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A Mal'cev characterization of tolerance regularity

IVAN CHAJDA

A variety $\mathcal{V}$ of algebras is *regular* if it contains only regular algebras, i.e. if any two congruences on $\mathfrak{A}\in\mathcal{V}$ coincide whenever they have a congruence class in common. The regularity of varieties is a Mal'cev condition, see [7], [11], [12]. A *tolerance* T on an algebra $\mathfrak{A}=(A, F)$ is a reflexive and symmetric binary relation on A satisfying the Substitution Property with respect to all operations of $\mathfrak{A}$; this means that

$$\langle f(a_1, ..., a_n), f(b_1, ..., b_n)\rangle\in T$$

for each n-ary $f\in F$ whenever $\langle a_i, b_i\rangle\in T$ for $a_i, b_i\in A$ $(i=1, ..., n)$. This notion comes from that of congruence by omitting the requirement of transitivity. The set $\mathrm{LT}(\mathfrak{A})$ of all tolerances on an algebra $\mathfrak{A}$ forms an algebraic lattice with respect to set inclusion (see [2], [6]). Hence, we can introduce the following concepts. If $a, b\in A$ and $M\subseteq A\times A$, denote by $T(a, b)$ or $T(M)$ the least tolerance on $\mathfrak{A}$ containing the pair $\langle a, b\rangle$ or the set M, respectively.

Let $T\in\mathrm{LT}(\mathfrak{A})$. Call $[a]_T=\{b\in A; \langle a, b\rangle\in T\}$ the *tolerance class of T containing $a\in A$*. This generalizes the concept of a congruence class; other generalizations can be found in [3], [4], [5].

Definition 1. An algebra $\mathfrak{A}$ is *tolerance regular* if any two tolerances on $\mathfrak{A}$ coincide whenever they have a tolerance class in common. A variety $\mathcal{V}$ of algebras is *tolerance regular* if each $\mathfrak{A}\in\mathcal{V}$ has this property.

Let $T\in\mathrm{LT}(\mathfrak{A})$ and let $[a]_T$ be a tolerance class of T. Denote by $\mathrm{Tol}\{[a]_T\}$ the least tolerance on $\mathfrak{A}$ having a tolerance class equal to $[a]_T$. Clearly $\mathrm{Tol}\{[a]_T\}=$ $=T(M)$ for $M=\{a\}\times[a]_T$.

Lemma 1. *Let $\mathfrak{A}=(A, F)$ be an algebra and $M\subseteq A\times A$. Then $\langle x, y\rangle\in T(M)$ if and only if there exist a $(k+m+n)$-ary polynomial p over $\mathfrak{A}$ and $x_i, y_i\in A$ $(i=1, ..., k+m+n)$ with $x_i=y_i$ for $i\leqq k$, $\langle x_i, y_i\rangle\in M$ for $k<i\leqq k+m$ and*

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$\langle y_i, x_i\rangle \in M$ for $k+m<i\leqq k+m+n$ such that

$$x = p(x_1, \dots, x_{k+m+n}), \quad y = p(y_1, \dots, y_{k+m+n}).$$

Proof. Let R be the set of all $\langle x, y\rangle$ such that there exist a $(k+m+n)$-ary p and x_i, y_i with the prescribed properties. Clearly R is reflexive and symmetric and $M\subseteq R$. The Substitution Property for R can be shown easily by induction on the rank of polynomials, thus $R\in \mathrm{LT}(\mathfrak{A})$ and $T(M)\subseteq R$. If $S\in \mathrm{LT}(\mathfrak{A})$ and $M\subseteq S$ then $\langle x_i, y_i\rangle \in S$ whether $x_i=y_i$, $\langle x_i, y\rangle_i \in M$ or $\langle y_i, x_i\rangle \in M$, hence by the Substitution Property for S, we also have $\langle x, y\rangle \in S$ for $x=p(x_1, \dots, x_{k+m+n})$, $y=p(y_1, \dots, y_{k+m+n})$. Hence $R\subseteq S$, implying that $R=T(M)$.

Lemma 2. *Let $\mathfrak{A}=(A, F)$ and $x, y\in A$. Then $\langle a, b\rangle \in T(x, y)$ if and only if there exists a binary algebraic function φ over $\mathfrak{A}$ such that $a=\varphi(x, y)$ and $b=\varphi(y, x)$.*

This follows immediately from Lemma 1.

Theorem. *Let $\mathcal{V}$ be a variety of algebras. The following conditions are equivalent:*

(1) *$\mathcal{V}$ is tolerance regular;*

(2) *there exist a $(3+m+n)$-ary polynomial p and 5-ary polynomials q_j such that $q_j(y, x, x, y, z)=z$ and*

$$x = p(x, y, z, q_1(x, y, x, y, z), \dots, q_m(x, y, x, y, z), z, \dots, z),$$

$$y = p(x, y, z, z, \dots, z, q_{m+1}(x, y, x, y, z), \dots, q_{m+n}(x, y, x, y, z)).$$

Proof. (1)$\Rightarrow$(2). Let $\mathcal{V}$ be a tolerance regular variety and $\mathfrak{A}=(A, F)= =\mathfrak{F}_3(x, y, z)$ the free algebra of $\mathcal{V}$ with free generators $\{x, y, z\}$. Put $T=T(x, y)$. Since $\mathfrak{A}$ is tolerance regular, we have $T=\mathrm{Tol}\,\{[z]_T\}$, where $[z]_T$ is a tolerance class of T containing z. However, $\mathrm{Tol}\,\{[z]_T\}=T(M)$ for $M=\{z\}\times[z]_T$, thus $T(x, y)=T(M)$. Since $\langle x, y\rangle \in T(M)$, Lemma 1 implies the existence of a polynomial p^* and elements $x_i, y_i\in A$ with

$$x_i = y_i \quad \text{for} \quad i = 1, \dots, k,$$

$$x_i\in[z]_T, \quad y_i = z \quad \text{for} \quad k< i \leqq k+m,$$

$$x_i = z, \quad y_i\in[z]_T \quad \text{for} \quad k+m < i \leqq k+m+n$$

such that

$$x = p^*(x_1, \dots, x_{k+m+n}), \quad y = p^*(y_1, \dots, y_{k+m+n}).$$

Since $\mathfrak{A}=\mathfrak{F}_3(x, y, z)$, there exist ternary polynomials v_i such that $x_i=y_i= =v_i(x, y, z)$ for $i=1, \dots, k$, i.e. there exists a $(3+m+n)$-ary polynomial p such that

$$\text{(a)} \qquad x = p^*(x_1, \dots, x_{k+m+n}) = p(x, y, z, u_1, \dots, u_{m+n}),$$

$$y = p^*(y_1, \dots, y_{k+m+n}) = p(x, y, z, w_1, \dots, w_{m+n}),$$

where $u_j=x_{j+k}$, $w_j=y_{j+k}$ $(j=1, ..., m+n)$, i.e.

(b) $$u_j\in[z]_T, \quad w_j = z \quad \text{for} \quad j = 1, ..., m,$$

$$u_j = z, \quad w_j\in[z]_T \quad \text{for} \quad j = m+1, ..., m+n.$$

If $u_j\in[z]_T$, $w_j=z$ then $\langle u_j, z\rangle=\langle u_j, w_j\rangle\in \text{Tol}\,\{[z]_T\}=T(x, y)$. By Lemma 2, there exists a binary algebraic function φ_j such that $u_j=\varphi_j(x, y)$ and $z=\varphi_j(y, x)$. Since $\mathfrak{A}=\mathfrak{F}_3(x, y, z)$, there exists a 5-ary polynomial q_j over $\mathcal{V}$ such that $\varphi_j(r, s)=q_j(r, s, x, y, z)$, i.e.

(c) $$u_j = q_j(x, y, x, y, z) \quad \text{and} \quad z = q_j(y, x, x, y, z).$$

We can proceed analogously if $u_j=z, w_j\in[z]_T$. Thus (a), (b), (c) imply (2).

(2)$\Rightarrow$(1). Let $\mathfrak{A}=(A, F)\in\mathcal{V}$, $T_1, T_2\in\text{LT}\,(\mathfrak{A})$ and let $[z]_{T_1}=[z]_{T_2}$ be a common tolerance class of T_1, T_2. Suppose $\langle x, y\rangle\in T_1$. Then also

$$\langle q_j(x, y, x, y, z), z\rangle = \langle q_j(x, y, x, y, z),\ q_j(y, x, x, y, z)\rangle\in T_1,$$

i.e. $q_j(x, y, x, y, z)\in[z]_{T_1}=[z]_{T_2}$. Therefore $\langle q_j(x, y, x, y, z), z\rangle\in T_2$ for $j=1, ...$ $..., m+n$. By (2), we have $\langle x, y\rangle\in T_2$, i.e. $T_1\subseteq T_2$. The converse inclusion can be proved analogously, thus $\mathfrak{A}$ and also $\mathcal{V}$ is tolerance regular.

Remark. Since every congruence is a tolerance, tolerance regularity of $\mathcal{V}$ implies regularity of $\mathcal{V}$, i.e. (2) of the theorem is a sufficient condition for the regularity of $\mathcal{V}$. If $\mathcal{V}$ is, moreover, congruence-permutable, then Werner's Theorem in [9] implies $\text{LT}\,(\mathfrak{A})=\text{Con}\,(\mathfrak{A})$ for each $\mathfrak{A}\in\mathcal{V}$, thus (2) of the Theorem is also necessary. A simpler Mal'cev characterization of permutability and regularity is given by the author in [10].

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