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On sets which contain sum sets

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Introduction. Let $\mathbf{Z}$ be the set of integers and let $E\subset\mathbf{Z}$. For $A\subset\mathbf{Z}$ denote by $|A|$ the cardinal number of A. For $n\geqq 1$, define $\varrho_n(E)=\sup\{\min(|A_1|, |A_2|, \ldots \ldots, |A_n|): A_1+A_2+\ldots+A_n\subset E, A_i\subset\mathbf{Z}\}$. For $n\geqq 2$, E is called a ϱ_n set if $\varrho_n(E)<\infty$ and $\varrho_{n-1}(E)=\infty$.

Many authors have studied ϱ_n sets in various connections; see, for example, [1] and [2]. A set D of integers is called *dissociate* if every integer has at most one representation of the form $\varepsilon_1 d_1+\ldots+\varepsilon_m d_m$ with $\varepsilon_j=\pm 1$ $(1\leqq j\leqq m)$ and $d_1<d_2<\ldots \ldots<d_m$ are in D. For $n\geqq 1$, suppose D_i, $i=1, 2, \ldots, n$ are such that $D_k\cap D_l=\emptyset$ if $k\neq l$, $|D_i|=\infty$ for $i=1, 2, \ldots, n$, and $\bigcup_{i=1}^{n} D_i$ is a dissociate set. It then follows from [3] that $D_1+D_2+\ldots+D_n$ is a ϱ_{n+1} set which is not a ϱ_n set. The proof of this fact uses the techniques of harmonic analysis. Our purpose here is to indicate, for any $n\geqq 2$, a very simple construction (which uses only the definition of a ϱ_n set) to obtain a new class of sets which are ϱ_{n+1} sets but not ϱ_n sets. We will actually construct a set, $\mathscr{S}$, of positive integers which is the sum of two infinite sets of positive integers such that $\mathscr{S}$ is a ϱ_3 set. It is clear from the construction we give how to construct, for any $n\geqq 2$, a class of sets, $\mathscr{S}_n$, such that $\mathscr{S}_n$ is the sum of n infinite sets of positive integers and $\mathscr{S}_n$ is a ϱ_{n+1} set. Furthermore, it is not hard to see that in our construction $\mathbf{Z}$ may be replaced, with appropriate modifications, by any infinite abelian group; cf. [3].

Moreover, although the sets we construct are not necessarily of the form $D_1+D_2+\ldots+D_n$, where $\bigcup_{i=1}^{n} D_i$ is dissociate, our proof can be easily modified to prove the result for any such sets.

The Construction. Before we begin our construction of the set $\mathscr{S}$, we observe that, since $\mathscr{S}$ will be a set of positive integers, and we will be concerned with showing that $\sup\{\min(|X|, |Y|, |W|): X+Y+W\subset S;\ X, Y, W\subset\mathbf{Z}\}$ is finite, it suffices throughout to consider only sets $X, Y,$ and W of positive integers.

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First take two singletons of positive integers $A_0=\{a_0\}$ and $B_0=\{b_0\}$. Next choose $a_1>3(a_0+b_0)$ and $b_1>\frac{4}{3}a_1$. Write $A_1=A_0\cup\{a_1\}$ and $B_1=B_0\cup\{b_1\}$. In general, if A_n and B_n have already been constructed then take $a_{n+1}>3\max(A_n+B_n)$ and $b_{n+1}\geqq\frac{4}{3}a_{n+1}$ and write $A_{n+1}=A_n\cup\{a_{n+1}\}$ and $B_{n+1}=B_n\cup\{b_{n+1}\}$. We define $\mathscr{S}=\bigcup_{n=0}^{\infty}(A_n+B_n)$.

Observe first, that for any n, $A_{n+1}+B_{n+1}=(A_n\cup\{a_{n+1}\})+(B_n\cup\{b_{n+1}\})=(A_n+B_n)\cup(B_n+\{a_{n+1}\})\cup(A_n+\{b_{n+1}\})\cup\{a_{n+1}+b_{n+1}\}$. Notice that $b_{n+1}>\frac{4}{3}a_{n+1}>a_{n+1}+\max(A_n+B_n)>\max(a_{n+1}+B_n)$ so that each element of $A_n+\{b_{n+1}\}$ is greater than each element of $B_n+\{a_{n+1}\}$. Also, neither $A_n, B_n, B_n+\{a_{n+1}\}$, nor $A_n+\{b_{n+1}\}$ contains a sum of two doubletons or a translate of a sum of two doubletons. For example, if $x'+(\{y_1, y_2\}+\{w_1, w_2\})\subset a_{n+1}+B_n$ with $y_1<y_2$ and $w_1<w_2$ then for some $b'<b''<b'''<b''''$ in B_n we have

$$x'+y_1+w_1=a_{n+1}+b',\quad x'+y_1+w_2=a_{n+1}+b'',\quad x'+y_2+w_1=a_{n+1}+b''',$$
$$x'+y_2+w_2=a_{n+1}+b'''',$$

or

$$x'+y_1+w_1=a_{n+1}+b',\quad x'+y_2+w_1=a_{n+1}+b'',\quad x'+y_1+w_2=a_{n+1}+b''',$$
$$x'+y_2+w_2=a_{n+1}+b''''.$$

In either case, we obtain $b''''-b'''=b''-b'$ which is impossible since each member of B_n is more than twice its predecessor.

Now suppose that, for some n, if X, Y, and W are sets of positive integers such that if $X+Y+W\subset A_n+B_n$, then $\min(|X|, |Y|, |W|)<4$. Suppose also that X', Y', W' are sets of positive integers with $|X'|, |Y'|, |W'|$ each at least 4 and $D=X'+Y'+W'\subset A_{n+1}+B_{n+1}$. By the induction hypothesis, $D\not\subset A_n+B_n$ and so $D\cap(B_n+\{a_{n+1}\})\neq\emptyset$ or $D\cap(A_n+\{b_{n+1}\})\neq\emptyset$ or $D\cap\{a_{n+1}+b_{n+1}\}\neq\emptyset$. Thus, some element of X' or Y' or W' must be greater than $\max(A_n+B_n)$ because $3\max(A_n+B_n)<a_{n+1}$. Without loss of generality, call this element $x'\in X'$.

Now, if $|Y'|\geqq 4$ and $|W'|\geqq 4$, then we can see that either $B_n+\{a_{n+1}\}$ or $A_n+\{b_{n+1}\}$ must contain a translate of a sum of two doubletons as follows:

Say $y_1<y_2<y_3<y_4$ are the four smallest elements of Y' and $w_1<w_2<w_3<w_4$ are the four smallest elements of W'. Look at $u_1=x'+y_1+w_1$, $u_2=x'+y_1+w_2$, $u_3=x'+y_2+w_1$, and $u_4=x'+y_2+w_2$. If $u_1\in A_n+\{b_{n+1}\}$, then clearly $u_1, u_2, u_3, u_4\in A_n+\{b_{n+1}\}$. If $u_1\in B_n+\{a_{n+1}\}$, then we're done unless $u_4\in A_n+\{b_{n+1}\}$. But then $x'+(\{y_2, y_3\}+\{w_2, w_3\})\subset A_n+\{b_{n+1}\}$ and we are done. We now have a contradiction and so it follows that $\min(|X'|, |Y'|, |W'|)<4$.

Thus, by induction, for any n, if X, Y, and W are three sets of positive integers with $X+Y+W\subset A_n+B_n$, then $\min(|X|, |Y|, |W|)<4$.

Clearly $\mathcal{S}=\bigcup_{n=0}^{\infty}(A_n+B_n)$ is not ϱ_2. However, if X, Y, and W are three finite sets of positive integers with $X+Y+W\subset\mathcal{S}$, then $X+Y+W\subset A_n+B_n$ for some n and so $\min(|X|, |Y|, |W|)<4$. Thus, $\mathcal{S}$ is a ϱ_3 set. Finally, $\mathcal{S}=\bigcup_{n=0}^{\infty}(A_n+B_n)=\left(\bigcup_{n=0}^{\infty}A_n\right)+\left(\bigcup_{n=0}^{\infty}B_n\right)$ because $A_n\subset A_{n+1}$ and $B_n\subset B_{n+1}$ for all n and so $\mathcal{S}$ is a sum of two infinite sets.

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