Acta Sci. Math., 42 (1980), 183—188

The maximal function of a contraction

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1. Let $\mathfrak{E}$ be a separable Hilbert space. An $\mathscr{L}(\mathfrak{E})$-valued *semi-spectral measure* F on the unit circle $\mathbf{T}$ is a map from the family of the Borel sets $\mathscr{B}(\mathbf{T})$ of the unit circle into $\mathscr{L}(\mathfrak{E})$, such that for any $a\in\mathfrak{E}$, $\sigma\to(F(\sigma)a, a)$ is a positive Borel measure. A semi-spectral measure E is *spectral* if for any σ_1, σ_2 in $\mathscr{B}(\mathbf{T})$ we have $E(\sigma_1\cap\sigma_2)=E(\sigma_1)E(\sigma_2)$ and $E(\mathbf{T})=I_{\mathfrak{E}}$.

By the Naimark dilation theorem, for any $\mathscr{L}(\mathfrak{E})$-valued semi-spectral measure F there exists a spectral dilation $[\mathfrak{K}, V, E]$, i.e., a Hilbert space $\mathfrak{K}$, a bounded operator V from $\mathfrak{E}$ into $\mathfrak{K}$ and an $\mathscr{L}(\mathfrak{K})$-valued spectral measure E on $\mathbf{T}$ such that for any $\sigma\in\mathscr{B}(\mathbf{T})$

$$F(\sigma) = V^* E(\sigma) V. \tag{1.1}$$

For a Hilbert space $\mathfrak{F}$, we denote by $E_{\mathfrak{F}}^{\times}$ the spectral measure corresponding to the multiplication by e^{it} on $L^2(\mathfrak{F})$.

An $\mathscr{L}(\mathfrak{E})$-valued semi-spectral measure F is of *analytic type* if it admits a spectral dilation of the form $[L^2(\mathfrak{F}), V, E_{\mathfrak{F}}^{\times}]$ such that $V\mathfrak{E}\subset L^2_+(\mathfrak{F})$. The name is justified by the fact that there exists an analytic function $\{\mathfrak{E}, \mathfrak{F}, \Theta(\lambda)\}$ (see [4], [5]) such that for each $a\in\mathfrak{E}$

$$\Theta(\lambda)a = (Va)(\lambda) \quad (\lambda\in\mathbf{D}). \tag{1.2}$$

Moreover, $\{\mathfrak{E}, \mathfrak{F}, \Theta(\lambda)\}$ is an L^2*-bounded analytic function*, i.e., there exists $M>0$ such that

$$\sup_{0\leq r<1} \frac{1}{2\pi}\int_0^{2\pi} \|\Theta(re^{it})a\|^2\,dt \leq M^2\|a\|^2 \quad (a\in\mathfrak{E}). \tag{1.3}$$

Conversely, to any L^2-bounded analytic function $\{\mathfrak{E}, \mathfrak{F}, \Theta(\lambda)\}$ it corresponds an analytic type semi-spectral measure F_Θ, with a dilation as $\{L^2(\mathfrak{F}), V_\Theta, E_{\mathfrak{F}}^{\times}\}$, such that (1.2) is verified.

Received November 1, 1978.

An L^2-bounded analytic function $\{\mathfrak{E}, \mathfrak{F}, \Theta(\lambda)\}$ is called *outer* if

$$\bigvee_0^\infty e^{int} V_\Theta \mathfrak{E} = L_+^2(\mathfrak{F}). \tag{1.4}$$

To an arbitrary $\mathcal{L}(\mathfrak{E})$-valued semi-spectral measure F on $\mathbf{T}$ a unique outer L^2-bounded analytic function $\{\mathfrak{E}, \mathfrak{F}_1, \Theta_1(\lambda)\}$ was attached in [4] such that the corresponding semi-spectral measure F_{Θ_1} is maximal among the $\mathcal{L}(\mathfrak{E})$-valued semi-spectral measures of analytic type dominated by F. This unique outer L^2-bounded analytic function is called the *maximal function* of the semi-spectral measure F. In the present note, for the semi-spectral measure F corresponding to a contraction T, some specific properties of the maximal function, in connection with the Sz.-Nagy—Foiaş model for T, are obtained.

2. Let T be a contraction on a Hilbert space $\mathfrak{H}$, and let U be its minimal unitary dilation acting on $\mathfrak{K}$. If E is the spectral measure of U, then the *semi-spectral measure of the contraction* T is the $\mathcal{L}(\mathfrak{H})$-valued semi-spectral measure obtained by the compression of E to $\mathfrak{H}$, i.e.

$$F_T(\sigma) = P_{\mathfrak{H}} E(\sigma)|\mathfrak{H} \quad (\sigma \in \mathcal{B}(\mathbf{T})). \tag{2.1}$$

Now, let us sketch the way to obtain the maximal function of F_T. If we put

$$\mathfrak{K}_+ = \bigvee_0^\infty U^n \mathfrak{H}, \tag{2.2}$$

then $U_+ = U|\mathfrak{K}_+$ is an isometry on $\mathfrak{K}_+$. Taking the Wold decomposition of U_+ on $\mathfrak{K}_+$ it follows that

$$\mathfrak{K}_+ = M_+(\mathfrak{L}_*) \oplus \mathfrak{R}, \tag{2.3}$$

where $\mathfrak{L}_* = \mathfrak{K}_+ \ominus U_+ \mathfrak{K}_+$, $M_+(\mathfrak{L}_*) = \bigoplus_0^\infty U_+^n \mathfrak{L}_*$ and $\mathfrak{R} = \bigcap_{n=0}^\infty U_+^n \mathfrak{K}_+$. Let $P^{\mathfrak{L}_*}$ be the orthogonal projection of $\mathfrak{K}_+$ onto $M_+(\mathfrak{L}_*)$, $\Phi^{\mathfrak{L}_*}$ the Fourier representation of $M_+(\mathfrak{L}_*)$ onto $L_+^2(\mathfrak{L}_*)$, and V_1 the bounded linear operator from $\mathfrak{H}$ into $L_+^2(\mathfrak{L}_*)$ defined by

$$V_1 = \Phi^{\mathfrak{L}_*} P^{\mathfrak{L}_*} | \mathfrak{H}. \tag{2.4}$$

Then the $\mathcal{L}(\mathfrak{H})$-valued semi-spectral measure defined by

$$F_1(\sigma) = V_1^* E_{\mathfrak{L}_*}^{\times}(\sigma) V_1 \quad (\sigma \in \mathcal{B}(\mathbf{T})) \tag{2.5}$$

is of analytic type. The L^2-bounded analytic function $\{\mathfrak{H}, \mathfrak{L}_*, \Theta_1(\lambda)\}$ attached to F_1, as in section 1, is the maximal function of F_T and is called the *maximal function of the contraction* T.

In the next Proposition (suggested by C. Foiaş) an explicit form of the maximal function of T is given.

Proposition 1. *The maximal function* $\{\mathfrak{H}, \mathfrak{L}_*, \Theta_1(\lambda)\}$ *of the contraction* T *on* $\mathfrak{H}$ *coincides with* $\{\mathfrak{H}, \mathfrak{D}_{T*}, \Theta(\lambda)\}$ *where*

$$\Theta(\lambda) = D_{T^*}(I-\lambda T^*)^{-1} \quad (\lambda\in\mathbf{D}). \tag{2.6}$$

Proof. We shall show that for any $\lambda\in\mathbf{D}$ and $h\in\mathfrak{H}$

$$\Theta(\lambda)h = \omega_*\Theta_1(\lambda)h \quad (\lambda\in\mathbf{D}), \tag{2.7}$$

where ω_* is the unitary operator from $\mathfrak{L}_*$ into $\mathfrak{D}_{T*}$ defined by

$$\omega_*(I_{\mathfrak{R}}-UT^*)h = D_{T^*}h. \tag{2.8}$$

If Θ_n: $\mathfrak{H}\to\mathfrak{L}_*$ are the Taylor coefficients of the maximal function $\{\mathfrak{H}, \mathfrak{L}_*, \Theta_1(\lambda)\}$, then for any $h\in\mathfrak{H}$ and $l_*\in\mathfrak{L}_*$ we have

$$(\Theta_n h, l_*)_{\mathfrak{L}_*} = \frac{1}{2\pi}\int_0^{2\pi} ((V_1h)(e^{it}), e^{int}l_*)_{\mathfrak{L}_*}dt = (V_1h, e^{int}\Phi^{\mathfrak{L}_*}l_*)_{L^2(\mathfrak{L}_*)} =$$

$$= (\Phi^{\mathfrak{L}_*}P^{\mathfrak{L}_*}h, \Phi^{\mathfrak{L}_*}U^n l_*)_{L^2(\mathfrak{L}_*)} = (P^{\mathfrak{L}_*}h, U^n l_*)_{\mathfrak{R}} = (U^{*n}P^{\mathfrak{L}_*}h, l_*)_{\mathfrak{R}} = (P^{\mathfrak{L}_*}U^{*n}h, l_*)_{\mathfrak{L}_*}.$$

Hence, the coefficients of $\Theta_1(\lambda)$ are of the form

$$\Theta_n = P^{\mathfrak{L}_*}U^{*n}|\mathfrak{H}. \tag{2.9}$$

In order to prove (2.7) it is enough to show that

$$\Theta_n h = (I-UT^*)T^{*n}h \quad \text{for} \quad n\geqq 0,$$

or, by (2.9), that

$$U^{*n}h-(I-UT^*)T^{*n}h\perp\mathfrak{L}_*. \tag{2.10}$$

But, for any $h, h'\in\mathfrak{H}$ we have

$$(U^{*n}h-(I-UT^*)T^{*n}h, (I-UT^*)h') =$$
$$= (U^{*n}h-T^{*n}h+UT^{*n+1}h, h')-(U^{*n+1}h-U^*T^{*n}h+T^{*n+1}h, T^*h') =$$
$$= (T^{*n}h-T^{*n}h+TT^{*n+1}h, h')-(T^{*n+1}h-T^{*n+1}h+T^{*n+1}h, T^*h') =$$
$$= (TT^{*n+1}h, h')-(T^{*n+1}h, T^*h') = 0.$$

Hence (2.10) is valid and then so is (2.7). Thus the proof of Proposition 1 is done.

Remark that the maximal function of T is not zero unless T is a coisometric operator.

From the fact that the characteristic function of T

$\Theta_T(\lambda) = [-T+\lambda D_{T^*}(I-\lambda T^*)^{-1}D_T]|\mathfrak{D}_T$ verifies $\Theta_T(\lambda)D_T = D_{T^*}(I-\lambda T^*)^{-1}(\lambda I-T)$

(see [5]) it results the following relation between the maximal function and the characteristic function of the contraction T:

$$\Theta_T(\lambda)D_T = \Theta(\lambda)(\lambda I-T) \quad (\lambda\in\mathbf{D}). \tag{2.11}$$

If the contraction T belongs to the class $C_{\cdot 0}$ (i.e. $T^{*n} \to 0$ as $n \to \infty$) then $\mathfrak{K} = M(\mathfrak{L}_*)$, and thus by (2.3)—(2.5) it results that the semi-spectral measure of T is of analytic type if and only if $T \in C_{\cdot 0}$. In this case the contraction T is uniquely determined (up to unitary equivalence) by its maximal function. Moreover, in the $C_{\cdot 0}$ case, the maximal function gives an explicit form of the imbedding of $\mathfrak{H}$ into the space $\mathbf{H} = H^2(\mathfrak{D}_{T^*}) \ominus \Theta_T H^2(\mathfrak{D}_T)$ of the Sz.-Nagy—Foiaş functional model for T.

Proposition 2. *Let T be a contraction of the class $C_{\cdot 0}$ on the Hilbert space $\mathfrak{H}$ and let $\{\mathfrak{H}, \mathfrak{D}_{T^*}, \Theta(\lambda)\}$ be its maximal function. The image of an element $h \in \mathfrak{H}$ in the space of the functional model* $\mathbf{H}$ *is the function $u \in H^2(\mathfrak{D}_{T^*})$ defined by*

$$u(\lambda) = \Theta(\lambda) h \quad (\lambda \in \mathbf{D}). \tag{2.12}$$

Proof. The functional model (see [5], Ch. VI) is obtained by a unitary imbedding Φ of the dilation space $\mathfrak{K}$ of T into a functional space.

In the $C_{\cdot 0}$ case $\mathfrak{K} = M(\mathfrak{L}_*)$, $\Phi = \Phi^{\mathfrak{D}_T^*}$ and it follows that

$$\mathbf{H} = \Phi \mathfrak{H} = \Phi^{\mathfrak{D}_T^*} \mathfrak{H}.$$

From (2.4) and (2.7) it results that $\mathbf{H} = V_\Theta \mathfrak{H}$ and, consequently, $u \in \mathbf{H}$ is given by

$$u(\lambda) = (V_\Theta h)(\lambda) = \Theta(\lambda) h \quad (\lambda \in \mathbf{D}).$$

The proof is finished.

3. In general, the maximal function of a contraction is not bounded. If $\{\mathfrak{H}, \mathfrak{L}_*, \Theta_1(\lambda)\}$ is bounded, then there exists $\Theta_1(e^{it})$ a.e. as non-tangential strong limit of $\Theta_1(\lambda)$ and

$$dF_{\Theta_1} = \frac{1}{2\pi} \Theta_1(e^{it})^* \Theta_1(e^{it})\, dt \quad \text{a.e.} \tag{3.1}$$

Concerning the boundedness of $\Theta_1(\lambda)$, we have the following

Proposition 3. *The L^2-bounded analytic function $\{\mathfrak{E}, \mathfrak{F}, \Theta(\lambda)\}$ is bounded if and only if the corresponding semi-spectral measure F_Θ is boundedly dominated by the Lebesgue measure dt on* $\mathbf{T}$.

Proof. If $\{\mathfrak{E}, \mathfrak{F}, \Theta(\lambda)\}$ is bounded, then for any analytic polynomial p and for $a \in \mathfrak{E}$

$$\int_0^{2\pi} |p|^2 d(F_\Theta(t)a, a) = \frac{1}{2\pi} \int_0^{2\pi} |p|^2 (\Theta(e^{it})^* \Theta(e^{it}) a, a)\, dt =$$

$$= \frac{1}{2\pi} \int_0^{2\pi} |p|^2 \|\Theta(e^{it}) a\|^2 dt \leqq M^2 \frac{1}{2\pi} \int_0^{2\pi} |p|^2 \|a\|^2 dt.$$

It results that $dF_\Theta \leqq M^2 \frac{1}{2\pi}\, dt$. Conversely, if $dF_\Theta \leqq M^2 \frac{1}{2\pi}\, dt$, then

$$\int_0^{2\pi} |p|^2 \|(V_\Theta a)(t)\|^2\, dt = \|pV_\Theta a\|^2_{L^2(\mathfrak{F})} = \int_0^{2\pi} |p|^2 d(E_{\mathfrak{F}}^{\times} V_\Theta a, V_\Theta a) =$$

$$= \int_0^{2\pi} |p|^2 d\big(F_\Theta(t)a, a\big) \leqq M^2 \frac{1}{2\pi} \int_0^{2\pi} |p|^2 \|a\|^2\, dt.$$

It follows that

$$\|(V_\Theta a)(t)\| \leqq M\|a\| \quad \text{a.e.} \tag{3.2}$$

Using the Poisson integral of $\Theta(\lambda)$ and (3.2), it results that

$$\|\Theta(\lambda)a\| = \left\| \frac{1}{2\pi} \int_0^{2\pi} P_\lambda(t)(V_\Theta a)(t)\, dt \right\| \leqq$$

$$\leqq \frac{1}{2\pi} \int_0^{2\pi} P_\lambda(t)\|(V_\Theta a)(t)\|\, dt \leqq M\|a\| \frac{1}{2\pi} \int_0^{2\pi} P_\lambda(t)\, dt = M\|a\|$$

and the proof is finished.

It is known [2] that the contraction T with the spectral radius $\varrho(T) < 1$ is characterized by the fact that the semi-spectral measure F_T has bounded derivative. Therefore the following holds.

Corollary. *If the spectrum of the contraction T is in the open unit disc, then the maximal function $\{\mathfrak{H}, \mathfrak{L}_*, \Theta_1(\lambda)\}$ is bounded.*

Moreover, the above quoted result of Schreiber can be completed in the following manner.

Proposition 4. *A contraction T on a Hilbert space $\mathfrak{H}$ has the semi-spectral measure F_T of the form $dF_T = \Theta(e^{it})^* \Theta(e^{it})dt$, with $\{\mathfrak{H}, \mathfrak{F}, \Theta(\lambda)\}$ a bounded analytic function, if and only if $T \in C_{\cdot 0}$ and $\varrho(T) < 1$. Moreover, the bounded analytic function $\{\mathfrak{H}, \mathfrak{F}, \Theta(\lambda)\}$ has a bounded inverse if and only if T is a strict contraction.*

Proof. By the form of dF_T it follows that F_T is of analytic type i.e. $T \in C_{\cdot 0}$. The boundedness of $\Theta(\lambda)$ implies that F_T has bounded derivative and from [2] it results that $\varrho(T) < 1$.

Conversely, if $T \in C_{\cdot 0}$ and $\varrho(T) < 1$, then $F_T = F_\Theta$, and using the above Corollary the function $\Theta(\lambda)$ is bounded and $dF_T = \Theta(e^{it})^* \Theta(e^{it})dt$.

If, moreover, $dF_T = \Theta(e^{it})^* \Theta(e^{it})dt$ and $\{\mathfrak{H}, \mathfrak{F}, \Theta(\lambda)\}$ has a bounded inverse, then the associated operator Θ from $L^2_+(\mathfrak{H})$ into $L^2_+(\mathfrak{L}_*)$ defined by

$$(\Theta u)(e^{it}) = \Theta(e^{it})u(t) \quad \big(u \in L^2_+(\mathfrak{H})\big) \tag{3.3}$$

is boundedly invertible, and for any trigonometric polynomial p and for $h \in \mathfrak{H}$ we have

$$\int_0^{2\pi} |p(e^{it})|^2 d(F_T(t)h, h) = \int_0^{2\pi} |p(e^{it})|^2 (\Theta(e^{it})^* \Theta(e^{it})h, h)\, dt =$$

$$= \int_0^{2\pi} \|\Theta(e^{it}) p(e^{it}) h\|^2 dt \leqq \|\Theta\|^2 \int_0^{2\pi} |p(e^{it})|^2 \|h\|^2 dt.$$

Also, we have

$$\int_0^{2\pi} \|\Theta(e^{it}) p(e^{it}) h\|^2 dt \geqq \|\Theta^{-1}\|^{-2} \int_0^{2\pi} \|\Theta(e^{it})^{-1} \Theta(e^{it}) p(e^{it}) h\|^2 dt =$$

$$= \|\Theta^{-1}\|^{-2} \int_0^{2\pi} |p(e^{it})|^2 \|h\|^2 dt.$$

For any positive continuous function φ on $\mathbf{T}$ it follows that

$$\|\Theta^{-1}\|^{-2} \int_0^{2\pi} \varphi\, dt \leqq \int_0^{2\pi} \varphi\, dF_T(t) \leqq \|\Theta\|^2 \int_0^{2\pi} \varphi\, dt.$$

Therefore, there exists a positive constant c such that

$$c\, dt \leqq dF_T \leqq c^{-1} dt. \tag{3.4}$$

But (3.4) holds (see [1], [3]) if and only if T is a strict contraction.

Now, let us suppose that T is a strict contraction. Then F_T is of analytic type, $F_T = F_\Theta$ where $\Theta(\lambda)$ is the maximal function of T, and (3.4) implies that the bounded operator Θ defined by (3.3) has a bounded inverse. By the fact that Θ intertwines the shift operators in $L^2_+(\mathfrak{H})$ and $L^2_+(\mathfrak{L}_*)$, using Lemma 3.2 from [5] it follows that $\Theta(\lambda)$ has a bounded inverse.

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