*Acta Sci. Math.*, 42 (1980), 247—255

# A Hausdorff—Young type inequality and necessary multiplier conditions for Jacobi expansions

GEORGE GASPER and WALTER TREBELS

**1. Introduction.** We shall show how necessary conditions for Jacobi multipliers can be derived from certain Hausdorff—Young type inequalities.

In order to become more precise we first have to introduce the following notation. Fix $\alpha \geqq \beta \geqq -\frac{1}{2}$ and let $L^p_{(\sigma,\tau)} = L^p_{(\sigma,\tau;\alpha,\beta)}$, $1 \leqq p < \infty$, denote the space of measurable functions on $[0, \pi]$ such that

$$\|f\|_{p;\sigma,\tau} = \left(\int_0^\pi \left|\left(\sin\frac{\theta}{2}\right)^\sigma \left(\cos\frac{\theta}{2}\right)^\tau f(\theta)\right|^p d\mu(\theta)\right)^{1/p}$$

is finite where

$$d\mu(\theta) = d\mu^{(\alpha,\beta)}(\theta) = \left(\sin\frac{\theta}{2}\right)^{2\alpha+1} \left(\cos\frac{\theta}{2}\right)^{2\beta+1} d\theta.$$

If $\tau = 0$ we write $L^p_{(\sigma,0)} = L^p_\sigma$, $\|\cdot\|_{p;\sigma,0} = \|\cdot\|_{p,\sigma}$ and if, additionally, $\sigma = 0$ we use the standard notations $L^p$, $\|\cdot\|_p$. Note that $L^\infty \subset L^p_{(\sigma,\tau)} \subset L^1$ if $-(2\alpha+2) < \sigma p < (2\alpha+2)(p-1)$, $-(2\beta+2) < \tau p < (2\beta+2)(p-1)$. Here, as elsewhere, the inclusion sign means that the identity map is continuous. Each $f \in L^1$ has an expansion of the form

$$f(\theta) \sim \sum_{k=0}^\infty f^\wedge(k)\, h_k R_k(\cos\theta)$$

where $R_k(x) = R_k^{(\alpha,\beta)}(x) = P_k^{(\alpha,\beta)}(x)/P_k^{(\alpha,\beta)}(1)$, $P_k^{(\alpha,\beta)}(x)$ being the Jacobi polynomial of degree $k$ and order $(\alpha, \beta)$, [8]. Also the $k$-th Fourier—Jacobi coefficient $f^\wedge(k)$ is defined by

$$f^\wedge(k) = \int_0^\pi f(\theta) R_k(\cos\theta)\, d\mu(\theta)$$

[1]) Supported in part by NSF Grant MCS 76—06635 A01.

Received May 21, 1979.

and $h_k=h_k^{(\alpha,\beta)}=\|R_k(\cos\theta)\|_2^{-2}\approx k^{2\alpha+1}$, where the $\approx$ sign means that there are positive constants $C, C'$ such that $C'h_k\leqq k^{2\alpha+1}\leqq Ch_k$ ($k^{2\alpha+1}$ must be replaced by 1 when $k=0$).

A sequence $m=\{m_k\}_0^\infty\in l^\infty$ is called a multiplier from $L^p_{(\sigma,\tau)}$ into $L^q_{(\sigma,\tau)}$, notation $m\in M_p^q(\sigma,\tau)=M_p^q(\sigma,\tau;\alpha,\beta)$, if for each $f\in L^p_{(\sigma,\tau)}$ there exists a function $Mf\in L^q_{(\sigma,\tau)}$ with

$$(1.1)\qquad Mf(\theta)\sim\sum_{k=0}^\infty m_k f^\wedge(k)h_kR_k(\cos\theta),\quad \|Mf\|_{q;\sigma,\tau}\leqq C\|f\|_{p;\sigma,\tau}.$$

The smallest constant $C$ independent of $f$ for which this holds is called the multiplier norm of $m$ and it is denoted by $\|m\|_{M_p^q(\sigma,\tau)}$. If $\tau=0$ we write $M_p^q(\sigma,0)=M_p^q(\sigma)$.

The derivation of sharp sufficient multiplier conditions (see e.g. [2], [3], [6]) relies heavily upon the following Parseval type inequality ($f(\theta)$ being a polynomial in $\cos\theta$)

$$(1.2)\qquad \left\|\left(\sin\frac{\theta}{2}\right)^\gamma f(\theta)\right\|_2^2\leqq C\sum_{k=0}^\infty|\Delta^\gamma f^\wedge(k)|^2h_k,\quad -\frac{1}{2}<\gamma<\alpha+2,$$

where the fractional difference operator $\Delta^\gamma$, $\gamma\in R$, is defined by

$$\Delta^\gamma m_k=\sum_{j=k}^\infty A_{j-k}^{-\gamma-1}m_j,\quad A_k^\gamma=\binom{k+\gamma}{k}=\frac{\Gamma(k+\gamma+1)}{\Gamma(k+1)\Gamma(\gamma+1)},$$

whenever the series converges. So one can expect that the converse of (1.2)

$$(1.3)\qquad \sum_{k=0}^\infty|\Delta^\gamma f^\wedge(k)|^2h_k\leqq C\left\|\left(\sin\frac{\theta}{2}\right)^\gamma f(\theta)\right\|_2^2,\quad \gamma>-1,$$

proved in [7] for functions $f(\theta)$ which are polynomials in $\cos\theta$, will yield necessary multiplier conditions; this will turn out to be true on $L^2_\sigma$. However, to obtain necessary conditions also on $L^p_\sigma$, $p\neq 2$, we shall need a Hausdorff—Young type variant of (1.3).

The plan of this paper is as follows. In Sec. 2 we derive for the special case $\alpha=\beta=-\frac{1}{2}$, i.e. for cosine expansions, the desired Hausdorff—Young type inequality and deduce from it necessary multiplier conditions on $L^p_\sigma$. Then in Sec. 3 we consider the general case $\alpha\geqq\beta\geqq-\frac{1}{2}$, $\alpha>-\frac{1}{2}$, and derive the corresponding Hausdorff—Young type inequality and necessary multiplier conditions. Finally we close with several remarks concerning our results.

**2. Necessary multiplier conditions for cosine expansions in weighted Lebesgue spaces.** Consider $f\in L^1$ and observe that, since $R_k^{(-1/2,-1/2)}(\cos\theta)=\cos k\theta$,

$$f^\wedge(k)=\int_0^\pi f(\theta)\cos k\theta\,d\theta=\frac{1}{2}\int_0^\pi f(\theta)(e^{ik\theta}+e^{-ik\theta})\,d\theta$$

and hence

$$\Delta^\gamma f^\wedge(k) = \frac{1}{2}\int_0^\pi f(\theta)\sum_{j=k}^\infty A_{j-k}^{-\gamma-1}(e^{ij\theta}+e^{-ij\theta})\,d\theta =$$

$$= \frac{1}{2}\int_0^\pi f(\theta)\{e^{ik\theta}(1-e^{i\theta})^\gamma + e^{-ik\theta}(1-e^{-i\theta})^\gamma\}\,d\theta.$$

Thus we obtain

(2.1) $$\sup_k |\Delta^\gamma f^\wedge(k)| \leqq C\int_0^\pi \left|\left(\sin\frac{\theta}{2}\right)^\gamma f(\theta)\right| d\theta$$

and hence, by applying the Riesz—Thorin interpolation theorem to (1.3) and (2.1)

(2.2) $$\left(\sum_{k=0}^\infty |\Delta^\gamma f^\wedge(k)|^{p'}\right)^{1/p'} \leqq C\|f\|_{p,\gamma}, \quad 1 \leqq p \leqq 2,\ \gamma \geqq 0,$$

for polynomials $f(\theta)$ in $\cos\theta$, where $p'$ is defined by $\frac{1}{p}+\frac{1}{p'}=1$. To state our necessary conditions for cosine multipliers we need to use the following sequence spaces of weak bounded variation (see [4]): for $1\leqq q\leqq\infty$, $\gamma>0$,

$$wbv_{q,\gamma} = \{m\in l^\infty:\ \|m\|_{q,\gamma;w} < \infty\}$$

where

$$\|m\|_{q,\gamma;w} = \|m\|_\infty + \sup_{j\in N}\left(\sum_{k=2^{j-1}}^{2^j-1} k^{-1}|k^\gamma \Delta^\gamma m_k|^q\right)^{1/q}$$

for $q<\infty$ and, in case $q=\infty$,

$$\|m\|_{\infty,\gamma;w} = \|m\|_\infty + \sup_{k\in N}|k^\gamma \Delta^\gamma m_k|.$$

Theorem 1. *If* $0<\gamma<1-1/p$, $1<p\leqq 2$, *and* $m\in M_p^p(\gamma)$, *then* $m\in wbv_{p',\gamma}$ *and* $\|m\|_{p',\gamma;w}\leqq C\|m\|_{M_p^p(\gamma)}$, *i.e.*, $M_p^p(\gamma)\subset wbv_{p',\gamma}$.

Remark. At the AMS Summer Institute in Williamstown, Mass., 1978, Muckenhoupt, Wheeden and Wo-Sang Young announced necessary and sufficient conditions for a sequence to belong to $M_2^2(\gamma)$ when $\gamma>1/2$. Here we treat the case $-\frac{1}{2}<\gamma<\frac{1}{2}$, $\gamma\neq 0$ (note: $M_2^2(\gamma)=M_2^2(-\gamma)$). By combining the sufficient condition in [6] and the present necessary one it follows for $0<|\gamma|<\frac{1}{2}$ that

$$wbv_{q,\delta}\subset M_2^2(\gamma)\subset wbv_{2,|\gamma|},\quad \delta>\max\left\{\frac{1}{q},\ |\gamma|\right\}.$$

Proof of Theorem 1. For the Dirichlet kernel

$$D_n(\theta) = 1+2\sum_{k=1}^{n}\cos k\theta = \frac{\sin\left(n+\frac{1}{2}\right)\theta}{\sin\frac{\theta}{2}},$$

which we will use as a testfunction, it is easy to check that $\|D_n\|_{p,\gamma} \leqq Cn^{1-\frac{1}{p}-\gamma}$, $-\frac{1}{p}<\gamma<1-\frac{1}{p}$. In order to apply our Hausdorff—Young type inequality (2.2) we need a decomposition of $\{m_k\}$ into a set of sequences with finite support. In fact, setting

$$E_k(l) = \begin{cases} 1, & 2^{l-1} \leqq k < 2^l, \\ 0, & \text{otherwise,} \end{cases}$$

each sequence $E(l)=\{E_k(l)\}_{k=0}^{\infty}$ has support on a dyadic block and so by (2.2)

$$(2.3) \qquad \left(\sum_{2^{j-1}}^{2^j-1} |\Delta^\gamma m_k|^{p'}\right)^{1/p'} \leqq$$

$$\leqq \left(\sum_{k=2^{j-1}}^{2^j-1} |\Delta^\gamma(m_k(E_k(j)+E_k(j+1)))|^{p'}\right)^{1/p'} + \sum_{l=j+1}^{\infty}\left(\sum_{k=2^{j-1}}^{2^j-1} |\Delta^\gamma(m_k E_k(l))|^{p'}\right)^{1/p'} \leqq$$

$$\leqq C\|m\|_{M_p^p(\gamma)}\|D_{2^{j+1}-1}-D_{2^{j-1}-1}\|_{p,\gamma}+\|m\|_\infty \sum_{l=j+1}^{\infty}\left(\sum_{k=2^{j-1}}^{2^j-1}(2^l)^{-\gamma p'}\right)^{1/p'}$$

since for $2^{j-1}\leqq k<2^j$, $l\geqq j+1$,

$$\Delta^\gamma(m_k E_k(l)) = \sum_{n=2^{l-1}}^{2^l-1} A_{n-k}^{-1-\gamma} m_n \leqq C\|m\|_\infty (2^l)^{-\gamma}.$$

Now observe that $\|m\|_\infty \leqq C\|m\|_{M_p^p(\gamma)}$ and multiply both sides of (2.3) by $(2^j)^{\gamma+\frac{1}{p}-1}$ to obtain

$$\left(\sum_{2^{j-1}}^{2^j-1} k^{-1}|k^\gamma \Delta^\gamma m_k|^{p'}\right)^{1/p'} \leqq C\|m\|_{M_p^p(\gamma)}\left\{1+(2^j)^\gamma\sum_{l=j+1}^{\infty}(2^l)^{-\gamma}\right\} \leqq C\|m\|_{M_p^p(\gamma)},$$

which establishes the theorem since $C$ is independent of $j$.

**3. The case** $\alpha\geqq\beta\geqq-\frac{1}{2}, \alpha>-\frac{1}{2}$. In [7, Sec. 2] it was shown that if $\gamma>-1, f(\theta)$ is a polynomial in $\cos\theta$, and

$$d_k = \int_0^\pi f(\theta) R_k^{(\alpha+\gamma,\beta)}(\cos\theta)\left(\sin\frac{\theta}{2}\right)^{2\alpha+2\gamma+1}\left(\cos\frac{\theta}{2}\right)^{2\beta+1} d\theta$$

then

$$(3.1) \qquad \Delta^\gamma f^\wedge(n) = \sum_{k=n}^{\infty} B_k(n) d_k.$$

where $B_k(n)=B_k(n;\alpha,\beta,\gamma)=O(k^{\gamma-1})$ for $k\geqq n+1$ and $B_n(n)=O(n^\gamma)$. Since, by Szegő [8; Theorem 7.32.3],

$$\left|\sqrt{h_k^{(\alpha,\beta)}}\,R_k^{(\alpha,\beta)}(\cos\theta)\left(\sin\frac{\theta}{2}\right)^{\alpha+\frac{1}{2}}\left(\cos\frac{\theta}{2}\right)^{\beta+\frac{1}{2}}\right|\leqq C$$

we have

$$d_k\sqrt{h_k^{(\alpha+\gamma,\beta)}}\leqq C\int_0^\pi |f(\theta)|\left(\sin\frac{\theta}{2}\right)^{\alpha+\gamma+\frac{1}{2}}\left(\cos\frac{\theta}{2}\right)^{\beta+\frac{1}{2}}d\theta$$

and so from (3.1) and the fact that $h_k^{(\alpha+\gamma,\beta)}\approx k^{2\alpha+2\gamma+1}$ it follows that

$$\sup_n\left|\sqrt{h_n}\,\Delta^\gamma f^\wedge(n)\right|\leqq C\int_0^\pi\left|\left(\sin\frac{\theta}{2}\right)^{\alpha+\gamma+\frac{1}{2}}\left(\cos\frac{\theta}{2}\right)^{\beta+\frac{1}{2}}f(\theta)\right|d\theta. \tag{3.2}$$

Application of the Riesz—Thorin theorem to (3.2) and our previous result [7; Theorem 1b]

$$\left(\sum_{n=0}^\infty\left|\sqrt{h_n}\,\Delta^\gamma f^\wedge(n)\right|^2\right)^{1/2}\leqq C\left(\int_0^\pi\left|\left(\sin\frac{\theta}{2}\right)^{\alpha+\gamma+\frac{1}{2}}\left(\cos\frac{\theta}{2}\right)^{\beta+\frac{1}{2}}f(\theta)\right|^2d\theta\right)^{\frac{1}{2}} \tag{3.3}$$

then gives (in combination with (2.2))

Theorem 2. *Let* $1\leqq p\leqq 2$, $\gamma\geqq 0$, $\alpha\geqq\beta\geqq-\frac{1}{2}$, $\alpha>-\frac{1}{2}$ *and let* $f(\theta)$ *be a polynomial in* $\cos\theta$. *Then there exists a constant* $C$ *independent of* $f$ *such that*

$$\left(\sum_{n=0}^\infty\left|\sqrt{h_n}\,\Delta^\gamma f^\wedge(n)\right|^{p'}\right)^{1/p'}\leqq C\|f\|_{p;\sigma,\tau}$$

*where* $\sigma=\gamma+(2\alpha+1)\left(\frac{1}{2}-\frac{1}{p}\right)$ *and* $\tau=(2\beta+1)\left(\frac{1}{2}-\frac{1}{p}\right)$.

Unfortunately, when $\alpha>-\frac{1}{2}$ the Dirichlet kernel for Jacobi series is too bad a test function to obtain necessary conditions for multipliers on $L^p_{(\sigma,\tau;\alpha,\beta)}$ analogous to Theorem 1. In order to estimate test functions with "nice" Jacobi coefficients we shall need

Lemma 1. *Let* $\alpha,\beta,p,\sigma,\tau$ *be as in Theorem 2 and let* $\{g_k\}_{k=0}^\infty\in l^\infty$ *have compact support*; *set*

$$I_\lambda:=\sum_{k=0}^\infty k^{\lambda+\alpha+\frac{3}{2}-\gamma-\frac{1}{p}}\left|\Delta^{\lambda+1}g_k\right|. \tag{3.4}$$

*Then, for some integer* $\lambda>\alpha+\frac{1}{2}$,

$$\left(\int_0^\pi \left|\sum_{k=0}^\infty g_k h_k R_k(\cos\theta)\left(\sin\frac{\theta}{2}\right)^{\gamma+\alpha+\frac{1}{2}}\left(\cos\frac{\theta}{2}\right)^{\beta+\frac{1}{2}}\right|^p d\theta\right)^{\frac{1}{p}} \leqq CI_\lambda.$$

Proof. Set

$$g(\theta)=\sum_{k=0}^\infty \Delta^{\lambda+1}g_k\sum_{j=0}^k A_{k-j}^\lambda h_j R_j(\cos\theta). \tag{3.5}$$

Then, by Szegő [8, Sec. 9.41], (3.5) and the substitution $x=\cos\theta$,

$$\|g\|_{p;\sigma,\tau}=\left(\int_0^\pi \left|g(\theta)\left(\sin\frac{\theta}{2}\right)^{\gamma+\alpha+\frac{1}{2}}\left(\cos\frac{\theta}{2}\right)^{\beta+\frac{1}{2}}\right|^p d\theta\right)^{1/p} \leqq$$

$$\leqq C\sum_{k=0}^\infty |\Delta^{\gamma+1}g_k|\sum_{j=0}^k j^{\alpha+\lambda+2}G_j(k,\lambda)\Big(\int_{-1}^1 |P_j^{(\alpha+\lambda+1,\beta)}(x)(1-x)^{\left(\frac{\gamma}{2}+\frac{\alpha}{2}+\frac{1}{4}-\frac{1}{2p}\right)}\cdot$$

$$\cdot(1+x)^{\left(\frac{\beta}{2}+\frac{1}{4}-\frac{1}{2p}\right)}|^p\,dx\Big)^{1/p}$$

where $G_j(k,\lambda)$ is defined as in [8, (9.4.6)]. The above integral can be estimated with the aid of [8; Ex. 91] by $O(j^{\lambda-\gamma+1/2-1/p})$ and so, by [8; (9.41.7)],

$$\|g\|_{p;\sigma,\tau}\leqq C\sum_{k=0}^\infty |\Delta^{\lambda+1}g_k|\,k^{\lambda+\alpha+\frac{3}{2}-\gamma-\frac{1}{p}}.$$

Finally a standard argument shows that $g\hat{\ }(k)=g_k$.

Next we have to use Lemma 1 to estimate the $L^p_{(\sigma,\tau)}$ norm of certain functions $\Phi_n(\theta)$ arising from a partition of the unit sequence $\{1,1,\ldots\}$. Consider $\varphi_0\in C^\infty$ with compact support such that $0\leqq\varphi_0(t)\leqq1$,

$$\varphi_0(t)=\begin{cases}1, & 2^{-1/3}\leqq t\leqq 2^{1/3},\\ 0, & t\leqq 2^{-2/3}\ \text{or}\ t\geqq 2^{2/3},\end{cases}$$

and $\sum_{n=0}^\infty \varphi_n(t)=1$ for $t\geqq1$ where we set $\varphi_n(t)=\varphi_0(2^{-n}t)$. Now define

$$\Phi_n(\theta)=\sum_{k=0}^\infty \varphi_n(k)h_kR_k(\cos\theta).$$

Then, for integer $\lambda>\alpha+\frac{1}{2}$,

$$\sum_{k=0}^\infty k^{\lambda+\alpha+\frac{3}{2}-\gamma-\frac{1}{p}}|\Delta^{\lambda+1}\varphi_n(k)|=O\left(2^{n\left(\alpha+\frac{3}{2}-\gamma-\frac{1}{p}\right)}\right)$$

and so, by Lemma 1,

$$(3.6)\qquad \left(\int_0^\pi \left|\Phi_n(\theta)\left(\sin\frac{\theta}{2}\right)^{\gamma+\alpha+\frac{1}{2}}\left(\cos\frac{\theta}{2}\right)^{\beta+\frac{1}{2}}\right|^p d\theta\right)^{\frac{1}{p}} = O\left(2^{n\left(\alpha+\frac{3}{2}-\gamma-\frac{1}{p}\right)}\right).$$

**Theorem 3.** *Let* $1\leqq p\leqq 2$, $0<\gamma<\alpha+\frac{3}{2}-\frac{1}{p}$, $\alpha>-\frac{1}{2}$ *and* $\alpha\geqq\beta\geqq-\frac{1}{2}$. *If* $\sigma=\gamma+(2\alpha+1)\left(\frac{1}{2}-\frac{1}{p}\right)$ *and* $\tau=(2\beta+1)\left(\frac{1}{2}-\frac{1}{p}\right)$ *then* $M_p^p(\sigma,\tau;\alpha,\beta)\subset wbv_{p',\gamma}$.

Proof. Let $m\in M_p^p(\sigma,\tau;\alpha,\beta)$. For any $j\in N$ Minkowski's inequality and (3.6) give

$$\left(\sum_{2^{j-1}}^{2^j-1}\left|\sqrt{h_k}\,\Delta^\gamma m_k\right|^{p'}\right)^{1/p'} \leqq \sum_{n=j-1}^{\infty}\left(\sum_{2^{j-1}}^{2^j-1}\left|\sqrt{h_k}\,\Delta^\gamma(m_k\varphi_n(k))\right|^{p'}\right)^{1/p'} \leqq$$

$$\leqq C\|m\|_{M_p^p(\sigma,\tau;\alpha,\beta)}2^{j\left(\alpha+\frac{3}{2}-\gamma-\frac{1}{p}\right)}+C\sum_{n=j+2}^{\infty}\left(\sum_{2^{j-1}}^{2^j-1}\left|\sqrt{h_k}\,\Delta^\gamma(m_k\varphi_n(k))\right|^{p'}\right)^{1/p'}$$

and therefore

$$(3.7)\qquad \left(\sum_{2^{j-1}}^{2^j-1}k^{-1}|k^\gamma\Delta^\gamma m_k|^{p'}\right)^{1/p'}\leqq$$

$$\leqq C\|m\|_{M_p^p(\sigma,\tau;\alpha,\beta)}+C2^{j\left(\gamma+\frac{1}{p}-\alpha-\frac{3}{2}\right)}\sum_{n=j+2}^{\infty}\left(\sum_{2^{j-1}}^{2^j-1}\left|\sqrt{h_k}\,\Delta^\gamma(m_k\varphi_n(k))\right|^{p'}\right)^{1/p'}.$$

But, since

$$\left|\Delta^\gamma(m_k\varphi_n(k))\right| = \left|\sum_{2^{n-2/3}\leqq l\leqq 2^{n+2/3}} A_{l-k}^{-\gamma-1}m_l\varphi_n(l)\right| = O(\|m\|_\infty 2^{-\gamma n})$$

the last term in (3.7) can be estimated by

$$C2^{j\left(\gamma+\frac{1}{p}-\alpha-\frac{3}{2}\right)}\|m\|_\infty 2^{-\gamma j}\left(\sum_{2^{j-1}}^{2^j-1}h_k^{p'/2}\right)^{1/p'}\leqq C\|m\|_\infty.$$

Noting that $\|m\|_\infty\leqq C\|m\|_{M_p^p(\sigma,\tau;\alpha,\beta)}$ we finally obtain

$$\left(\sum_{2^{j-1}}^{2^j-1}k^{-1}|k^\gamma\Delta^\gamma m_k|^{p'}\right)^{1/p'}\leqq C\|m\|_{M_p^p(\sigma,\tau;\alpha,\beta)}$$

uniformly in $j$, i.e. the assertion of Theorem 3.

Remarks. 1. In the unweighted case $\sigma=\tau=0$, we conclude that if $\alpha>-\frac{1}{2}$ and $1\leqq p<2$ then

$$(3.8)\qquad M_p^p\left(0,0;\alpha,-\frac{1}{2}\right)\subset wbv_{p',\gamma},\quad \gamma=(2\alpha+1)\left(\frac{1}{p}-\frac{1}{2}\right).$$

**2.** Application of (3.8) to

$$m_k^{(n)}=\begin{cases} A_{n-k}^{\delta}/A_n^{\delta}, & 0\leqq k\leqq n,\\ 0, & k>n\end{cases}$$

and duality shows that in order for $\|m^{(n)}\|_{M_p^p\left(0,0;\alpha,-\frac{1}{2}\right)}$, $1<p<\infty$, to be uniformly bounded it is necessary that $\delta>(2\alpha+2)\left|\frac{1}{2}-\frac{1}{p}\right|-\frac{1}{2}$; this is only non-trivial as long as $\delta\geqq 0$ (which is a consequence of the trivial necessary condition: $l^\infty\supset M_p^p$). Note that $\delta>(2\alpha+2)\left|\frac{1}{2}-\frac{1}{p}\right|-\frac{1}{2}$ is sufficient [4, Sec. 8] for $\|m^{(n)}\|_{M_p^p(0,0;\alpha,\beta)}=O(1)$, $1\leqq p\leqq\infty$; in the case $p=1$, (3.8) leads only to the restriction $\delta\geqq\alpha+\frac{1}{2}$.

**3.** In [6] it is proved for $1<p<\infty$ that

$$wbv_{2,\lambda}\subset M_p^p(0,0;\alpha,\beta),\quad \lambda>\max\left\{(2\alpha+2)\left|\frac{1}{p}-\frac{1}{2}\right|,\ \frac{1}{2}\right\}. \tag{3.9}$$

Thus, when $\beta=-\frac{1}{2}$ and $1<p<2$ the required smoothness parameter $\lambda$ in the sufficient condition (3.9) differs from the necessary smoothness parameter $\gamma$ in (3.8) by any positive number larger than $\left(\frac{1}{p}-\frac{1}{2}\right)$. But the exact difference $\left(\frac{1}{p}-\frac{1}{2}\right)$ is needed for the embedding, i.e. $wbv_{2,\lambda}\subset wbv_{p',\mu}$ holds if $\lambda-\mu\geqq\left(\frac{1}{p}-\frac{1}{2}\right)$, $1<p\leqq 2$; see [4; Theorem 5].

**4.** Application of Askey's [1] transplantation theorem to Theorem 3 yields

$$M_p^p(0,0;\alpha,\beta)\subset wbv_{p',\gamma},\quad 1<p<2,$$

provided that $0<\gamma=(2\alpha+1)\left(\frac{1}{p}-\frac{1}{2}\right)<1-\frac{1}{p}$ and $0\leqq(2\beta+1)\left(\frac{1}{p}-\frac{1}{2}\right)<1-\frac{1}{p}$.

**5.** A little modification of Theorem 3 allows us to give necessary conditions for $M_p^q$ multipliers, $1\leqq p<q\leqq 2$; for, arguing as in the proof of Theorem 3 it follows that

$$\left(\sum_{2^{j-1}}^{2^j-1}\left|\sqrt{h_k}\,\Delta^\gamma m_k\right|^{q'}\right)^{1/q'}\leqq C2^{j\left(\alpha+\frac{3}{2}-\gamma-\frac{1}{p}\right)}\|m\|_{M_p^q(\sigma,\tau;\alpha,\beta)}$$

which implies that

$$\left(\sum_{2^{j-1}}^{2^j-1}k^{-q'/p'}\left|k^\gamma\Delta^\gamma m_k\right|^{q'}\right)^{1/q'}\leqq C\|m\|_{M_p^q(\sigma,\tau;\alpha,\beta)};$$

that is, in the $wbv_{q,\gamma,\delta}$ notation of [5]

$$M_p^q(\sigma, \tau; \alpha, \beta) \subset wbv_{q',\gamma,1/q'-1/p'}, \quad 1 \leqq p < q \leqq 2,$$

when $\alpha \geqq \beta \geqq -\frac{1}{2}$, $\alpha > -\frac{1}{2}$, $0 < \gamma = \sigma + (2\alpha+1)\left(\frac{1}{q} - \frac{1}{2}\right)$, and $\tau = (2\beta+1)\left(\frac{1}{2} - \frac{1}{q}\right)$.

In particular,

$$M_p^q\left(0, 0; \alpha, -\frac{1}{2}\right) \subset wbv_{q',\gamma,1/q'-1/p'}, \quad 1 \leqq p < q \leqq 2,$$

when $0 < \gamma = (2\alpha+1)\left(\frac{1}{q} - \frac{1}{2}\right)$.

**6.** By analogous techniques one can also obtain necessary Hankel multiplier conditions.

## References


[1] R. Askey, A transplantation theorem for Jacobi series, *Illinois J. Math.*, **13** (1969), 583—590.

[2] R. R. Coifman and G. Weiss, *Analyse harmonique non-commutative sur certains espaces homogenes*, Springer-Verlag (Berlin, 1971).

[3] W. C. Connett and A. L. Schwartz, A multiplier theorem for Jacobi expansions, *Studia Math.*, **52** (1975), 243—261.

[4] G. Gasper and W. Trebels, A characterization of localized Bessel potential spaces and applications to Jacobi and Hankel multipliers, *Studia Math.*, **65** (1979), 243—278.

[5] G. Gasper and W. Trebels, Jacobi and Hankel multipliers of type $(p, q)$, $1 < p < q < \infty$, *Math. Annalen*, **237** (1978), 243—251.

[6] G. Gasper and W. Trebels, Multiplier criteria of Hörmander type for Fourier series and applications to Jacobi series and Hankel transforms, *Math. Annalen*, **242** (1979), 225—240.

[7] G. Gasper and W. Trebels, Multipliers and Parseval type formulas for Jacobi series, in *Proc. Symposia in Pure Math.* vol. 35, *Harmonic Analysis in Euclidean Spaces and Related Topics*, Amer. Math. Soc. (Providence, R. I.), 1979, Part 2, 417—427.

[8] G. Szegő, *Orthogonal Polynomials*, Amer. Math. Soc. (Providence, R. I., 1975).



(G. G.)
NORTHWESTERN UNIVERSITY
EVANSTON, ILLINOIS 60201
USA

(W. T.)
TH DARMSTADT
D—6100 DARMSTADT
WEST GERMANY