and A^2 dominate each other, the restriction $Z|_{\mathrm{ran}(A)^-}$ is an invertible operator on $\mathrm{ran}(A)^-$. In particular,

(13) $$B_A = AB^{-1}A \quad \text{for invertible } B.$$

The following properties of complement can be derived easily from definition (e.g. [7]).

(14) $$B_A \leqq C_A \quad \text{if} \quad B \geqq C \quad \text{and } C \text{ dominates } A^2.$$

(15) $$B_n \downarrow B \quad \text{implies} \quad (B_n)_A \uparrow B_A \quad \text{if } B \text{ dominates } A^2.$$

As a consequence of (15), B dominates A^2 if and only if $A(B+\varepsilon I)^{-1}A$ is bounded above for $\varepsilon>0$. In this case

(16) $$B_A = \lim_{\varepsilon\downarrow 0} A(B+\varepsilon I)^{-1}A.$$

A little more effort will show

(17) $$B_A = \text{weak-}\lim_{\varepsilon\downarrow 0}(A+\varepsilon I)(B+\varepsilon I)^{-1}(A+\varepsilon I).$$

The following property and calculation rules of complement can be derived easily by (16), (2) and (4) through approximation.

(18) $$[A^2](B_A) = B_A \quad \text{if } B \text{ dominates } A^2.$$

(19) $$(A:B)_C = A_C + B_C \quad \text{if both } A \text{ and } B \text{ dominate } C^2.$$

(20) $$(A+B)_C \leqq A_C : B_C \quad \text{if both } A \text{ and } B \text{ dominate } C^2.$$

Lemma 2. (Pekarev and Smuljan [6]) *Let A, B be positive operators. Then B dominates A if and only if B does $A+B$. In this case, the following identity holds*

$$B_{A+B} = B_A + 2A + B.$$

Proof. The first assertion is immediate from definition. The expected identity is true when B is invertible. In fact, by (13)

$$B_{A+B} = (A+B)B^{-1}(A+B) = AB^{-1}A+2A+B = B_A+2A+B.$$

The general case results by (17) through approximation. This completes the proof.

Lemma 3. *The following three conditions for positive operators A, B and C are mutually equivalent,*

(i) $A:B \geqq C$,

(ii) $A-C \geqq (B-C)_C$,

(iii) $A-C \geqq (A+B)_A$.

If equality holds anywhere in (i) *or* (ii) *or* (iii) *then equality holds everywhere. In particular, the following identity holds*

$$A:B = A-(A+B)_A.$$

Proof. By definition of parallel addition, (i) is equivalent to

$$\begin{pmatrix} A-C & A \\ A & A+B \end{pmatrix} \geqq 0,$$

which is equivalent to (iii) by definition of complement. On the other hand, the identity

$$\begin{pmatrix} I & 0 \\ -I & I \end{pmatrix}\begin{pmatrix} A-C & A \\ A & A+B \end{pmatrix}\begin{pmatrix} I & -I \\ 0 & I \end{pmatrix} = \begin{pmatrix} A-C & C \\ C & B-C \end{pmatrix}$$

implies the equivalence of (ii) and (iii). This completes the proof.

Given positive operators A, B there is the maximum of all positive operator C such that

$$\begin{pmatrix} A & C \\ C & B \end{pmatrix} \geqq 0.$$

This maximum is called the *geometric mean* of A and B, and is denoted by $A \# B$. The following properties of geometric mean can be derived from definition (e.g. [3], [7]).

(21) $$A \# B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}}A^{\frac{1}{2}} \quad \text{for invertible } A.$$

For general A, the geometric mean $A \# B$ can be computed by approximation.

(22) $$A_n \downarrow A \quad \text{and} \quad B_n \downarrow B \quad \text{imply} \quad A_n \# B_n \downarrow A \# B.$$

(23) $$(CAC) \# (CBC) \geqq C(A \# B)C.$$

A consequence of (23) is

(24) $$(CAC) \# (CBC) = C(A \# B)C \quad \text{for invertible } C.$$

Lemma 4. *For any positive operators A, B the following indentity holds*

$$A_{(A\#B)} \# B_{(A\#B)} = A \# B.$$

Proof. Let $C = A \# B$. Clearly (see the second sentence after the proof of Lemma 1) A, B and A_C dominate C^2 and moreover $A \geqq B_C$, $B \geqq A_C$ hold. This implies $A \# B \geqq A_C \# B_C$. Reverse inequality follows immediately from the inequality

$$A_C \# B_C \geqq (A \# B)_C,$$

because $(A \# B)_{A\#B} = A \# B$. This inequality is surely guaranteed whenever both A and B dominate C^2. In fact (13), (14) and (23) will yield

$$\begin{aligned} A_C \# B_C &\geqq C(A+\varepsilon I)^{-1}C \# C(B+\varepsilon I)^{-1}C \\ &\geqq C\big((A+\varepsilon I)^{-1} \# (B+\varepsilon I)^{-1}\big)C \\ &= C\big((A+\varepsilon I) \# (B+\varepsilon I)\big)^{-1}C \quad \text{for} \quad \varepsilon > 0, \end{aligned}$$

the last equality resulting from (21), in which by (22) and (15) the last term $C\big((A+\varepsilon I) \# (B+\varepsilon I)\big)^{-1}C$ converges increasingly to $(A \# B)_C$ on taking limit $\varepsilon \to 0$. This completes the proof.

Relations between parallel sum and geometric mean are gathered in the following lemma.

Lemma 5. *The following relations hold for parallel sum and geometric mean.*

(i) $2^{-1}(A \# B) \geqq A:B \geqq \|A+B\|^{-1}(A \# B)^2$.

(ii) $(A + A \# B):(B + A \# B) = A \# B$.

Proof. By using approximation, A can be assumed to be invertible. Let $C = A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$. Then by (5) and (24) the first inequality of (i) is equivalent to

$$2^{-1}(I \# C) \geqq I:C,$$

which is, in turn, equivalent to

$$2^{-1}C^{\frac{1}{2}} \geqq C(I+C)^{-1}$$

by (21) and Lemma 3. But the last inequality is surely guaranteed by arithmetic-geometric mean inequality for a positive operator. In the same way the second inequality of (i) is equivalent to

$$C(I+C)^{-1} \geqq \|A+B\|^{-1}C^{\frac{1}{2}}AC^{\frac{1}{2}},$$

hence to

$$(I+C)^{-1} \geqq \|A+B\|^{-1}A.$$

Since the inverse map converts order-relation, this last inequality is equivalent to

$$\|A+B\|\, I \geqq A^{\frac{1}{2}}(I+C)A^{\frac{1}{2}},$$

which is surely guaranteed. (ii) results from the identity

$$(I+C^{\frac{1}{2}}):(C+C^{\frac{1}{2}}) = C^{\frac{1}{2}},$$

which is guaranteed by simple computation. This completes the proof.

3. Theorems

Theorem 1. *For any positive operators A, B*

$$[A]B = \lim_{\varepsilon \downarrow 0} \left((B+\varepsilon A)_{A^{\frac{1}{2}}}\right)_{A^{\frac{1}{2}}}.$$

If B dominates A, then

$$[A]B = (B_{A^{\frac{1}{2}}})_{A^{\frac{1}{2}}}.$$

Proof. Suppose first that A and B dominate each other. Then by (12)

$$B_{A^{\frac{1}{2}}} = Z^*Z,$$

where Z is an operator such that $\ker(Z)=\ker(A^{\frac{1}{2}})$, $A^{\frac{1}{2}}=B^{\frac{1}{2}}Z$ and $Z|_{\operatorname{ran}(A)^-}$ is an invertible operator on $\operatorname{ran}(A)^-$. This implies that

$$V(B_{A^{\frac{1}{2}}})^{\frac{1}{2}} = Z,$$

where V is a unitary operator on $\operatorname{ran}(A)^-$, and hence

$$(B_{A^{\frac{1}{2}}})^{\frac{1}{2}} V^* B^{\frac{1}{2}} = Z^* B^{\frac{1}{2}} = A^{\frac{1}{2}}.$$

Again by (12) the last identity leads to

$$(B_{A^{\frac{1}{2}}})_{A^{\frac{1}{2}}} = B^{\frac{1}{2}} V V^* B^{\frac{1}{2}} = B = [A]B,$$

the last equality resulting from domination by A. Thus the assertion is true in case A and B dominate each other.

Suppose next that B dominates A. Then for each n the operator $(nA):B$ and A dominate each other, hence

$$(nA):B = [A]\big((nA):B\big) = \big((nA):B\big)_{A^{\frac{1}{2}}}\big)_{A^{\frac{1}{2}}} \leqq (B_{A^{\frac{1}{2}}})_{A^{\frac{1}{2}}},$$

which implies, by definition of $[A]B$,

$$[A]B \leqq (B_{A^{\frac{1}{2}}})_{A^{\frac{1}{2}}}.$$

Since $(B_{A^{\frac{1}{2}}})_{A^{\frac{1}{2}}}$ is A-absolute continuous by (18), and $[A]B$ is the maximum of all A-absolutely continuous C with $C \leqq B$, the reverse inequality holds too, proving the second assertion of the theorem.

To prove the first assertion, remark that for any positive B and $\varepsilon>0$, $B+\varepsilon A$ dominates A. Therefore by Lemma 1 (i) and the second assertion already proved

$$[A]B+\varepsilon A = [A](B+\varepsilon A) = ((B+\varepsilon A)_{A^{\frac{1}{2}}})_{A^{\frac{1}{2}}},$$

which leads to the first assertion on taking limit $\varepsilon\to 0$. This completes the proof.

Theorem 2. *For any positive operators A, B*

$$[A]B=[A:B]B=[(A:B)^2]B=(A-A:B)_{A:B}+A:B.$$

Proof. Let $C=A:B$. Since $A, B\geqq C$ and C dominates C^2, by (9)

$$[A]B\geqq [C]B\geqq [C^2]B.$$

Further (10) (concavity) implies

$$[C^2]B\geqq [C^2](B-C)+[C^2]C=[C^2](B-C)+C,$$

the last equality resulting from Lemma 1 (ii). On the other hand, by Lemma 3

$$B-C\geqq (A-C)_C,$$

which together with (11) and C^2-absolute continuity of $(A-C)_C$ implies

$$[C^2](B-C)\geqq (A-C)_C.$$

Now it remains to prove the relation

$$(A-C)_C+C=[A]B.$$

To this end, for each n let $B_n=(nC):B$ and $C_n=A:B_n$. Since A and $A+B_n$ dominate each other, Lemma 1 (ii) with $p=\frac{1}{2}$ and with A^2 instead of A shows

$$[A^2](A+B_n)=A+B_n,$$

and hence by Theorem 1

$$((A+B_n)_A)_A=A+B_n.$$

On the other hand, since the relation by Lemma 3

$$C_n=A:B_n=A-(A+B_n)_A$$

yields

$$((A+B_n)_A)_A=(A-C_n)_A,$$

combination of these two relations leads to

$$A+B_n=(A-C_n)_A.$$

Further Lemma 2, with $A-C_n$ and C_n instead of B and A respectively, shows

$$(A-C_n)_A=(A-C_n)_{C_n}+C_n+A.$$

Therefore the following relation has been proved

$$B_n = (A - C_n)_{C_n} + C_n.$$

Since (3) (associativity and commutativity) implies

$$B_n = \frac{n}{n+1}(((n+1)A):B) \quad \text{and} \quad C_n = \frac{n}{n+1}C,$$

the established relation becomes

$$((n+1)A):B = \frac{n}{n+1}\left(A - \frac{n}{n+1}C\right)_C + C,$$

which leads to

$$[A]B = (A-C)_C + C$$

by (15) because $A - \frac{n}{n+1}C$ converges decreasingly to $A-C$. This completes the proof.

Corollary 3. *For any positive operators A, B*

$$A:B = A:[A]B.$$

Proof. Let $C = A:B$. Then definitely both A and B dominate C^2. By Theorem 2 it suffices to show $A:[C^2]B = C$. Calculation rules (19), (20) and Theorem 1 will yield

$$C = [C^2]C = ((A:B)_C)_C = (A_C + B_C)_C \leqq (A_C)_C:(B_C)_C \leqq A:[C^2]B \leqq A:B = C.$$

This completes the proof.

Theorem 4. *For any positive operators A, B*

$$[A]B = [A \# B]B = [(A \# B)^2]B = A_{A \# B}.$$

Proof. With $C = A \# B$ Lemma 5 (ii) shows

$$(A+C):(B+C) = C.$$

Then, in the proof of Theorem 2, replacement of A and B by $A+C$ and $B+C$, respectively yields

$$[C](B+C) \geqq [C^2]B + C \geqq A_C + C = [C](B+C),$$

which implies by Lemma 1 (i)

$$[C]B = [C^2]B = A_C.$$

On the other hand, Lemma 5 (i) together with (9) implies

$$[C]B \geqq [A:B]B \geqq [C^2]B,$$

which completes the proof, because $[A:B]B = [A]B$ by Theorem 2.

Corollary 5. *For any positive operators A, B*

$$A \# B = A \# [A] B.$$

Proof. Let $C = A \# B$. By Theorem 4 it suffices to prove $A \# [C^2] B = C$. Twice applications of Lemma 4 will show

$$C = A_C \# B_C = (A_C)_C \# (B_C)_C,$$

hence by Theorem 1

$$C = [C^2] A \# [C^2] B \leqq A \# [C^2] B \leqq A \# B = C.$$

Therefore $A \# [C^2] B = C$. This completes the proof.

Acknowledgment. The author would like to thank Professor T. Ando for suggesting this research and stimulating discussions.

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