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On empirical Prékopa processes

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1. Introduction. In 1963 PRÉKOPA [8] considered the following inventory model. Given a time period T during which we observe the production of a factory A. In its production factory A uses a type of a material with constant intensity c. So in the given period T it needs the amount cT of this material. This should be supplied by factory B on the basis of the following contract.

a) For a fixed number λ, $0\leqq\lambda\leqq 1$, B will deliver the λ-portion λcT of the whole amount cT at n time-points $t_1, \ldots, t_n$, each time the amount $\lambda\dfrac{cT}{n}$. These instants t_i are independent random variables (r.v.'s) uniformly distributed on $(0, T)$.

b) The remaining portion $(1-\lambda)cT$ will be delivered at n time-points $r_1, \ldots, r_n$ which are again independent r.v.'s uniformly distributed on $(0, T)$, in amounts $s_1, \ldots, s_n$, respectively. These amounts are also r.v.'s, they are uniform spacings of the interval $(0, (1-\lambda)cT)$, and the sequence $(s_1, \ldots, s_n)$ is independent of both $(r_1, \ldots, r_n)$ and $(t_1, \ldots, t_n)$.

As usual, the spacing variables are constructed as follows. Divide the interval $(0, (1-\lambda)cT)$ into n subintervals by $n-1$ independent, uniformly distributed r.v.'s. Then $s_1, \ldots, s_n$ are the resulting lengths of these subintervals, and they will be referred to as "random additions".

Factory A wishes to avoid lack of material, so it needs an initial stock M_λ (at $t=0$) to balance with high prescribed probability the uncertainty in the delivery. In order to formulate exactly what M_λ is, we have to introduce the following quantities. Let $q_1, \ldots, q_{n-1}$ be independent r.v.'s uniformly distributed on $(0, cT)$, and let $0\leqq t_1^*\leqq\ldots\leqq t_n^*\leqq T$, $0\leqq r_1^*\leqq\ldots\leqq r_n^*\leqq T$ and $0\leqq q_1^*\leqq\ldots\leqq q_{n-1}^*\leqq cT$ denote the respective order statistics corresponding to the sequences t, r, and q. If we

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introduce the stochastic processes

$$R_n(t) = \begin{cases} 0, & \text{for} \quad 0 \leqq t \leqq r_1^*, \\ q_k^*, & \text{for} \quad r_k^* < t \leqq r_{k+1}^* \quad (k = 1, 2, \ldots, n-1), \\ cT, & \text{for} \quad r_n^* < t \leqq T, \end{cases}$$

$$K_n(t; \lambda, c, T) = \begin{cases} (1-\lambda) R_n(t), & \text{for} \quad 0 \leqq t \leqq t_1^*, \\ \lambda \dfrac{k}{n} cT + (1-\lambda) R_n(t), & \text{for} \quad t_k^* < t \leqq t_{k+1}^* \quad (k = 1, 2, \ldots, n-1), \\ \lambda cT + (1-\lambda) R_n(t), & \text{for} \quad t_n^* < t \leqq T, \end{cases}$$

then $(1-\lambda)R_n(t)$ and $K_n(t; \lambda, c, T)$ represent the amount of random additions and the total amount delivered up to time t, respectively. Let $\varepsilon > 0$. Prékopa's problem is: what initial stock $M_\lambda = M(\varepsilon, \lambda, c, T, n)$ should A posses to ensure the continuous production with probability $1-\varepsilon$. In order to obtain a solution we therefore need to know the probability

$$p_n(\lambda) = P\big(\sup_{0 \leqq t \leqq T} (ct - K_n(t; \lambda, c, T)) < M_\lambda\big) =$$

$$= P\left(\sup_{0 \leqq t \leqq T} \left(\frac{t}{T} - \frac{1}{cT} K_n(t; \lambda, c, T)\right) < \frac{M_\lambda}{cT}\right)$$

to find at least an asymptotic solution M_λ of the reliability equation

$$p_n(\lambda) = 1 - \varepsilon. \tag{1.1}$$

2. Summary. The form of the latter probability suggests to simplify the whole model. Following PRÉKOPA [8], let

$$X = (X_1, X_2, \ldots), \quad Y = (Y_1, Y_2, \ldots), \quad Z = (Z_1, Z_2, \ldots)$$

be three sequences of independent r.v.'s uniformly distributed on $(0, 1)$. For fixed n, $X_1^* \leqq \ldots \leqq X_n^*$, $Y_1^* \leqq \ldots \leqq X_n^*$, $Z_1^* \leqq \ldots \leqq Z_n^*$ are the three corresponding ordered samples. Sequences $X, Y,$ and Z will correspondingly play the role of the former sequences $(t_1, t_2, \ldots)$, $(r_1, r_2, \ldots)$, $(q_1, q_2, \ldots)$.

In the original model of Prékopa the delivery times of the fixed amounts and the random additions were identical $(t_1 = r_1, \ldots, t_n = r_n)$, and consequently in the simplified model he had $X = Y$. CSÖRGŐ [5] considered the possibility $X \neq Y$, assuming that X and Y are independent. The aim of the present paper is to study the general model when X and Y can depend on each other, i.e., to "bridge" the two extreme cases considered by Prékopa and later by Csörgő.

Denote by $F_n(t; X)$, $F_n(t; Y)$ and $F_n(t; Z)$ the n-th stage empirical distribution functions corresponding to the sequences X, Y and Z, respectively. If

$$\psi(x) = \begin{cases} 0, & \text{for} \quad x < 0, \\ 1, & \text{for} \quad x \geqq 0 \end{cases}$$

then, for instance, $F_n(t;X)=\frac{1}{n}\sum_{i=1}^{n}\psi(t-X_i)$. The following equivalent form for F_n will be used later. Clearly $\psi(x)=\frac{1}{2}\left(\frac{|x|}{x}+1\right)$, if $x\neq 0$. Since the distribution function of the X_k variables is continuous, for each fixed $t\in[0,1]$ we have almost surely that

$$F_n(t;X)=\frac{1}{2n}\sum_{i=1}^{n}\left(\frac{|t-X_i|}{t-X_i}+1\right).$$

Define the stochastic process

$$I_n(t)=\begin{cases}0, & \text{if } \ 0\leqq t\leqq X_1^*,\\ Z_k^*, & \text{if } \ X_k^*<t\leqq X_{k+1}^* \quad (k=1,2,\ldots,n-1),\\ 1, & \text{if } \ X_n^*<t\leqq 1,\end{cases}$$

and for an arbitrarily fixed λ $(0\leqq\lambda\leqq 1)$ consider

$$X_n^{f(\lambda)}(t)=\left(n/f(\lambda)\right)^{1/2}\left(t-K_n(t;\lambda)\right)=\left(n/f(\lambda)\right)^{1/2}\left(t-\lambda F_n(t;Y)-(1-\lambda)I_n(t)\right),$$

where $f(\lambda)$ is an arbitrary function on the interval $[0,1]$ such that

$$\inf_{0\leqq\lambda\leqq 1} f(\lambda)=\lambda^*>0. \tag{2.1}$$

In his first paper Prékopa [8] made an assertion (if $X=Y$) concerning the limit distribution of $\sup\limits_{0\leqq t\leqq 1} X_n^{1+(1-\lambda)^2}(t)$, which reduced to Smirnov's classical result when $\lambda=1$. Later in [9] he proved more, namely that

$$X_n^{1+(1-\lambda)^2}(\cdot)\xrightarrow{\mathcal{D}}B(\cdot),\quad X=Y, \tag{2.2}$$

where $\xrightarrow{\mathcal{D}}$ denotes weak convergence in Skorohod's $D[0,1]$ space and $B(t)$, $0\leqq t\leqq 1$, is the Brownian bridge process (cf. Billingsley [1]). Csörgő [5] noticed that the $X_n^{f(\lambda)}(t)$ process admits the following more convenient representation:

$$\begin{aligned}X_n^{f(\lambda)}(t)=\lambda\left(n/f(\lambda)\right)^{1/2}\left(t-F_n(t;Y)\right)+(1-\lambda)\left(n/f(\lambda)\right)^{1/2}\left(t-F_n(t;X)\right)+\\ +(1-\lambda)\left(f(\lambda)\right)^{-1/2}q_n\left(F_n(t;X);Z^{-1}\right)\end{aligned} \tag{2.3}$$

where

$$q_n(t;Z^{-1})=\sqrt{n}\left(t-F_n(t;Z^{-1})\right)$$

is the uniform quantile process, i.e.,

$$F_n(t;Z^{-1})=\inf\left(x\in[0,1]:\ F_n(x;Z)\geqq t\right).$$

Using (2.3) he gave an easy proof of (2.2) and also proved that

$$X_n^{\lambda^2+2(1-\lambda)^2}(\cdot)\xrightarrow{\mathcal{D}}B(\cdot)\quad\text{with}\quad X,Y\ \text{independent}. \tag{2.4}$$

Assuming a general condition on the dependency structure of sequences X and Y, we prove in Section 3 a general weak convergence theorem. The limit process is Gaussian, but it is not always a Brownian bridge. A necessary and sufficient condition is given to ensure that the limit process be the Brownian bridge. So (2.2) and (2.4) become corollaries of the general theorem. In Section 4 we apply the general weak convergence result to answer the original question, i.e., to determine (asymptotically) the required initial stock M_λ for the continuous production. Following Prékopa we generalize our general model in Section 5 to the case when the consumption of the delivered material in factory A is the same type random process as the delivery process. In Section 6 we come back to the two special cases in (2.2), (2.4) and apply recent strong approximation results to approximate $X_n^{f(\lambda)}$ by a sequences of appropriate Brownian bridges. This result will provide information about the accuracy of the asymptotic solutions of our reliability equations.

3. Weak convergence of the process $X_{n(\cdot)}^{f(\lambda)}$. According to our assumption in the original (non-simplified) model, we assume throughout that the sequence Z is independent of both sequences X and Y. It is also assumed throughout that the two dimensional random vectors (X_1, Y_1), (X_2, Y_2)... are independent. Define the distribution function of the pair (X_i, Y_i) $(i=1, 2, ...)$

$$P(X_i < t,\ Y_i < s) = G_i(t, s).$$

Theorem 3.1. *Suppose*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \big(G_i(t, s)+G_i(s, t)\big) = G(t, s)$$

exists for every $t, s \in [0, 1]$. *Then*

$$X_n^{f(\lambda)}(\cdot) \xrightarrow{\mathcal{D}} X^{f(\lambda)}(\cdot)$$

where $X^{f(\lambda)}$ *is a Gaussian process on* $[0, 1]$ *with* $EX^{f(\lambda)}(t)=0$ *and*

$$EX^{f(\lambda)}(t)X^{f(\lambda)}(s) = (f(\lambda))^{-1}\big[\lambda(1-\lambda)\big(G(t, s)-2t\big)+t(1-s)+(1-\lambda)^2 t(1-s)\big], \quad t \leqq s.$$

Proof. Since Z is independent of X and Y, the limit process of the third term of (2.3) and that of the sum of the first two terms are independent. As it is well known, the uniform quantile process $q_n(t; Z^{-1})$ converges weakly to the Brownian bridge, therefore it is enough to study the limit behaviour of the process

$$\tilde{X}_n^{f(\lambda)}(t) = (n/f(\lambda))^{1/2}\big[\lambda(t-F_n(t; Y))+(1-\lambda)(t-F_n(t; X))\big].$$

First we prove that the finite dimensional distributions of $\tilde{X}_n^{f(\lambda)}$ converge to those of the Gaussian process $\tilde{X}^{f(\lambda)}$ with $E\tilde{X}^{f(\lambda)}(t)=0$ and

$$E\tilde{X}^{f(\lambda)}(t)\tilde{X}^{f(\lambda)}(s) = (f(\lambda))^{-1}\big[\lambda(1-\lambda)\big(G(t, s)-2t\big)+t(1-s)\big], \quad t \leqq s.$$

In order to do this we need the following easy equation

$$P\big((t-X_i)(s-Y_i)>0\big)+P\big((s-X_i)(t-Y_i)>0\big)= \tag{3.1}$$
$$=2[G_i(t,s)+G_i(s,t)]+2-2(t+s),\quad t\leqq s,$$

and the identity

$$\tilde{X}_n^{f(\lambda)}(t)=(nf(\lambda))^{-1/2}\left(\sqrt{n}t-\sum_{i=1}^{n}\left(\frac{\lambda}{2}\left(\frac{|t-Y_i|}{t-Y_i}+1\right)+\frac{1-\lambda}{2}\left(\frac{|t-X_i|}{t-X_i}+1\right)\right)\right)$$

taking place with probability one. We need to know the following expectations computed by (3.1)

$$E\frac{|t-X_i|}{t-X_i}=E\frac{|t-Y_i|}{t-Y_i}=2t-1, \tag{3.2}$$

$$E\frac{|t_j-X_i|}{t_j-X_i}\cdot\frac{|t_l-X_i|}{t_l-X_i}=E\frac{|t_j-Y_i|}{t_j-Y_i}\cdot\frac{|t_l-Y_i|}{t_l-Y_i}=2P\big((t_j-X_i)(t_l-X_i)>0\big)-1=$$
$$=2(t_j-t_l)+1,\quad t_j\leqq t_l,$$

$$E\frac{|t_j-Y_i|}{t_j-Y_i}\cdot\frac{|t_l-X_i|}{t_l-X_i}=2P\big((t_j-Y_i)(t_l-X_i)>0\big)-1.$$

Taking the time-points $t_1, \ldots, t_k$ $(0\leqq t_1<t_2<\ldots<t_k\leqq 1)$ and real numbers $a_1, \ldots, a_k$ by the Cramér—Wold device (Billingsley [1], p. 49) we must show that

$$\sum_{j=1}^{k}a_j\tilde{X}_n^{f(\lambda)}(t_j)\xrightarrow{\mathcal{D}}\sum_{j=1}^{k}a_j\tilde{X}^{f(\lambda)}(t_j). \tag{3.3}$$

Here $\xrightarrow{\mathcal{D}}$ stands, naturally, for convergence in distribution on the real line. The left hand side of (3.3) can be written as

$$(nf(\lambda))^{1/2}\sum_{j=1}^{k}a_j\left[\sqrt{n}t_j-\sum_{i=1}^{n}\left(\frac{\lambda}{2}\left(\frac{|t_j-Y_i|}{t_j-Y_i}+1\right)+\frac{1-\lambda}{2}\left(\frac{|t_j-X_i|}{t_j-X_i}+1\right)\right)\right]=$$
$$=(f(\lambda))^{-1/2}\sum_{j=1}^{k}a_jt_j-(nf(\lambda))^{-1/2}\sum_{i=1}^{n}\alpha_i,$$

where the r.v. α_i is defined by

$$\alpha_i=\sum_{j=1}^{k}a_j\left(\frac{\lambda}{2}\left(\frac{|t_j-Y_i|}{t_j-Y_i}+1\right)+\frac{1-\lambda}{2}\left(\frac{|t_j-X_i|}{t_j-X_i}+1\right)\right)=\sum_{j=1}^{k}a_j\gamma_{j,i}.$$

$\alpha_1, \alpha_2, \ldots$ are independent because the random vectors $(X_1, Y_1), (X_2, Y_2), \ldots$ are independent. Applying the equalities in (3.2), we get $E\alpha_i=\sum_{j=1}^{k}a_jt_j$ $(i=1, 2, \ldots)$. Also, if $t_j<t_l$, then by (3.1) and (3.2) we find for the products of the terms in α_i the following

$$E\gamma_{j,i}\gamma_{l,i}=\lambda^2t_j+(1-\lambda)^2t_j+\lambda(1-\lambda)[G_i(t_j,t_l)+G_i(t_l,t_j)].$$

Consequently,

$$D^2\left((nf(\lambda))^{-1/2}\sum_{i=1}^{k}\alpha_i\right) = (nf(\lambda))^{-1}\sum_{i=1}^{n}D^2\left(\sum_{j=1}^{k}a_j\gamma_{j,i}\right) =$$

$$= (nf(\lambda))^{-1}\sum_{i=1}^{n}\left(\sum_{j=1}^{k}a_j^2D^2\gamma_{j,i}+2\sum_{l<j}a_la_j(E\gamma_{l,i}\gamma_{j,i}-E\gamma_{l,i}E\gamma_{j,i})\right) =$$

$$= (f(\lambda))^{-1}\sum_{j=1}^{k}a_j^2\left[2\lambda(1-\lambda)\left(n^{-1}\sum_{i=1}^{n}G_i(t_j,t_j)-t_j\right)+t_j-t_j^2\right]+$$

$$+2(f(\lambda))^{-1}\sum_{l<j}a_la_j\left(\frac{\lambda(1-\lambda)}{n}\sum_{i=1}^{n}(G_i(t_l,t_j)+G_i(t_j,t_l))+t_j(1-t_l)-2\lambda(1-\lambda)t_j\right).$$

We saw that $\alpha_1, \alpha_2, \ldots$ are independent, and it is easy to see that the moments $E|\alpha_i-E\alpha_i|^3$ are bounded. Hence the central limit theorem ([10], p. 442) can be used to finish the proof of (3.3).

Now we show that the sequence $X_n^{f(\lambda)}$ is tight. Since $X_n^{f(\lambda)}(0)=0$, it is enough to prove (cf. Theorem 15.5 of Billingsley [1]) that for each positive ε we have

$$\lim_{c\to 0}\overline{\lim_{n\to\infty}}\,P\Big(\sup_{|s-t|\le c}|X_n^{f(\lambda)}(s)-X_n^{f(\lambda)}(t)|>\varepsilon\Big)=0. \tag{3.4}$$

Besides the already introduced quantile process, let us also introduce the empirical processes $\beta_n(t;U)=\sqrt{n}\,(F_n(t;U)-t)$, $U=X, Y$. We have

$$P\Big(\sup_{|s-t|\le c}|X_n^{f(\lambda)}(s)-X_n^{f(\lambda)}(t)|>\varepsilon\Big)\le P\left(\sup_{|s-t|\le c}|\beta_n(s;Y)-\beta_n(t;Y)|>\frac{\varepsilon}{2}\sqrt{\lambda^*}\right)+$$

$$+P\left(\sup_{|s-t|\le c}|\beta_n(s;X)-\beta_n(t;X)|>\frac{\varepsilon}{4}\sqrt{\lambda^*}\right)$$

$$+P\left(\sup_{|s-t|\le 2c}|q_n(s;Z^{-1})-q_n(t;Z^{-1})|>\frac{\varepsilon}{4}\sqrt{\lambda^*}\right)+$$

$$+P\Big(\sup_{0\le u\le 1}|F_n(u;X)-u|>c\Big),$$

where λ^* is of (2.1). Using the Glivenko—Cantelli theorem and the fact that the empirical and quantile processes satisfy condition (3.4), the tightness of $X_n^{f(\lambda)}$ is clear.

Having the form of the covariance function given in Theorem 3.1, one obtains the following

Corollary 3.2. *Under the conditions of Theorem 3.1, the process $X_n^{f(\lambda)}$ converges weakly to the Brownian bridge for every λ, $0\le\lambda\le 1$ if and only if the following two conditions are satisfied*

(i) $$G(t,s)=kt(1-s)+2t,\quad t\le s$$

where k *is a fixed real number with* $-2 \leqq k \leqq 0$

(ii) $$f(\lambda) = 1+k\lambda(1-\lambda)+(1-\lambda)^2.$$

The following simple example shows that one can indeed have a limit process in Theorem 3.1 which is not a Brownian bridge. If $G_i(t, s) = t - t^2(1-s)$ $(t \leqq s)$, then

$$EX^{f(\lambda)}(t)X^{f(\lambda)}(s) = t(1-s)\big(1+(1-\lambda)^2-2\lambda(1-\lambda)t\big)\big(f(\lambda)\big)^{-1}, \quad t \leqq s,$$

and clearly there is no such $f(\lambda)$, for which the latter would be the covariance function of the Brownian bridge.

If the variables X_i, Y_i are identical then $k=0$, if the variables X_i, Y_i are independent then $k=-2$, thus (2.2) of PRÉKOPA [9] and (2.4) of CSÖRGŐ [5] follow from Theorem 3.1. Also, it follows from Theorem 3.1 that the processes $X_n^{f(1)}$ and $X_n^{f(0)}$ converge to a Brownian bridge, if $f(1)=1$ and $f(0)=2$ for every function $G(t, s)$. Indeed, in the first case our process is merely the classical empirical process (on the Y sequences), while in the second case "the empirical process with random jumps". It can also be noted here that LÁSZLÓ [7] managed to compute the exact distribution of the supremum of the process $X_n^{f(0)}$, $f(0)=2$.

4. Application to the solution of the reliability equation. The result of the preceding section can be applied to obtain an asymptotic solution of the reliability equation (1.1).

Theorem 4.1. *If the conditions of Corollary* 3.2 *are true, then the asymptotic solution of the reliability equation is*

$$M_\lambda \approx cT\left(\frac{f(\lambda)}{2n}\log\frac{1}{\varepsilon}\right)^{1/2}.$$

Proof. The reliability equation can be written in the form

$$P\left(\sup_{0\leqq t\leqq 1}\left(\frac{n}{f(\lambda)}\right)^{1/2}(t-K_n(t;\lambda)) < \frac{M_\lambda}{cT}\left(\frac{n}{f(\lambda)}\right)^{1/2}\right) = 1-\varepsilon.$$

By Corollary 3.2 $P\left(\sup\limits_{0\leqq t\leqq 1}\left(\frac{n}{f(\lambda)}\right)^{1/2}(t-K_n(t;\lambda))<x\right)$ converges to

$$P\big(\sup_{0\leqq t\leqq 1} B(t) < x\big) = \begin{cases} 0, & \text{if } x \leqq 0 \\ 1-\exp(-2x^2), & \text{if } x > 0, \end{cases}$$

Therefore, for large enough n, the reliability equation is $1-\exp\left(-2\left(\frac{M_\lambda}{cT}\right)^2\frac{n}{f(\lambda)}\right)=$ $=1-\varepsilon$, and hence the theorem.

Let us imagine that the "dependency constant" k, $-2\leqq k\leqq 0$, (between the transportation instants of the fixed amounts and the random additions) has already been determined (probably by some independent statistical procedure). The minimum of $f(\lambda)=f_k(\lambda)$ is attained at $\lambda_k=(2-k)/2(1-k)$, and $f_k(\lambda_k)=$ $=(k+5-1/(1-k))/4$. Regarding $f_k(\lambda_k)$ as a function of k we see that it strictly increases on $[-2,0]$. Consequently, the initial stock is minimal if the random variables X_i, Y_i are independent, and it is maximal if the r.v.'s X_i, Y_i are identical. At first it may seem surprising that the initial stock is minimal, when the delivery process is "most unorganized". On the other hand this is intuitively clear if we think that there are $2n$ independent deliveries in this case. Now we also have a concrete measure of this intuitive feeling. Since the maximum is $M_{\lambda_0}=cT\left(\log\frac{1}{\varepsilon}\Big/2n\right)^{1/2}$ and the minimum is $M_{\lambda_{-2}}=cT\left(\log\frac{1}{\varepsilon}\Big/3n\right)^{1/2}$, the proportion of the minimal and the maximal stock is $(2/3)^{1/2}\approx 0{,}82$.

5. Random consumption. We interpret the consumption process $T_n(t;\mu)$ similarly as we interpreted the arrival process $K_n(t;\lambda)$ in Section 1. The function $H_i(t,s)$ and $H(t,s)$ are defined as we defined the functions $G_i(t,s)$ and $G(t,s)$ there, respectively. We assume that the process $T_n(t;\mu)$ and $K_n(t;\lambda)$ are independent for every n. In this case the reliability equation is

$$P\left(\sup\left(T_n(t;\mu)-K_n(t;\lambda)\right)<\frac{M_{\lambda,\mu}}{K}\right)=1-\varepsilon \tag{5.1}$$

where K is the total amount of the material used by the factory in its production.

Theorem 5.1. *The process* $(n/(f(\lambda)+g(\mu)))^{1/2}(T_n(t;\mu)-K_n(t;\lambda))$ *converges weakly to the Gaussian process* $Z(t)$ *with* $EZ(t)=0$

$$\begin{aligned}EZ(t)Z(s) &= (f(\lambda)+g(\mu))^{-1}\left[\lambda(1-\lambda)(G(t,s)-2t)+2t(1-s)+\right.\\&\left.+(1-\lambda)^2t(1-s)+\mu(1-\mu)(H(t,s)-2t)+(1-\mu)^2t(1-s)\right],\quad t\leqq s.\end{aligned}$$

Proof. Because the limit processes of $(n/g(\mu))^{1/2}(t-T_n(t;\mu))$ and $(n/f(\lambda))^{1/2}(t-K_n(t;\lambda))$ are independent, the proof follows from Theorem 3.1. The same way as in the preceding section we have

Corollary 5.2. *The process* $(n/(f(\lambda)+g(\mu)))^{1/2}(T_n(t;\mu)-K_n(t;\lambda))$ *converges weakly to the Brownian bridge for every* λ *and for every* μ $0\leqq\lambda,\mu\leqq 1$ *if and only if*

(i) $G(t,s)=k_1t(1-s)+2t$

$H(t,s)=k_2t(1-s)+2t,\quad t\leqq s$

where k_1, k_2 *are fixed real numbers with* $-2\leqq k_1, k_2\leqq 0$,

(ii) $f(\lambda)=k_1\lambda(1-\lambda)+1+(1-\lambda)^2$

$g(\mu)=k_2\mu(1-\mu)+1+(1-\mu)^2$.

Under the conditions of Corollary 5.2 the asymptotic solution of the reliability equation (5.1) is

$$M_{\lambda,\mu} \approx K\left(\left(k_1\lambda(1-\lambda)+k_2\mu(1-\mu)+(1-\lambda)^2+(1-\mu)^2+2\right)\log\frac{1}{\varepsilon}\Big/2n\right)^{1/2}.$$

The proportion of the minimal and maximal initial stock is again $(2/3)^{1/2}$.

6. Strong approximation of the process $X_n^{f(\lambda)}$. When talking about approximation of the empirical and quantile process by appropriate Gaussian processes, we think of constructing the latter on the probability space of the former so that they should be near to each other with probability one. This can be done if this probability space is rich enough in the sense that an infinite independent sequence of Wiener processes can be defined on it, which is also independent of the originally given i.i.d. sequence (cf. M. Csörgő—P. Révész [3] and Komlós—Major—Tusnády [6]). It will be assumed that the underlying probability space is rich enough in this sense.

Theorem 6.1. *If X, Y are independent or $X=Y$ then one can define, for each n, a Brownian bridge $\{B_n(t),\ 0\le t\le 1\}$ such that we have*

$$P\big(\sup_{0\le t\le 1}|X_n^{f(\lambda)}(t)-B_n(t)| > K(\log n)^{3/4}n^{-1/4}\big) < Ln^{-2}$$

where $f(\lambda)=\lambda^2+2(1-\lambda)^2$ (X, Y are independent) or $f(\lambda)=1+(1-\lambda)^2$ ($X=Y$), and K, L are appropriate positive absolute constants.

Proof. Using the celebrated approximation result of Komlós—Major—Tusnády [6] for the empirical process and that for the uniform quantile process of M. Csörgő and P. Révész [3], there exist Brownian bridges $B_n^{(1)}(t), B_n^{(2)}(t), B_n^{(3)}(t)$, which are independent (if X, Y are independent) or $B_n^{(1)}(t)=B_n^{(2)}(t)$ ($X=Y$) and they are near to $\beta_n(t; X)$, $\beta_n(t; Y)$, $q_n(t; Z^{-1})$. The representation (2.3) and the precise form of the Komlós—Major—Tusnády approximation easily gives

$$P\big(\sup_{0\le t\le 1}|X_n^{f(\lambda)}(t)-B_n(t)| > K(\log n)^{3/4}n^{-1/4}\big) \le$$

$$\le L_1n^{-2}+P\left(\sup_{0\le t\le 1}\left|q_n(F_n(t; X); Z^{-1})-B_n^{(3)}(t)\right| > \frac{K}{3}(\log n)^{3/4}n^{-1/4}\right)$$

$$\le L_1n^{-2}+P\left(\sup_{0\le t\le 1}\sup_{|s|\le(\log n/n)^{1/2}}|q_n(t+s; Z^{-1})-B_n^{(3)}(t)| > \frac{K}{3}(\log n)^{3/4}n^{-1/4}\right)+$$

$$+P\big(\sup_{0\le t\le 1}|F_n(t; X)-t| > (\log n/n)^{1/2}\big) \le$$

$$\le L_1n^{-2}+P\left(\sup_{0\le t\le 1}|q_n(t; Z^{-1})-B_n^{(3)}(t)| > \frac{K}{6}(\log n)^{3/4}n^{-1/4}\right)+$$

$$+2P\left(\sup_{0\le t\le 1}\sup_{0\le s\le(\log n/n)^{1/2}}|B(t+s)-B(t)| > \frac{K}{6}(\log n)^{3/4}n^{-1/4}\right)+$$

$$+P\big(\sup_{0\le t\le 1}|F_n(t; X)-t| > (\log n/n)^{1/2}\big)$$

for any K, and a suitable $L_1>0$. Let now $K=\max(6A, 6\sqrt{30})$ where A is an appropriate constant to make the first probability here less then L_2n^{-2}, using the quantile process approximation with a suitable $L_2>0$. Using the lemma of DVORETZKY—KIEFER—WOLFOWITZ [2], the third probability is again smaller than L_3n^{-2}, where $L_3>0$ is some constant. Therefore the only problem now is to show that the second probability behaves the same way. Since $B(t)=W(t)-tW(1)$ with a standard Wiener process, this probability is not greater than

$$P\left(\sup_{0\le t\le 1}\sup_{0\le s\le(\log n/n)^{1/2}}|W(t+s)-W(t)|>\frac{1}{2}\sqrt{30}(\log n)^{3/4}n^{-1/4}\right)+$$
$$+P\left((\log n/n)^{1/2}|W(1)|>\frac{1}{2}\sqrt{30}(\log n)^{3/4}n^{-1/4}\right)\le$$
$$\le 40(n/\log n)^{1/2}n^{-5/2}+2(15\pi)^{-1}(n\log n)^{1/4}\exp\left(-15(n\log n)^{1/2}/2\right)\le L_4n^{-2}$$

where we have used the routine tail estimation of a normal variable, and, for the first term, Lemma 1 of M. CSÖRGŐ and P. RÉVÉSZ [4].

It follows (among others) under the conditions of Theorem 6.1 that

$$\sup_{-\infty<x<\infty}\left|P\left(\sup_{0\le t\le 1}X_n^{f(\lambda)}(t)<x\right)-P\left(\sup_{0\le t\le 1}B(t)<x\right)\right|=O\left((\log n)^{3/4}n^{-1/4}\right).$$

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