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Almost periodic functions and functional equations

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1. Introduction. In this paper we deal with bounded solutions of a class of functional equations defined on topological groups. Our results are based on the fact that all characters of a group are almost periodic functions (see e.g. MAAK [5], [6]). This can be restated by saying that all bounded solutions of the functional equation $f(xy)=f(x)f(y)$ are almost periodic functions. In this work this result is generalized for the functional equation (1) which has been dealt by many authors (see [1], [7], [8], [9], [11], [12]) but has not been completely solved. Using our results we give all bounded solutions of (1) on commutative groups. Our other main result is the proof of the fact that all bounded solutions of (2), and in particular of (3), are almost periodic functions. Concerning these equations see [1].

We note that some of our results remain valid on topological semigroups as well. On the other hand the method used in Section 3 to solve equation (1) can be used successfully to solve other similar equations ([10]).

2. Preliminary facts and results. Let G be a group and X a uniform space. A function $f\colon G\to X$ is said to be *almost periodic* if for every X-entourage R there exists a finite covering $A_1, \ldots, A_n$ of G such that $\big(f(xz), f(yz)\big)\in R$ whenever $z\in G$, $x, y\in A_i$ $(i=1, \ldots, n)$.

Let H be a set and X a uniform space. A function $f\colon H\to X$ is said to be *totally bounded* if for every X-entourage R there exists a finite covering $B_1, \ldots, B_n$ of $\operatorname{ran} f$, the range of f, such that $(x, y)\in R$ whenever $x, y\in B_i$ $(i=1, \ldots, n)$.

If G is a group, X is a uniform space and $f\colon G\to X$ is an almost periodic function, then f is totally bounded. Indeed, if R is any X-entourage and $A_1, \ldots, A_n$ is a finite covering of G for which $\big(f(xz), f(yz)\big)\in R$ holds whenever $z\in G$, $x, y\in A_i$ $(i=1, \ldots, n)$ then $B_i=f(A_i)$ $(i=1, \ldots, n)$ yields an appropriate covering of $\operatorname{ran} f$.

If G is a topological group, X is a Banach space, then the continuous function $f\colon G\to X$ is almost periodic if and only if the orbit of f is relatively compact in the Banach space of all continuous, bounded X-valued functions on G. (The

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orbit of f is the set of all right translates of f. It can be proved that this is equivalent to the relative compactness of the set of all left translates of f.)

For more about almost periodic functions see e.g. [2], [3], [4], [5], [6].

3. Bounded solutions of functional equations.

Theorem 3.1. *Let G be a topological group, n a positive integer, and a_k, b_k: $G \to \mathbf{C}$ bounded functions, where the a_k's are continuous $(k=1, \ldots, n)$. If f: $G \to \mathbf{C}$ is a function for which*

$$f(xy) = \sum_{k=1}^{n} a_k(x)\, b_k(y) \tag{1}$$

holds whenever $x, y \in G$, then f is a continuous almost periodic function. If the a_k's are linearly independent, then the b_k's are also continuous almost periodic functions.

Proof. Let $B(G)$ denote the set of all complex valued continuous bounded functions on G equipped with the pointwise operations and sup-norm. $B(G)$ is a Banach space. Let

$$A_k = \{b_k(y)\, a_k \colon y \in G\} \quad (k = 1, \ldots, n).$$

As a_k is continuous bounded and b_k is bounded, hence A_k is relatively compact in $B(G)$ $(k=1, \ldots, n)$. Let F denote the orbit of f in $B(G)$, then by (1) we see that F is a subset of the set $A_1 + \ldots + A_n$, which is a continuous image of the relatively compact set $A_1 \times \ldots \times A_n$. Hence f is almost periodic. The continuity of f follows directly from (1) by the substitution $y=e$ (the unit element).

If the a_k's are linearly independent then there are elements $x_1, \ldots, x_n$ of G for which the matrix $(a_i(x_j))$ is regular (see e.g. [11]). Substituting the x_i's into (1) in place of x, for fixed y we get that the numbers $b_k(y)$ satisfy a system of linear equations, the matrix of which is regular. Hence the functions b_k can be represented as a linear combination of some translates of f and thus they are continuous almost periodic functions.

Theorem 3.1 can be generalized as follows:

Theorem 3.2. *Let G be a topological group, L, M, N normed spaces, g: $G \to L$ a totally bounded continuous function, h: $G \to M$ a bounded function and F: $L \times M \to N$ a bounded bilinear operator. If f: $G \to N$ is a function for which*

$$f(xy) = F(g(x), h(y)) \tag{2}$$

holds whenever $x, y \in G$, then f is a continuous almost periodic function.

Proof. The continuity of f follows by substituting $y=e$. Let $\varepsilon > 0$ be arbitrary and let K be a bound for h. As g is totally bounded, there exists a finite covering $L_1, \ldots, L_n$ of $\operatorname{ran} g$ such that $\|u-v\| < \varepsilon$ whenever $u, v \in L_i$ $(i=1, \ldots, n)$. Let $A_i = g^{-1}(L_i)$ $(i=1, \ldots, n)$ then $A_1, \ldots, A_n$ is a finite covering of G. If $x, y \in A_i$

$(i=1, \ldots, n)$ then $g(x), g(y) \in L_i$ hence $\|g(x)-g(y)\|<\varepsilon$ which implies for every $z \in G$

$$\|f(xz)-f(yz)\| = \big\|F\big(g(x), h(z)\big)-F\big(g(y), h(z)\big)\big\| =$$
$$= \big\|F\big(g(x)-g(y), h(z)\big)\big\| \leqq C\|g(x)-g(y)\| \cdot K \leqq C \cdot \varepsilon \cdot K$$

that is f is almost periodic.

The linearity of F in (2) can be replaced by uniform continuity. Namely, we have

Theorem 3.3. *Let G be a topological group, L, M, N uniform spaces, $g\colon G \to L$ a totally bounded continuous function, $h\colon G \to M$ a bounded function and $F\colon L \times M \to N$ a uniformly continuous function. If $f\colon G \to N$ is a function for which* (2) *holds whenever $x, y \in G$ then f is a continuous almost periodic function.*

Proof. The continuity of f follows by substituting $y=e$. Let R be an arbitrary X-entourage. By the uniform continuity of F there exists an $L \times M$-entourage S, for which $\big((u, v), (u', v')\big) \in S$ implies $\big(F(u, v), F(u', v')\big) \in R$. Further there exists an L-entourage T such that $(u, u') \in T$ and $v \in M$ implies $\big((u, v), (u', v')\big) \in S$. By the totally boundedness of g there exists a finite covering $L_1, \ldots, L_n$ of ran g such that $u, u' \in L_i$ implies $(u, u') \in T$ $(i=1, \ldots, n)$.

Let $A_i = g^{-1}(L_i)$ $(i=1, \ldots, n)$, then $A_1, \ldots, A_n$ is a finite covering of G, and for $x, y \in A_i$ $(i=1, \ldots, n)$ we have $g(x), g(y) \in L_i$, hence for $z \in G$

$$\big(f(xz), f(yz)\big) = \big(F(g(x), h(z)), F(g(y), h(z))\big) \in R,$$

that is f is almost periodic.

Remark 3.4. The conditions of Theorem 3.3 are satisfied for instance if g, h are bounded functions with values in finite dimensional vector spaces (or, more generally, in Montel spaces), L, M denote the closures of their ranges respectively, and F is continuous on $L \times M$. Hence we have the corollaries:

Corollary 3.5. *Let G be a topological group, let $g, h\colon G \to \mathbf{C}$ (the set of complex numbers) be bounded functions, and let g be continuous. Let $F\colon$ $(\operatorname{ran} g \times \operatorname{ran} h)^- \to \mathbf{C}$ be a continuous function. If $f\colon G \to \mathbf{C}$ is a function for which* (2) *holds whenever $x, y \in G$, then f is a continuous almost periodic function.*

Corollary 3.6. *Let G be a topological group, $f\colon G \to \mathbf{C}$ be a continuous bounded function. Let $F\colon$ $(\operatorname{ran} f \times \operatorname{ran} f)^- \to \mathbf{C}$ be a continuous function. If the equality*

$$f(xy) = F\big(f(x), f(y)\big) \tag{3}$$

holds whenever $x, y \in G$, then f is almost periodic.

4. Bounded solutions of equation (1). In this section we exhibit all bounded solutions of equation (1) on commutative groups. More exactly, we show that f is a trigonometric polynomial and so are the functions a_k, b_k whenever the a_k's and also the b_k's are linearly independent. By trigonometric polynomial we mean a linear combination of continuous characters. Here the number of different characters is called the degree of the trigonometric polynomial.

In what follows we assume that G is a commutative topological group with sufficiently many continuous characters, that is any two elements of G can be separated by continuous character. For instance all locally compact Hausdorff groups posses this property and so does the additive group of any locally convex topological vectorspace. Then the Fourier transform of almost periodic functions can be defined as an injective mapping by the formula

$$\hat{f}(\gamma) = \int f\bar{\gamma}$$

(where $\int$ denotes the invariant mean on almost periodic functions) whenever γ is a continuous character of G (see [5], [6]).

Theorem 4.1. *Let G be a commutative group with sufficiently many continuous characters, n a positive integer and a_k, b_k, f: $G \to \mathbf{C}$ $(k=1, \ldots, n)$ functions. If f is a continuous bounded function, then it is a trigonometric polynomial of degree at most n.*

Proof. First we assume that the a_k's and also the b_k's are linearly independent. Then there are elements $x_1, \ldots, x_n$ of G for which the matrix $(a_i(x_j))$ is regular. As in Theorem 3.1 we obtain that the b_k's are continuous bounded functions. Similarly, we get the same for the a_k's.

By Theorem 3.1, f, a_k, b_k are almost periodic functions. On the other hand, the linear independence of the a_k's implies the same for their Fourier transforms.

Now let y be fixed and compute the Fourier transforms of both sides of (1) as functions of x. We obtain

$$\hat{f}(\gamma)\gamma(y) = \sum_{k=1}^{n} \hat{a}_k(\gamma) b_k(y) \tag{4}$$

where $y \in G$ and γ is a character of G. Now compute the Fourier transforms of both sides of (4) as functions of y. We obtain

$$\hat{f}(\gamma)\hat{\gamma}(\tau) = \sum_{k=1}^{n} \hat{a}_k(\gamma)\hat{b}_k(\tau) \tag{5}$$

where γ, τ are characters of G. Let $\gamma_1, \ldots, \gamma_n$ be characters of G for which the matrix $(\hat{a}_k(\gamma_j))$ is regular. Substituting in (5) γ_j for γ we get that the numbers $\hat{b}_k(\tau)$ for $\tau \neq \gamma_j$ $(j=1, \ldots, n)$ satisfy a homogeneous linear system of equations, the matrix of which is regular, hence $\hat{b}_k(\tau)=0$ for $\tau \neq \gamma_j$ $(j=1, \ldots, n,\ k=1, \ldots, n)$. Thus the Fourier transform of $b_k - \sum_{j=1}^{n} \hat{b}_k(\gamma_j)\gamma_j$ vanishes, and hence b_k is a trigonometric polynomial of degree at most n $(k=1, \ldots, n)$. Similarly we get the statement for a_k, f.

In the general case, when the a_k's or the b_k's are linearly dependent, then by the successive decreasing of n we can make the a_k's and the b_k's simultaneously linearly independent and hence the statement remains valid for f.

Corollary 4.2. *Let G be a locally compact topological group. Then any finite dimensional translation invariant subspace of the Banach space of all continuous bounded complex valued functions on G consists of almost periodic functions. If G is commutative then this subspace consists of trigonometric polynomials.*

Proof. Let M be the subspace in question and let $a_1, \dots, a_n$ be a basis of M. Then for every $f \in M$ we have

$$f(xy) = \sum_{k=1}^{n} a_k(x)\, b_k(y) \tag{6}$$

whenever $x, y \in G$. Since the a_k's are linearly independent, hence the b_k's are continuous bounded functions. This implies that f is almost periodic. If G is commutative then, as in the proof of Theorem 4.1, we obtain that the b_k's are trigonometric polynomials and hence substituting $x=e$ in (6) we see that f is a trigonometric polynomial. In particular, the a_k's are trigonometric polynomials.

References

[1] J. Aczél, *Lectures on functional equations and their applications*, Academic Press (New York—London, 1966).

[2] S. Bochner—J. von Neumann, Almost periodic functions in groups. II, *Trans. Amer. Math. Soc.*, **37** (1935), 21—50.

[3] W. F. Eberlein, Abstract ergodic theorems and weak almost periodic functions, *Trans. Amer. Math. Soc.*, **67** (1949), 217—240.

[4] E. Hewitt—K. Ross, *Abstract Harmonic Analysis* I, II, Springer (Berlin—Heidelberg—New York, 1963, 1970).

[5] W. Maak, *Darstellungstheorie unendlicher Gruppen und fastperiodische Funktionen*, Enzyklopädie d. math. Wiss. I. 1, 2. Aufl., Heft 7, I.

[6] W. Maak, *Fastperiodische Funktionen*, Springer (Berlin—Göttingen—Heidelberg, 1950).

[7] M. A. McKiernan, Equations of the form $H(x \circ y) = \sum_i f_i(x) g_i(y)$, *Aequationes Math*, **16** (1977), 51—58.

[8] M. A. McKiernan, The matrix equation $a(x \circ y) = a(x) + a(x)a(y) + a(y)$, *Aequationes Math.*, **15** (1977), 213—223.

[9] T. A. O'Connor, A solution of the functional equation $\varphi(x-y) = \sum_i^n a_j(x)\overline{a_j(y)}$ on a locally compact Abelian group, *Aequationes Math.*, **15** (1977), 113.

[10] L. Székelyhidi, Almost periodic solutions of linear functional equations, *to appear*.

[11] E. Vincze, Eine allgemeinere Methode in der Theorie der Funktionalgleichungen. I, II, *Publ. Math. Debrecen*, **9** (1962), 149—163, 314—323.

[12] E. Vincze, Eine allgemeinere Methode in der Theorie der Funktionalgleichungen. III, *Publ. Math. Debrecen*, **10** (1963), 283—318.

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