Acta Sci. Math., 42 (1980), 257—263

On the lattice of quasivarieties of distributive lattices with pseudocomplementation

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1. Introduction. In G. GRÄTZER and H. LAKSER [6] it was shown that not every quasivariety (implicational class) of distributive lattices with pseudocomplementation is a variety (equational class). In G. GRÄTZER [5] it was conjectured that there are $2^{\aleph_0}$ quasivarieties of distributive lattices with pseudocomplementation. This was proved to be the case by M. E. ADAMS [1] and, independently, by A. WROŃSKI [10].

In [4], V. A. GORBUNOV asked (Question 6) whether the lattice of quasivarieties of distributive lattices with pseudocomplementation is distributive.

K. B. LEE [7] showed that the lattice of varieties of distributive lattices with pseudocomplementation is a countable chain $\mathbf{B}_{-1} \subsetneqq \mathbf{B}_0 \subsetneqq \mathbf{B}_1 \subsetneqq \ldots \subsetneqq \mathbf{B}_n \subsetneqq \ldots \subsetneqq \mathbf{B}_\omega$. If B_n denotes the n-atom finite Boolean lattice and $\bar{B}_n$ is B_n with a new unit element, then $\bar{B}_n$, regarded as a distributive lattice with pseudocomplementation, generates the variety $\mathbf{B}_n$, $n \geqq 1$. $\mathbf{B}_0$ is the variety of Boolean algebras and $\mathbf{B}_{-1}$ is the trivial variety.

In this paper we sharpen the result of Adams and Wroński by showing that there are $2^{\aleph_0}$ quasivarieties in the variety $\mathbf{B}_3$. We also show that the lattice of quasivarieties in $\mathbf{B}_3$ is nonmodular, thereby answering Gorbunov's question.

Since the distributive lattices with pseudocomplementation $\bar{B}_0, \bar{B}_1$, and $\bar{B}_2$ are projective, all quasivarieties in $\mathbf{B}_2$ are varieties. Our results are thus optimal.

2. The Priestley duality. H. A. PRIESTLEY [8] established a duality between distributive lattices and certain partially-ordered (Hausdorff) topological spaces. M. E. ADAMS [1] investigated this duality for distributive lattices with pseudocomplementation. In this paper we need only consider finite distributive lattices with pseudocomplementation and so we can dispense with any considerations of topology.

Received August 28, 1979.

The research of all the authors was supported by the N.S.E.R.C. of Canada.

Let **L** denote the category of finite distributive lattices with pseudocomplementation; the morphisms preserve 0, 1, and the operation of pseudocomplementation *. We denote by **P** the category whose objects are the finite posets and whose morphisms are isotone maps with the following property.

(*) If P_0, P_1 are posets, then $f\colon P_0 \to P_1$ is isotone and, for each $x \in P_0$ and $y \in P_1$, if $y \leqq f(x)$ and y is minimal in P_1, then there is a $z \in P_0$ with $z \leqq x$ and $f(z) = y$.

We call maps satisfying (*) *admissible maps*. There are a pair of contravariant functors $L\colon \mathbf{P} \to \mathbf{L}$ and $S\colon \mathbf{L} \to \mathbf{P}$. With each distributive lattice with pseudocomplementation L we associate the poset $S(L)$ of join-irreducible elements of L and for each homomorphism $f\colon L_0 \to L_1$ we define $S(f)\colon S(L_1) \to S(L_0)$ by setting

$$S(f)(x) = \sup \{y \in L_0 | f(y) \leqq x\}.$$

With each finite poset P we associate the distributive lattice with pseudocomplementation $L(P)$ consisting of all hereditary subsets (including the empty set) of P. If $f\colon P_0 \to P_1$ is admissible we define $L(f)\colon L(P_1) \to L(P_0)$ by setting $L(f)(x) = f^{-1}(x)$, where x is a hereditary subset of P_1.

Lemma 1 (M. E. Adams [1]). *$L\colon \mathbf{P} \to \mathbf{L}$ and $S\colon \mathbf{L} \to \mathbf{P}$ are contravariant functors, $HL\colon \mathbf{P} \to \mathbf{P}$ is naturally isomorphic to the identity functor and $LH\colon \mathbf{L} \to \mathbf{L}$ is naturally isomorphic to the identity functor.*

Lemma 2 (H. A. Priestley [8]). *Let P_0 and P_1 be finite posets and let $f\colon P_0 \to P_1$ be admissible. Then $L(f)$ is one-one if and only if f is surjective and $L(f)$ is surjective if and only if f is an embedding.*

Given two finite posets P_0, P_1 we denote their disjoint union (coproduct in **P**) by $P_0 \dot{\cup} P_1$. The following lemma is immediate.

Lemma 3. *If P_0, P_1 are finite posets then $L(P_0 \dot{\cup} P_1)$ is naturally isomorphic to $L(P_0) \times L(P_1)$. If $f_i\colon P_i \to P_0 \dot{\cup} P_1$, $i=0$ or 1, is the natural embedding, then $L(f_i)$ corresponds to the projection $L(P_0) \times L(P_1) \to L(P_i)$.*

3. Steiner quasigroups. We shall consider posets that derive from partial Steiner triple systems. We proceed in an algebraic manner. A *partial Steiner quasigroup* is a partial algebra A endowed with a binary partial operation $\cdot$ satisfying the following three conditions.

i) For all $x \in A$, x^2 is defined and $x^2 = x$;

ii) for $x, y \in A$, xy is defined if and only if yx is defined, and in this case $xy = yx$;

iii) if $x, y \in A$ and xy is defined then so is $x(xy)$ and $x(xy) = y$.

We say that A is a *Steiner quasigroup* if the binary operation is defined everywhere.

With each partial Steiner quasigroup is associated, in an obvious way, a partial Steiner triple system — see, e.g., the Introduction to [9].

A mapping $f: A \to B$ of partial Steiner quasigroups is said to be a *homomorphism* if, whenever $x, y \in A$ and xy is defined, then $f(x) \cdot f(y)$ is defined and $f(x) \cdot f(y) = f(xy)$.

With each partial Steiner quasigroup A we associate a poset $P(A)$ of height 1. The elements of $P(A)$ are all subsets of A of the form $\{x, y, xy\}$ (i.e., all singletons and certain triples, namely, the blocks of the corresponding triple system), and the partial order is set containment, $\subseteqq$. If $f: A \to B$ is a homomorphism we define $P(f): P(A) \to P(B)$ by setting

$$P(f)(\{x, y, z\}) = \{f(x), f(y), f(z)\}.$$

Lemma 4. *If A and B are finite partial Steiner quasigroups and $f: A \to B$ is a homomorphism, then $P(f): P(A) \to P(B)$ is an admissible map.*

Proof. $P(f)$ clearly maps $P(A)$ to $P(B)$ and it is equally clearly isotone. The minimal elements of $P(B)$ are the singletons $\{b\}$, $b \in B$. Let $\{x, y, z\} \in P(A)$ and let $b \in B$ with

$$\{b\} \subseteqq P(f)(\{x, y, z\}) = \{f(x), f(y), f(z)\},$$

that is, with $b = f(x)$, say. Then $\{x\} \subseteqq \{x, y, z\}$ and $P(f)(\{x\}) = \{b\}$; thus $P(f)$ is admissible, concluding the proof.

As a converse to Lemma 4 we have the following lemma.

Lemma 5. *Let A and B be finite partial Steiner quasigroups and let $g: P(A) \to P(B)$ be an admissible map. Then there is a unique homomorphism $f: A \to B$ with $g = P(f)$.*

Proof. Since g is admissible it maps minimal elements of $P(A)$ to minimal elements of $P(B)$. If $x \in A$, define $f(x) \in B$ by setting $\{f(x)\} = g(\{x\})$. The uniqueness of f is immediate, and we need only show that f is a homomorphism.

Let $x_0, x_1 \in A$ with $x_0 x_1$ defined; set $x_2 = x_0 x_1$. For each $i = 0, 1, 2$, $g(\{x_i\}) \subseteqq g(\{x_0, x_1, x_2\})$ by the isotonicity of g. Thus $\{f(x_0), f(x_1), f(x_2)\} \subseteqq g(\{x_0, x_1, x_2\})$. Let $y \in g(\{x_0, x_1, x_2\})$; by the admissibility of g there is an $i = 0, 1$, or 2 with $f(x_i) = y$. Consequently, $g(\{x_0, x_1, x_2\}) = \{f(x_0), f(x_1), f(x_2)\}$, and so $f(x_2) = f(x_0) \cdot f(x_1)$, showing that f is a homomorphism and concluding the proof.

4. The theorems. In the proof of our theorems we consider distributive lattices with pseudocomplementation of the form $LP(A)$, where A is a finite partial

Steiner quasigroup. We first show that such distributive lattices with pseudocomplementation reside in the correct variety.

Lemma 6. *If A is a finite partial Steiner quasigroup, then the distributive lattice with pseudocomplementation $LP(A)$ is a member of $\mathbf{B}_3$.*

Proof. Let T be the Steiner quasigroup consisting of three elements u, v, w with $uv=w$. Let $I=\{\langle a, b\rangle \in A^2 | ab$ is defined$\}$. For each $\langle a, b\rangle \in I$, define the homomorphism $f_{a,b}: T \to A$ by setting $f_{a,b}(u)=a, f_{a,b}(v)=b, f_{a,b}(w)=ab$. We then get a homomorphism $f: \dot{\bigcup}(T|I) \to A$, where $\dot{\bigcup}(T|I)$ denotes the partial Steiner quasigroup consisting of $|I|$ copies of T indexed by I, with xy defined if and only if x and y lie in the same copy of T. The resulting admissible map $P(f)$: $P(\dot{\bigcup}(T|I)) \to P(A)$ is surjective; if $\{a, b, c\} \in P(A)$, then $\{a, b, c\}=P(f_{a,b})(\{u, v, w\})$. Since $P(\dot{\bigcup}(T|I))=\dot{\bigcup}(P(T)|I)$ we see, by Lemma 2 and 3, that $LP(A)$ is a subalgebra of $(LP(T))^I$. But $LP(T) \cong \bar{B}_3$; thus $LP(A) \in \mathbf{B}_3$, proving the lemma.

We now recall the characterization of the quasivariety generated by a class of algebras.

Lemma 7 (G. Grätzer and H. Lakser [6]). *Let $\mathbf{K}$ be a class of algebras. An algebra A is a member of the quasivariety generated by $\mathbf{K}$ if and only if A is isomorphic to a subalgebra of a product of ultraproducts of families of algebras in $\mathbf{K}$.*

As immediate corollaries we get the following two lemmas.

Lemma 8. *Let $(\mathbf{K}_i|i \in I)$ be a family of quasivarieties of distributive lattices with pseudocomplementation, and let P be a finite poset. The distributive lattice with pseudocomplementation $L(P)$ is a member of the quasivariety $\mathbf{K}$ generated by the family $(\mathbf{K}_i|i \in I)$ if and only if there are a finite subset $I_0 \subseteq I$, finite posets P_i, $i \in I_0$, with $L(P_i) \in \mathbf{K}_i$, and an admissible surjection f: $\dot{\bigcup}(P_i|i \in I_0) \to P$.*

Proof. Since $L(P)$ is finite, Lemma 7 implies that $L(P) \in \mathbf{K}$ if and only if there are a finite subset $I_0 \subseteq I$ and finite distributive lattices with pseudocomplementation $L_i \in \mathbf{K}_i$, $i \in I_0$, such that $L(P)$ is a subalgebra of $\Pi(L_i|i \in I_0)$. The lemma is then immediate from Lemmas 1, 2 and 3.

Lemma 9. *Let $(P_i|i \in I)$ be a family of finite posets and let P be a finite poset. The distributive lattice with pseudocomplementation $L(P)$ is a member of the quasivariety generated by $(L(P_i)|i \in I)$ if and only if there is a finite subset $I_0 \subseteq I$ and an admissible surjection f: $\dot{\bigcup}(P_i|i \in I_0) \to P$.*

A Steiner quasigroup is said to be *planar* if it has at least four elements and any three distinct elements a, b, c with $ab \neq c$ generate the whole quasigroup. J. Doyen [3] showed that for each $n \geqq 7$ with $n \equiv 1$ or $3 \pmod 6$ there is a planar Steiner quasigroup of cardinality n. R. W. Quackenbush [9] proved that a finite planar Steiner quasigroup is simple if its cardinality is not 9. Since any four distinct elements of a planar Steiner quasigroup generates the quasigroup, we immediately get the following lemma.

Lemma 10. *Let A and B be nonisomorphic finite planar Steiner quasigroups with $|A|\neq 9$, and let $f\colon A\to B$ be a homomorphism. Then f is trivial, that is,* $\operatorname{Im} f$ *is a singleton.*

Theorem 1. *There are $2^{\aleph_0}$ quasivarieties of distributive lattices with pseudocomplementation in* $\mathbf{B}_3$.

Proof. Let $I=\{n\geqq 7 \mid n\equiv 1 \text{ or } 3 \pmod 6 \text{ and } n\neq 9\}$, and, for each $n\in I$, let A_n be a Steiner quasigroup of cardinality n. For each subset $J\subseteq I$ let $\mathbf{Q}(J)$ be the quasivariety of distributive lattices with pseudocomplementation generated by $(LP(A_n)|n\in J)$. By Lemma 6, $\mathbf{Q}(J)\subseteq \mathbf{B}_3$. We claim that if $J_0\neq J_1$ then $\mathbf{Q}(J_0)\neq \neq\mathbf{Q}(J_1)$. Indeed, let $i\in J_1-J_0$ and assume that $LP(A_i)\in\mathbf{Q}(J_0)$. Let $a_0, a_1, a_2\in A_i$ with $a_0\neq a_1$ and $a_0a_1=a_2$. By Lemma 9, there is an $n\in J_0$ and an admissible map $g\colon P(A_n)\to P(A_i)$ with $\{a_0, a_1, a_2\}\in \operatorname{Im} g$. By Lemma 5, there is a homomorphism $f\colon A_n\to A_i$ with $g=P(f)$. Now, f is not trivial since $\{a_0, a_1, a_2\}\subseteq \operatorname{Im} f$, contradicting Lemma 10.

Thus the quasivarieties $\mathbf{Q}(J)$ are distinct for distinct $J\subseteq I$, proving the theorem.

Theorem 2. *The lattice of quasivarieties of distributive lattices with pseudocomplementation in $\mathbf{B}_3$ is not modular.*

Proof. A. I. Budkin and V. A. Gorbunov [2] proved that the lattice of quasivarieties of algebras in a variety is modular if and only if it is distributive; thus we need only establish nondistributivity.

Let A_0, A_1 be nonisomorphic finite planar Steiner quasigroups, neither of cardinality 9; each is thus simple. For $i=0, 1$, let $a_i, b_i, c_i\in A_i$ be distinct with $c_i=a_ib_i$. Let B by the partial Steiner quasigroup obtained by amalgamating A_0 and A_1 over $\{a_i, b_i, c_i\}$. More specifically,

$$B=(A_0-\{a_0, b_0, c_0\})\,\dot{\cup}\,(A_1-\{a_1, b_1, c_1\})\,\dot{\cup}\,\{a, b, c\}$$

with a, b, c distinct and the operation defined as follows.

i) $ab=c$;

ii) if $x\in A_i-\{a_i, b_i, c_i\}$, $i=0$ or 1, then $xa=xa_i$ (in A_i), $xb=xb_i$, and $xc=xc_i$;

iii) if $x_0, x_1\in A_i-\{a_i, b_i, c_i\}$, $i=0$ or 1; then x_0x_1 is defined as in A_i;

iv) if $x_i\in A_i-\{a_i, b_i, c_i\}$, $i=0, 1$, then x_0x_1 is undefined.

Then B is a partial Steiner quasigroup and we have the obvious embeddings $A_i\to B$, $i=0, 1$; denote the images by A_i'. Thus $A_i'=(A_i-\{a_i, b_i, c_i\})\cup\{a, b, c\}$.

Claim 1. *Let* $i=0$ *or* 1 *and let* $f\colon A_i\to B$ *be a nontrivial homomorphism. Then* f *is one-one and* $\operatorname{Im} f=A'$.

By condition iv) every (total) subalgebra of B is a subalgebra of A'_0 or A'_1. The claim then follows by Lemma 10, recalling that A_i is simple.

Now, for $i=0,1$, let $\mathbf{A}_i$ be the quasivariety generated by $LP(A_i)$, and let $\mathbf{B}$ be the quasivariety generated by $LP(B)$.

Claim 2. $LP(B)\in\mathbf{A}_0\vee\mathbf{A}_1$ *(the join denoting the join in the lattice of quasi-varieties).*

The natural homomorphism of partial Steiner quasigroups $A_0\dot{\cup}A_1\to B$ yields an admissible map $P(A_0)\dot{\cup}P(A_1)\to P(B)$. This map is surjective by condition iv); consequently, $LP(B)$ is isomorphic to a subalgebra of $LP(A_0)\times LP(A_1)$, establishing the claim.

Claim 3. $LP(B)\notin(\mathbf{A}_0\wedge\mathbf{B})\vee(\mathbf{A}_1\wedge\mathbf{B})$.

Assume, to the contrary, that $LP(B)\in(\mathbf{A}_0\wedge\mathbf{B})\vee(\mathbf{A}_1\wedge\mathbf{B})$. By Lemma 8, there are finite posets P_0, P_1 with $L(P_i)\in\mathbf{A}_i\wedge\mathbf{B}$, $i=0,1$, and an admissible surjection $P_0\dot{\cup}P_1\to P(B)$. Thus, for some $i=0$ or 1, there is an admissible map $f\colon P_i\to P(B)$ with $\{a,b,c\}\in\operatorname{Im} f$, that is, $\{a,b,c\}=f(u)$ for some $u\in P_i$. We fix this i for the rest of the argument.

Since $L(P_i)\in\mathbf{B}$ we conclude, by Lemma 9, that there is an element $\{e_0,e_1,e_2\}\in$ $\in P(B)$ and an admissible map $g\colon P(B)\to P_i$ with $u=g(\{e_0,e_1,e_2\})$. Now there is a homomorphism $\varphi\colon B\to B$ with $P(\varphi)=fg$. Thus $\{\varphi e_0,\varphi e_1,\varphi e_2\}=\{a,b,c\}$ and so $\varphi e_0\neq\varphi e_1$. Since $e_0e_1\,(=e_2)$ is defined, both e_0 and e_1 are elements of $A'_j, j=0$ or 1, and so φ is nontrivial on A'_j. By Claim 1, φ is one-one on A'_j and, since $\{a,b,c\}\subseteq A'_0\cap A'_1$, φ is also nontrivial on A'_k, $k\neq j$. Thus, by Claim 1 again, $\varphi(A'_0)=A'_0$ and $\varphi(A'_1)=A'_1$. By condition iv) $fg=P(\varphi)\colon P(B)\to P(B)$ is surjective. Thus f is surjective.

Now let $x\in B-A'_i$; ax is defined and so there is a $v\in P_i$ with $f(v)=\{a,x,ax\}$. Since $L(P_i)$ is also a member of $\mathbf{A}_i$ there is, by Lemma 9, an admissible map $h\colon P(A_i)\to P_i$ with $v\in\operatorname{Im} h$. The admissible map $fh\colon P(A_i)\to P(B)$ is then of the form $P(\psi)$ for some homomorphism $\psi\colon A_i\to B$. Since $\{a,x,ax\}\subseteq\operatorname{Im}\psi$ $\psi\colon A_i\to B$ is nontrivial; since $x\notin A'_i$, we derive a contradiction to Claim 1. Thus our assumption that $LP(B)\in(\mathbf{A}_0\wedge\mathbf{B})\vee(\mathbf{A}_1\wedge\mathbf{B})$ is false, verifying Claim 3.

From Claim 2, $\mathbf{B}\leqq\mathbf{A}_0\vee\mathbf{A}_1$ and from Claim 3, $\mathbf{B}\not\leqq(\mathbf{A}_0\wedge\mathbf{B})\vee(\mathbf{A}_1\wedge\mathbf{B})$. Thus the lattice of quasivarieties in $\mathbf{B}_3$ is not distributive, concluding the proof of Theorem 2.

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