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The functional model of a contraction and the space L^1

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The present Note is a straightforward continuation of the recent paper [I]. Indeed, we have noticed subsequently that, under slightly changed assumptions, the results of that paper can be extended from the factor space L^1/H^1_0 to the space L^1 itself, and "localized" on parts of the unit circle C.

The ingredients of these extensions are mostly taken over, with some changes, from the paper [I], and so are the notations and the terminology. When referring to a specified lemma or formula of [I] we indicate it by the subscript I. Applications to the invariant subspace problem are to be given later.

1. Let us begin with some lemmas requiring little changes with respect to [I].

Lemma 1. *If $\{a_n\}$ converges weakly to 0 in $\mathfrak{E}_*$ then for any $\varphi\in H^2$ and $h\in\mathfrak{H}$ we have*

$$\|(\varphi\circ a_n)\cdot h^*\|_{L^1}\to 0 \quad \textit{and} \quad \|h\cdot(\varphi\circ a_n)^*\|_{L^1}\to 0 \quad \textit{as} \quad n\to\infty.$$

(This is a strengthening of Lemma $3_{\rm I}$, where only convergence in the factor space L^1/H^1_0 was established.)

Proof. For any $h, k\in\mathfrak{H}$ the function $k\cdot h^*$ is the complex conjugate of $h\cdot k^*$ so they have the same norm in L^1. Therefore it suffices to prove the first convergence. Now, by $(4.3)_{\rm I}$ and $(4.7)_{\rm I}$ we have

$$\|(\varphi\circ a_n)\cdot h^*\|_{L^1}\leqq\|\varphi(a_n,h)_{\mathfrak{E}_*}\|_{L^1}+\|([\Theta^*\varphi a_n]_+,h_2)_{\mathfrak{E}}\|_{L^1}\to 0 \quad \text{as} \quad n\to\infty.$$

Lemma 2. *For any $\varphi,\psi\in H^2$ and $a\in\mathfrak{E}_*$ we have*

$$\big\|(\psi\circ a)\cdot(\varphi\circ a)^*-\psi\bar\varphi\|a\|^2_{\mathfrak{E}_*}\big\|_{L^1}\leqq\|\psi a\|_{H^2(\mathfrak{E}_*)}\|[\Theta^*\varphi a]_+\|_{H^2(\mathfrak{E})}+\|[\Theta^*\psi a]_+\|_{H^2(\mathfrak{E})}\|\varphi a\|_{H^2(\mathfrak{E}_*)}.$$

(This takes over the role of Lemma $4_{\rm I}$, with the unpleasant difference that here we have to increase the right hand side of the inequality by a second term.)

Proof. It readily follows from $(4.2)_{\rm I}$ and $(4.3)_{\rm I}$ that

$$(\psi\circ a)\cdot(\varphi\circ a)^*=\psi\bar\varphi\|a\|^2_{\mathfrak{E}_*}-(\psi a,\Theta[\Theta^*\varphi a]_+)_{\mathfrak{E}_*}-(\Theta[\Theta^*\psi a]_+,\varphi a)_{\mathfrak{E}_*}+$$
$$+([\Theta^*\psi a]_+,[\Theta^*\varphi a]_+)_{\mathfrak{E}}=\psi\bar\varphi\|a\|^2_{\mathfrak{E}_*}-([\Theta^*\psi a]_-,[\Theta^*\varphi a]_+)_{\mathfrak{E}}-([\Theta^*\psi a]_+,\Theta^*\varphi a)_{\mathfrak{E}},$$

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where $[\cdot]_- = [\cdot] - [\cdot]_+$; hence,

$$\|(\psi \circ a) \cdot (\varphi \circ a)^* - \psi\bar{\varphi}\|a\|^2_{\mathfrak{E}_*}\|_{L^1} \leqq$$

$$\leqq \|[\Theta^* \psi a]_-\|_{L^2(\mathfrak{E})} \|[\Theta^* \varphi a]_+\|_{L^2(\mathfrak{E})} + \|[\Theta^* \psi a]_+\|_{L^2(\mathfrak{E})} \|\Theta^* \varphi a\|_{L^2(\mathfrak{E})}.$$

Since $\|[\cdot]_-\|_{L^2(\mathfrak{E})} \leqq \|[\cdot]\|_{L^2(\mathfrak{E})}$ and since Θ^* is also contractive, the proof is done.

Lemma 3. *Suppose $\mathfrak{E}_*$ is (countably) infinite dimensional, and let $h, k \in \mathfrak{H}$; $\varphi_1, \ldots, \varphi_r, \psi_1, \ldots, \psi_r \in H^2$ and $\varepsilon > 0$ be given. Then there exist $h', k' \in \mathfrak{H}$ such that*

$$\left\|(h+h') \cdot (k+k')^* - h \cdot k^* - \sum_1^r \psi_j \bar{\varphi}_j\right\|_{L^1} \leqq \sum_1^r \|\psi_j\|_{H^2} \|\varphi_j\|_{H^2} (\eta_\Theta(\psi_j) + \eta_\Theta(\varphi_j)) + \varepsilon,$$

$$\|h'\|^2 \leqq \sum_1^r \|\psi_j\|^2_{H^2}, \quad \|k'\|^2 \leqq \sum_1^r \|\varphi_j\|^2_{H^2}.$$

Remark. *One can choose h', k' even to run over sequences $h^{(n)}, k^{(n)}$ $(n = 1, 2, \ldots)$ such that, for every $l \in \mathfrak{H}$, $h^{(n)} \cdot l^*$ and $k^{(n)} \cdot l^*$ tend to 0 in L^1 as $n \to \infty$.*

Proofs. Almost identical with those of Lemma 5_{I} and $\mathrm{Remark}_{\mathrm{I}}$, by using Lemmas 1 and 2 in place of Lemmas 3_{I} and 4_{I}, and applying inequality $(5.3)_{\mathrm{I}}$ both to φ_j and ψ_j.

2. More essential change is needed with Lemma 2_{I}. Its role will be taken by

Lemma 4. *Given a subset S of the open unit disc $D = \{\lambda : |\lambda| < 1\}$ let s be the set of non-tangential limit points of S on the unit circle C.*[1] *Then for any $f \in L^1(s)$ and $\varepsilon > 0$ there exist $\mu_1, \ldots, \mu_n \in S$ and $c_1, \ldots, c_n \in \mathbf{C}$ such that*

$$\left\|f - \sum_1^n c_j P_{\mu_j}\right\|_{L^1(s)} < \varepsilon \quad \text{and} \quad \sum_1^n |c_j| \leqq \|f\|_{L^1(s)}, \tag{1}$$

where P_μ is the Poisson kernel function on C corresponding to the point $\mu (\in D)$, i.e.

$$P_\mu(e^{it}) = \frac{1 - |\mu|^2}{|1 - \bar{\mu} e^{it}|^2}. \tag{2}$$

Proof. Suppose there exist $f_0 \in L^1(s)$ and $\varepsilon_0 > 0$ for which the assertion does not hold, i.e. such that the open ball G in $L^1(s)$ with centre f_0 and radius ε_0 is disjoint from the set X of all finite linear combinations $\sum c_j P_{\mu_j}$ with $\mu_j \in S$, $c_j \in \mathbf{C}$, and $\sum |c_j| \leqq \|f_0\|_{L^1(s)}$. Since both G and X are convex, and G is open, there exist, by the Hahn—Banach separation theorem, a function $g_0 \in L^\infty(s)$ (the Banach dual of $L^1(s)$) and a real number α such that

$$\mathrm{Re} \int_s h g_0 \, dm \leqq \alpha < \mathrm{Re} \int_s f g_0 \, dm \tag{3}$$

[1]) For any $S \subset D$, the corresponding set $s \subset C$ is a Borel set, indeed an $F_{\sigma\delta\sigma}$.

for all $h \in X$ and $f \in G$ (in particular for $f=f_0$); m denotes normalized Lebesgue measure on C.

Thus if we set

$$\tilde{g}_0(\mu) = \int_s g_0(e^{it}) P_\mu(e^{it})\, dm \quad (\mu \in D)$$

and observe that

$$\|P_\mu\|_{L^1(s)} \leqq \|P_\mu\|_{L^1} = 1, \quad \text{and hence,} \quad \|f_0\|_{L^1(s)} P_\mu \in X,$$

the first inequality in (3) shows that

$$\|f_0\|_{L^1(s)} |\tilde{g}_0(\mu)| \leqq \alpha \quad \text{for all} \quad \mu \in S. \tag{4}$$

Since $\tilde{g}_0$ is a bounded harmonic function on D, by the Fatou theorem we infer from (4) that

$$\|f_0\|_{L^1(s)} |g_0(e^{it})| \leqq \alpha \quad \text{almost everywhere on} \quad s,$$

so that

$$\operatorname{Re} \int_s f_0 g_0\, dm \leqq \|f_0\|_{L^1(s)} \|g_0\|_{L^\infty(s)} \leqq \alpha.$$

This contradicts the second inequality (3). The proof of Lemma 4 is complete.

3. In the sequel the functional $\eta_\Theta(\varphi)$ defined in [I] will again play a basic part. Let us recall, in particular, that for $\varphi = p_\mu$, where

$$p_\mu(e^{it}) = (1-\bar{\mu}e^{it})^{-1} \quad (\mu \in D),$$

we have

$$\eta_\Theta(p_\mu) = \inf_{\mathfrak{A}} \|\Theta(\mu)^* | \mathfrak{A}\|,$$

where $\mathfrak{A}$ runs through the family of subspaces of $\mathfrak{E}_*$ of finite codimension; cf. $(2.6)_{\mathrm{I}}$.

For any number ϑ, $0 \leqq \vartheta < 1$, consider the subset

$$S_\vartheta = \{\mu \in D: \quad \eta_\Theta(p_\mu) \leqq \vartheta\} \tag{5}$$

of D, and the corresponding set s_ϑ of non-tangential limit points of S_ϑ on C.

We are going to prove the following substitute for Lemma 5_{I}:

Lemma 5. *Suppose $\mathfrak{E}_*$ is (countably) infinite dimensional and suppose $f \in L^1(s_\vartheta)$ and $h, k \in \mathfrak{H}$, and also $\varepsilon > 0$ are given. Then there exist $h', k' \in \mathfrak{H}$ such that*

$$\|(h+h') \cdot (k+k')^* - h \cdot k^* - f\|_{L^1(s_\vartheta)} \leqq 2\vartheta \|f\|_{L^1(s_\vartheta)} + 2\varepsilon,$$

$$\|h'\|, \|k'\| \leqq \|f\|_{L^1(s_\vartheta)}.$$

Proof. By Lemma 4 there exist $\mu_1, \ldots, \mu_n \in S$ and $c_1, \ldots, c_n \in \mathbf{C}$ satisfying (1) (with $s = s_\vartheta$). One can obviously assume that $c_j \neq 0$ for all j, so we can set

$$\varphi_j = |c_j|^{1/2}(1-|\mu_j|^2)^{1/2} p_{\mu_j}, \quad \psi_j = (\operatorname{sgn} c_j) \cdot \varphi_j$$

$(j=1, 2, \ldots, n)$. Then we have

$$\psi_j \bar{\varphi}_j = c_j P_{\mu_j}, \quad \|\psi_j\|_{H^2}^2 = \|\varphi_j\|_{H^2}^2 = |c_j|$$

so that by Lemma 3 we obtain $h', k' \in \mathfrak{H}$ such that

$$\left\| (h+h')\cdot(k+k')^* - h\cdot k^* - \sum_1^n c_j P_{\mu_j} \right\|_{L^1(s_\vartheta)} \leqq 2\vartheta \sum_1^m |c_j| + \varepsilon$$

and

$$\|h'\|^2, \|k'\|^2 \leqq \sum_1^n |c_j|.$$

Taking also account of (1) we conclude the proof.

4. Now we are ready to state the following:

Theorem. *Suppose $\{\mathfrak{E}, \mathfrak{E}_*, \Theta(\lambda)\}$ is a contractive analytic function, with separable $\mathfrak{E}$, $\mathfrak{E}_*$ and with* $\dim \mathfrak{E}_* = \infty$. *Suppose that for some ϑ, $0 \leqq \vartheta < \frac{1}{2}$, the set s_ϑ of non-tangential limit points of the set S_ϑ* (defined by (5)) *on C has positive Lebesgue measure. Then for every $f \in L^1(s_\vartheta)$ there exist $h, k \in \mathfrak{H}$ such that*

$$f = h\cdot k^* \quad \textit{almost everywhere on } s_\vartheta. \tag{6}$$

Proof. As in the proof of Theorem$_{\rm I}$ we choose a number ω such that $2\vartheta < \omega < 1$ and consider an $f \in L^1(s_\vartheta)$ with $\|f\|_{L^1(s_\vartheta)} \leqq 1$. Setting $h_{-1} = h_0 = k_{-1} = k_0 = 0$ (in $\mathfrak{H}$) we show by induction that there exist $h_n, k_n \in \mathfrak{H}$ $(n=1, 2, \ldots)$ such that

(7) $\|f - h_n\cdot k_n^*\|_{L(s_\vartheta)} \leqq \omega^n$ and $\|h_n - h_{n-1}\|^2$, $\|k_n - k_{n-1}\|^2 \leqq \omega^{n-1}$ $(n = 0, 1, \ldots)$.

This being obvious for $n=0$ we assume h_n, k_n to be already found for $n=0, \ldots, q$. Setting $f_q = f - h_q\cdot k_q^*$ and $\varepsilon_q = (\omega - 2\vartheta)\omega^q/2$, by Lemma 5 we infer that there exist $h_{q+1}, k_{q+1} \in \mathfrak{H}$ such that

$$\|h_{q+1}\cdot k_{q+1}^* - h_q\cdot k_q^* - f_q\|_{L^1(s_\vartheta)} \leqq 2\vartheta\cdot\|f_q\|_{L^1(s_\vartheta)} + 2\varepsilon_q$$

and

$$\|h_{q+1} - h_q\|^2, \quad \|k_{q+1} - k_q\|^2 \leqq \|f_q\|_{L^1(s_\vartheta)} \leqq \omega^q.$$

Then we have

$$\|f - h_{q+1}\cdot k_{q+1}^*\|_{L^1(s_\vartheta)} = \|(f_q + h_q\cdot k_q^*) - h_{q+1}\cdot k_{q+1}^*\|_{L^1(s_\vartheta)} \leqq 2\vartheta\cdot\omega^q + (\omega - 2\vartheta)\omega^q = \omega^{q+1},$$

and the proof of (7) by induction is done.

By account of (7), the sequences $\{h_n\}$, $\{k_n\}$ are strongly convergent (in $\mathfrak{H}$) and their respective limits h, k satisfy (6). Theorem is proved.

References

[I] B. Sz.-Nagy—C. Foiaş, The functional model of a contraction and the space L^1/H_1^0, *Acta Sci. Math.*, **41** (1979), 403—410.

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