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On p-weak contractions

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H. BERCOVICI and D. VOICULESCU [1] defined the algebraic adjoint of operators belonging to the class $I+\mathscr{C}_1$. In Sec. 2 we extend this notion to the operators of class $I+\mathscr{C}_p$, where $p\geqq 1$ is an arbitrary integer. In Sec. 3 we study contractions T such that $\sigma(T)\neq D^-$ and $I-T^*T\in\mathscr{C}_p$, where $p\geqq 1$ is an arbitrary real number. These contractions will be called *p-weak*. We show that their characteristic functions have (generally unbounded) scalar multiples. With the aid of this we characterize in Sec. 4 and 5 the spectra of p-weak contractions and some $C_{\cdot 1}$ contractions.

In [1] it was proved that a C_0 contraction is a weak if and only if its Jordan-model is a weak contraction. In Sec. 6 we study the validity of this statement for p-weak contractions.

1. Preliminaries

We shall consider separable Hilbert spaces over the complex field $\mathbf{C}$.

If A is a compact operator on the Hilbert space $\mathfrak{H}$, then $|A|=(A^*A)^{\frac{1}{2}}$ is a compact selfadjoint operator. So there exist a decreasing sequence of positive numbers $\{s_i\}$ (the *s-numbers of* A) and an orthonormal system $\{\varphi_i\}$ such that $\lim s_i=0$ and $|A|=\sum_i s_i\langle\cdot,\varphi_i\rangle\varphi_i$. If $p\geqq 1$ is an arbitrary real number, then the *Schatten class* $\mathscr{C}_p(\mathfrak{H})$ is the set of compact operators A such that $\sum_i (s_i(A))^p<\infty$. It can be shown that $\mathscr{C}_p(\mathfrak{H})$ is a two-sided self-adjoint ideal in $\mathscr{L}(\mathfrak{H})$ and the function $\|A\|_p:=[\sum_i s_i^p]^{\frac{1}{p}}$ is a Banach-norm in $\mathscr{C}_p$.

For arbitrary operators $A, C\in\mathscr{L}(\mathfrak{H})$ and $B\in\mathscr{C}_p(\mathfrak{H})$ we have $\|ABC\|_p\leqq \leqq\|A\|\,\|B\|_p\|C\|$. If $A_j\in\mathscr{C}_{p_j}$ $(j=1,\dots,n)$ and $\sum_{j=1}^n p_j^{-1}\leqq 1$ then $A=A_1\dots A_n\in\mathscr{C}_p$,

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where $p^{-1}=\sum_{j=1}^{n} p_j^{-1}$; moreover, $\|A\|_p \leqq \|A_1\|_{p_1} \dots \|A_n\|_{p_n}$. Let $\{B_n\}$ be a sequence of operators tending to the operator B in the strong sense and let A be an operator from $\mathscr{C}_p(\mathfrak{K})$. Then $\lim_{n\to\infty} \|B_nA-BA\|_p=\lim_{n\to\infty}\|AB_n-AB\|_p=\lim_{n\to\infty}\|B_nAB_n-BAB\|_p=0$.

The operators $A\in\mathscr{C}_1(\mathfrak{K})$ are those with finite trace, i.e. for which $\sum_{i=1}^{\infty}\langle A\varphi_i, \varphi_i\rangle$ is convergent for every orthonormal basis $\{\varphi_i\}$. This sum is independent from the choice of the basis and is called the *trace* of A, and is denoted by $\operatorname{tr} A$ or $\operatorname{sp} A$. The following properties hold. For every $A_1, A_2\in\mathscr{C}_1$ and $c_1, c_2\in\mathbf{C}$ we have $\operatorname{sp}(c_1A_1+c_2A_2)=c_1\operatorname{sp}A_1+c_2\operatorname{sp}A_2$. If $AB, BA\in\mathscr{C}_1$ and A or B is compact, then $\operatorname{sp}(AB)=\operatorname{sp}(BA)$. If $A\in\mathscr{C}_1$, then $\operatorname{sp}A^*=\overline{\operatorname{sp}A}$ and $|\operatorname{sp}A|\leqq\|A\|_1$.

Let us assume that the operator A has the form $A=I-X$ where $X\in\mathscr{C}_p$ ($p\geqq 1$ integer). Let $\{\lambda_j\}$ be the sequence of the characteristic values of X taking them according to their algebraic multiplicities and let $\{s_j\}$ be the sequence of the s-numbers of X. It can be proved that $\sum_j|\lambda_j|^p\leqq\sum_j s_j^p$, so $\sum_j|\lambda_j|^p<\infty$. Therefore we can define the *p-regulated determinant of* A by

$$\overset{(p)}{\det} A := \prod_j\left[(1-\lambda_j)\exp\left(\sum_{k=1}^{p-1}\frac{1}{k}\lambda_j^k\right)\right].$$

If $A\in I+\mathscr{C}_1$, then $\overset{(p)}{\det}A=(\det A)\exp\left[\sum_{k=1}^{p-1}\operatorname{sp}(I-A)^k\right]$, where $\det A=\overset{(1)}{\det}A=\prod_j(1-\lambda_j)$. The function $\overset{(p)}{\det}(\cdot)$ is continuous in the following sense. If A, A_n are operators from $I+\mathscr{C}_p$ $(n=1, 2, \dots)$ and $\lim_{n\to\infty}\|A_n-A\|_p=0$, then $\lim_{n\to\infty}\overset{(p)}{\det}A_n=\overset{(p)}{\det}A$.

For a detailed discussion of these facts see [2], ch. II, III, IV.

Let A be an arbitrary operator on the finite dimensional Hilbert space $\mathfrak{K}$, having the matrix $[a_{i,j}]_{i,j=1}^n$ in the orthonormal basis $\{e_i\}_{i=1}^n$. Let us denote by $b_{i,j}$ the determinant, multiplied by $(-1)^{i+j}$, of the matrix obtained from the matrix $[a_{i,j}]_{i,j=1}^n$ by deleting its ith column and jth line $(1\leqq i, j\leqq n)$. The *algebraic adjoint* A^{Ad} *of* A is defined as the operator having the matrix $[b_{i,j}]_{i,j=1}^n$ in the basis $\{e_i\}_{i=1}^n$. It can be shown that this definition does not depend on the choice of the basis $\{e_i\}_{i=1}^n$. For details we refer to [1], § 5.

For any two (separable) Hilbert spaces $\mathfrak{E}, \mathfrak{E}_*$ the operator-valued Hardy space $H^\infty(\mathscr{L}(\mathfrak{E}, \mathfrak{E}_*))$ is the set of all bounded, $\mathscr{L}(\mathfrak{E}, \mathfrak{E}_*)$-valued analytic functions in the unit disc $D=\{z\in\mathbf{C}\,|\,|z|<1\}$. A function $\Theta\in H^\infty(\mathscr{L}(\mathfrak{E}, \mathfrak{E}_*))$ is contractive if $\|\Theta(z)\|\leqq 1$, $z\in D$. It is purely contractive if moreover $\|\Theta(0)f\|<\|f\|$ for every $f\in\mathfrak{E}, f\neq 0$. We say that two functions $\Theta_i\in H^\infty(\mathscr{L}(\mathfrak{E}_i, \mathfrak{E}_{*i}))$ $(i=1, 2)$ *coincide* if there are unitary operators $U\colon\mathfrak{E}_1\to\mathfrak{E}_2$, $V\colon\mathfrak{E}_{*1}\to\mathfrak{E}_{*2}$ such that $\Theta_2(\lambda)U=V\Theta_1(\lambda)$ for all $\lambda\in D$. A function $\Theta\in H^\infty(\mathscr{L}(\mathfrak{E}, \mathfrak{E}_*))$ is *outer* if it has dense range as an element of $\mathscr{L}(H^2(\mathfrak{E}), H^2(\mathfrak{E}_*))$. The function $\Theta^\sim$ is defined by $\Theta^\sim(z)=(\Theta(\bar{z}))^*$, $z\in D$.

If $\Theta \in H^{\infty}(\mathscr{L}(\mathfrak{E}, \mathfrak{E}_*))$ is purely contractive, then $S(\Theta)$ is the operator acting on the Hilbert space

$$\mathbf{H} = [H^2(\mathfrak{E}_*) \oplus \overline{\Delta L^2(\mathfrak{E})}] \ominus \{\Theta w \oplus \Delta w | w \in H^2(\mathfrak{E})\},$$

where $\Delta(t) = (I - \Theta(e^{it})^* \Theta(e^{it}))^{\frac{1}{2}}$, and defined by

$$S(\Theta)^*(u_* \oplus v) = e^{-it}(u_*(e^{it}) - u_*(0)) \oplus e^{-it} v(t),$$

$u_* \oplus v \in \mathbf{H}$.

If T is a contraction on the Hilbert space $\mathfrak{H}$, then $D_T = (I - T^*T)^{\frac{1}{2}}$, $D_{T^*} = (I - TT^*)^{\frac{1}{2}}$ will be called the *defect operators,* and $\mathfrak{D}_T = (D_T \mathfrak{H})^-$, $\mathfrak{D}_{T^*} = (D_{T^*} \mathfrak{H})^-$ the *defect spaces of* T. The *characteristic function* of the contraction T is the puraly contractive function $\Theta_T \in H^{\infty}(\mathscr{L}(\mathfrak{D}_T, \mathfrak{D}_{T^*}))$ defined by

$$\Theta_T(\lambda) = [-T + \lambda D_{T^*}(I - \lambda T^*)^{-1} D_T] | \mathfrak{D}_T, \quad \lambda \in D.$$

A contraction T on $\mathfrak{H}$ is *completely non-unitary* (c.n.u.) if for no non-zero reducing subspace $\mathfrak{L}$ for T is $T|\mathfrak{L}$ a unitary operator. For these facts see [3], ch. V, VI.

If $\{m_n\}_n$ is a sequence of inner functions such that m_{n+1} divides m_n for all n, then we call the operator $\bigoplus_n S(m_n)$ a *Jordan-operator.* It was proved in [4] that for every C_0 operator T (cf. [3], ch. III) there exists a unique Jordan-operator S, the *Jordan-model of* T, such that T and S are quasi-similar.

2. Algebraic adjoints of operators of class $I + \mathscr{C}_p$

In this section we extend the notion of algebraic adjoint defined by H. BERCOVICI and D. VOICULESCU [1] in the case $p=1$ to the operators of class $I+\mathscr{C}_p$, where $p \geqq 1$ is an arbitrary integer. The definition will be introduced, as in [1], in three steps. Firstly we treat the finite dimensional case, after that the case when the operator belongs to the class $I+\mathscr{F}$ ($\mathscr{F}$ denotes the class of operators having finite rank) and at last the case when the operator is taken from the class $I+\mathscr{C}_p$.

2.1. The finite dimensional case. In this section let $\mathfrak{K}$ be a finite dimensional Hilbert space, $\dim \mathfrak{K} = n$.

Definition 2.1. If $A \in \mathscr{L}(\mathfrak{K})$ and $p \geqq 2$ is an integer, then the *p-regulated algebraic adjoint of* A is the operator defined by

$$\overset{(p)}{A^{\mathrm{Ad}}} := A^{\mathrm{Ad}} \exp\left[\sum_{k=1}^{p-1} \frac{1}{k} \operatorname{sp}(I-A)^k\right].$$

Proposition 2.2.

(i) $A\overset{(p)}{A^{\mathrm{Ad}}}=\overset{(p)}{A^{\mathrm{Ad}}}A=(\overset{(p)}{\det} A)I$;

(ii) *If* $\{e_i\}_{i=1}^n$ *is an orthonormal basis in* $\mathfrak{R}$, *and* $1\leqq i,j\leqq n$, *then*

$$\langle \overset{(p)}{A^{\mathrm{Ad}}}e_i, e_j\rangle = (\overset{(p)}{\det} A_{ij})\exp\left[\mathrm{sp}\left(q_p(A, U_{i,j}, P_j)P_j\right)\right],$$

where $P_j=\langle\cdot, e_j\rangle e_j$, $U_{i,j}=\langle\cdot, e_j\rangle e_i$, $A_{i,j}=U_{i,j}+A(I-P_j)$ *and* $q_p(x_1, x_2, x_3)$ *is a polynomial in its non-commuting variables (depending only on* p*).*

(iii) *There exist constants* $D_p>0$, $C_p>0$ *and* $\gamma_p>0$ *(*γ_p *being an integer) depending only on* p *such that*

$$\|\overset{(p)}{A^{\mathrm{Ad}}}\| \leqq D_p\exp[C_p\|I-A\|_p^{\gamma_p}].$$

Property (i) immediately follows from the definition. As for property (ii), we have

$$\langle \overset{(p)}{A^{\mathrm{Ad}}}e_i, e_j\rangle = \langle A^{\mathrm{Ad}}e_i, e_j\rangle\exp\left[\sum_{k=1}^{p-1}\frac{1}{k}\,\mathrm{sp}\,(I-A)^k\right] =$$

$$= (\overset{(p)}{\det} A_{i,j})\exp\left[\mathrm{sp}\left(\sum_{k=1}^{p-1}\frac{1}{k}\left((I-A)^k-(I-A_{i,j})^k\right)\right)\right] =$$

$$= (\overset{(p)}{\det} A_{i,j})\exp\left[\mathrm{sp}\left(\sum_{k=1}^{p-1}\frac{1}{k}\left((I-A)^k-[(I-A)+(A-U_{i,j})P_j]^k\right)\right)\right] =$$

$$= (\overset{(p)}{\det} A_{i,j})\exp\left[\mathrm{sp}\left(q_p(A, U_{i,j}, P_j)P_j\right)\right].$$

For proving property (iii) we need the next lemmas.

Lemma 2.3. *For every integer* $p\geqq 2$ *there exists a constant* $C_p^*>0$ *such that for all* $\lambda\in\mathbf{C}$ *we have*

$$\left|f_p(\lambda) = (1-\lambda)\exp\left[\lambda+\frac{1}{2}\lambda^2+\ldots+\frac{1}{p-1}\lambda^{p-1}\right]\right| \leqq \exp[C_p^*|\lambda|^p].$$

Proof. In the case $|\lambda|\leqq\frac{1}{2}$ we have

$$|f_p(\lambda)| = \left|\exp\left[\log^*(1-\lambda)+\lambda+\frac{1}{2}\lambda^2+\ldots+\frac{1}{p-1}\lambda^{p-1}\right]\right| = \left|\exp\left[-\sum_{n=p}^{\infty}\frac{\lambda^n}{n}\right]\right| \leqq \exp[|\lambda|^p].$$

If $|\lambda|\geqq p$, then $|f_p(\lambda)|\leqq\exp\left[2|\lambda|+\frac{1}{2}|\lambda|^2+\ldots+\frac{1}{p-1}|\lambda|^{p-1}\right]\leqq\exp[p|\lambda|^{p-1}]\leqq$ $\leqq\exp[|\lambda|^p]$. Now there exists a constant M_p, such that $|f_p(\lambda)|\leqq M_p$ if $\frac{1}{2}\leqq|\lambda|\leqq p$.

Choosing $C'_p \geqq 2^p \ln M_p$ we have $|f_p(\lambda)| \leqq M_p \leqq \exp\left[C'_p\left(\frac{1}{2}\right)^p\right] \leqq \exp[C'_p|\lambda|^p]$ when $\frac{1}{2} \leqq |\lambda| \leqq p$. Therefore $C^*_p = \max\{1, C'_p\}$ will be suitable.

Lemma 2.4. *There exist constants $D_{p,N}$ and $C_{p,N}$ such that for every normal operator A we have*

$$\|\overset{(p)}{A^{\mathrm{Ad}}}\| \leqq D_{p,N} \exp[C_{p,N}\|I-A\|_p^p]. \tag{2.1}$$

Proof. There exist an orthonormal basis $\{e_i\}_{i=1}^n$ and complex numbers $\{\lambda_i\}_{i=1}^n$ such that $I-A = \sum_{i=1}^{n} \lambda_i \langle \cdot, e_i\rangle e_i$. Then denoting by $A^{\wedge(n-1)}$ the exterior product, taking A $(n-1)$ times, we have

$$\|\overset{(p)}{A^{\mathrm{Ad}}}\| = \|A^{\wedge(n-1)}\| \left|\exp\left[\sum_{k=1}^{p-1} \frac{1}{k} \operatorname{sp}(I-A)^k\right]\right| =$$

$$= \prod_{\substack{i=1 \\ i \neq i_0}}^{n} \left|(1-\lambda_i)\exp\left[\sum_{k=1}^{p-1} \frac{1}{k} \lambda_i^k\right]\right| \exp\left[\operatorname{Re} \sum_{k=1}^{p-1} \frac{1}{k} \lambda_{i_0}^k\right]$$

for some index i_0. By virtue of Lemma 2.3 we have

$$\|\overset{(p)}{A^{\mathrm{Ad}}}\| \leqq \exp\left[C^*_p \sum_{i=1}^{n} |\lambda_i|^p\right] \cdot \exp\left[\operatorname{Re}\left(\sum_{k=1}^{p-1} \frac{1}{k} \lambda_{i_0}^k\right) - C^*_p|\lambda_{i_0}|^p\right] \leqq D^*_p \exp[C^*_p\|I-A\|_p^p],$$

where D^*_p is an upper bound of the second factor, when λ_{i_0} alters in $\mathbf{C}$. $D_{p,N} = D^*_p$ and $C_{p,N} = C^*_p$ will be suitable constants.

Lemma 2.5. *For arbitrary operators X, Y and integer $p \geqq 2$ $R_p(X, Y)$ denotes the operator given by*

$$R_p(X, Y) = \sum_{k=1}^{p-1} \frac{1}{k}[(X+Y-XY)^k - X^k - Y^k]. \tag{2.2}$$

Let $R'_p(X, Y)$ be the polynomial obtained from $R_p(X, Y)$ omitting the terms of degree less than p. Then

$$\operatorname{sp} R'_p(X, Y) = \operatorname{sp} R_p(X, Y). \tag{2.3}$$

Proof. We can assume that $p > 2$. Obviously

$$R_p(X, Y) = R'_p(X, Y) + \sum_{q=2}^{p-1} K_q(X, Y), \tag{2.4}$$

where $K_q(X, Y)$ represents the homogeneous part of degree q of R_p. So it will be enough to prove that

$$\operatorname{sp} K_q(X, Y) = 0 \quad \text{for} \quad q = 2, \ldots, p-1. \tag{2.5}$$

Let q be an arbitrary integer such that $2 \leqq q \leqq p-1$. All terms in K_q contain both X and Y.

Let T be the mapping on the formal products containing q factors of operators from $\mathcal{L}(\mathfrak{K})$ such that

$$T(A_1 \dots A_q) := A_2 A_3 \dots A_q A_1.$$

If $i \geqq 1$, $j \geqq 1$ and $i+j=q$, then let us denote $\mathcal{L}_{i,j}$ the set of $S(X, Y)$'s, where $S(X, Y)$ is a product containing X i-times and Y j-times. We call $S(X, Y)$ *equivalent to* $S'(X, Y)$ if there exists an integer $r \geqq 0$ such that $T^r(S(X, Y)) = S'(X, Y)$. It is clear that this is an equivalence relation on $\mathcal{L}_{i,j}$. If $S(X, Y)$ is equivalent to $S'(X, Y)$, then $\operatorname{sp} S(X, Y) = \operatorname{sp} S'(X, Y)$. Therefore to verify (2.5) it is enough to prove that in any case $i \geqq 1, j \geqq 1, i+j=q$, taking an arbitrary equivalence class of $\mathcal{L}_{i,j}$ the sum of the coefficients in K_q of the products belonging to this equivalence class is 0.

Let S be an arbitrary element in $\mathcal{L}_{i,j}$. We compute the coefficient of S in K_q. The factor Y is called *essential* in S if an X factor precedes it. Let us suppose that the number of essential Y's in S is j_0 $(0 \leqq j_0 \leqq j)$. We can get $S(X, Y)$ in K_q such that (XY) occurs s-times $(0 \leqq s \leqq j)$ as a factor only from $\frac{1}{q-s} \cdot (X+Y-XY)^{q-s}$. Since the number of the factors (XY) is j_0, the coefficient derived so is $\binom{j_0}{s}(-1)^s \frac{1}{q-s}$. We get that the coefficient of $S(X, Y)$ in K_q is

$$\sum_{s=0}^{j_0} (-1)^s \binom{j_0}{s} \frac{1}{q-s}. \tag{2.6}$$

We denote by $\hat{S}$ the equivalence class of S in $\mathcal{L}_{i,j}$. There exists a least positive integer r such that $T^r S = S$. We infer that r devides q, and $\hat{S} = \{S, TS, \dots, T^{r-1}S\}$. We may assume that the first factor in S is X, so $j_0 \geqq 1$. The number of essential Y's in $T^l S$ is $(j_0 - 1)$ if the first factor of $T^l S$ is an operator Y which has occurred in S as an essential factor. Otherwise this number is j_0. Therefore there exist $\frac{r}{q} j_0$ elements in $\hat{S}$ such that they have (j_0-1) essential Y's, and there exist $\frac{r}{q}(q-j_0)$ elements in $\hat{S}$ such that they have j_0 essential Y's. By virtue of (2.6) the sum of the coefficients of the elements of $\hat{S}$ in K_q is

$$\frac{r}{q} j_0 \sum_{s=0}^{j_0-1} (-1)^s \binom{j_0-1}{s} \frac{1}{q-s} + \frac{r}{q}(q-j_0) \sum_{s=0}^{j_0} (-1)^s \binom{j_0}{s} \frac{1}{q-s}. \tag{2.7}$$

A short computation shows that this sum is 0.

The Lemma is proved.

Lemma 2.6. *Let A be an arbitrary operator and $A=U|A|$ its polar decomposition. Let us denote $X=I-U$, $Y=I-|A|$ and $Z=I-A$. Then*

$$\|X\|_p \leqq \|Z\|_p^2+3\|Z\|_p \quad \text{and} \quad \|Y\|_p \leqq \|Z\|_p^2+2\|Z\|_p. \tag{2.8}$$

Proof. Because $|1-\sqrt{\lambda}|\leqq|1-\lambda|$ $(\lambda\geqq 0)$ we have

$$\|Y\|_p = \|I-|A|\|_p \leqq \|I-A^*A\|_p. \tag{2.9}$$

On the other hand from the identity $I-A^*A=-(I-A^*)(I-A)+(I-A)+(I-A^*)$ we can derive

$$\|I-A^*A\|_p \leqq \|I-A\|_p^2+2\|I-A\|_p. \tag{2.10}$$

The second relation of (2.8) follows from (2.9) and (2.10). The first one can be got from this and the inequality $\|X\|_p\leqq\|Y\|_p+\|Z\|_p$ derived from the equation $X^*=Y-U^*Z$.

We can now prove the property (iii) in Proposition 2.2. Let A be an arbitrary operator and let X, Y, Z be defined as in the Lemma 2.6. A short computation yields

$$\overset{(p)}{A^{\mathrm{Ad}}} = |A|^{\overset{(p)}{\mathrm{Ad}}}\overset{(p)}{U^{\mathrm{Ad}}} \exp[\operatorname{sp} R_p(X, Y)], \tag{2.11}$$

where $R_p(X, Y)$ is given by (2.2). Applying Lemmas 2.4 and 2.5 we infer that there exists a polynomial r_p with positive coefficients such that

$$\|\overset{(p)}{A^{\mathrm{Ad}}}\| \leqq \exp[r_p(\|X\|_p, \|Y\|_p)]. \tag{2.12}$$

On account of Lemma 2.6 property (iii) follows.

2.2. Definition and properties in the case $A\in I+\mathcal{F}$. There exists a decomposition $\mathfrak{K}=\mathfrak{K}_1\oplus\mathfrak{K}_2$ reducing A such that $\dim\mathfrak{K}_1=n<\infty$ and A has the form $A=A_1\oplus I_2$.

Definition 2.7. $\overset{(p)}{A^{\mathrm{Ad}}}:=\overset{(p)}{A_1^{\mathrm{Ad}}}\oplus(\overset{(p)}{\det} A_1)I_2$.

Proposition 2.8. *The properties in Proposition 2.2 hold.*

Proof. (i) is evident. (iii) follows from the same property of Proposition 2.2 and the inequality $|\overset{(p)}{\det} A_1| = \left|\prod_{i=1}^{n}(1-\lambda_i)\exp\left[\lambda_i+\frac{1}{2}\lambda_i^2+\dots+\frac{1}{p-1}\lambda_i^{p-1}\right]\right| \leqq$ $\leqq \exp\left[C_p^*\sum_{i=1}^{n}|\lambda_i|^p\right]=\exp[C_p^*\|I_1-A_1\|_p^p]$, where λ_i's are the characteristic roots of A_1. (We have used Lemma 2.3.)

We prove (ii) firstly in the special case when the basis $\{e_i\}$ is such that $e_1\oplus\dots\oplus e_n=\mathfrak{K}_1$. It can be easily verified that for every $1\leqq i, j$

$$\langle \overset{(p)}{A^{\mathrm{Ad}}} e_i, e_j\rangle = (\overset{(p)}{\det} A_{i,j})\exp\left[\operatorname{sp}\left(\sum_{k=1}^{p-1}\frac{1}{k}((I-A)^k-(I-A_{i,j})^k)\right)\right].$$

Then property (ii) follows as in the proof of Proposition 2.2. Now we can easily see that the definition of $\overset{(p)}{A^{\mathrm{Ad}}}$ does not depend on the decomposition $\mathfrak{K}=\mathfrak{K}_1\oplus\mathfrak{K}_2$. Therefore property (ii) is fulfilled in general.

2.3. Definition and properties when $A\in I+\mathscr{C}_p$. There exists a sequence of operators $\{A_n\}_n$ such that $A_n\in I+\mathscr{F}$ for all n, and $\lim_n \|A_n-A\|_p=0$. Let $\{e_k\}$ be an orthonormal basis in $\mathfrak{K}$, and $1\leqq i,j$. Since $\lim_n \|A_{i,j}^{(n)}-A_{i,j}\|_p=0$ also holds, we infer $\lim_n \overset{(p)}{\det} A_{i,j}^{(n)}=\overset{(p)}{\det} A_{i,j}$. On the other hand,

$$\lim_n \|q_p(A_n,\, U_{i,j},\, P_j)P_j-q_p(A,\, U_{i,j},\, P_j)P_j\|_1=0$$

for $\lim_n \|A_n-A\|=0$ and $\operatorname{rank} P_j=1$. So we can write $\lim_n \operatorname{sp}\big(q_p(A_n,\, U_{i,j},\, P_j)P_j\big)= =\operatorname{sp}\big(q_p(A,\, U_{i,j},\, P_j)P_j\big)$. Therefore by virtue of property (ii) of Proposition 2.8 we get

$$\lim_n \langle \overset{(p)}{A_n^{\mathrm{Ad}}} e_i,\, e_j\rangle = \overset{(p)}{\det} A_{i,j} \exp\big[\operatorname{sp}\big(q_p(A,\, U_{i,j}, P_j)P_j\big)\big].$$

Regarding property (iii) of Proposition 2.8 we see that $\{\|\overset{(p)}{A_n^{\mathrm{Ad}}}\|\}_n$ is bounded. So the operator sequence $\{\overset{(p)}{A_n^{\mathrm{Ad}}}\}_n$ is weakly convergent.

Definition 2.9. $\overset{(p)}{A^{\mathrm{Ad}}}:=\lim_n \overset{(p)}{A_n^{\mathrm{Ad}}}$, where the limit exists in the sense of weak operator convergence. We call $\overset{(p)}{A^{\mathrm{Ad}}}$ the *p-regulated algebraic adjoint of* A.

Theorem 2.10. *If* $A\in I+\mathscr{C}_p$ *is an arbitrary operator, then the properties of Proposition* 2.2 *are satisfied.*

Proof. We can infer these properties from the definition and the corresponding properties of Proposition 2.8.

Remark 2.11. We can define similarly p-regulated algebraic adjoints of higher order of operators belonging to $I+\mathscr{C}_p$.

3. p-weak contractions

In this section $p\geqq 1$ is an arbitrary real number.

Definition 3.1. A contraction T will be called a *p-weak contraction* if its spectrum $\sigma(T)$ does not fill the unit disc D and $I-T^*T$ belongs to the class $\mathscr{C}_p$.

Remark 3.2. We can easily see that if $I-T^*T$ is compact then there exist a maximal partial isometry U and a compact operator X such that $T=U+X$.

By reason of the properties of semi-Fredholm operators (cf. [5]) we infer that if $\sigma_p(T)\cap D\neq D$ and $\sigma_p(T^*)\cap D\neq D$ then $\sigma_p(T)\cap D=\overline{\sigma_p(T^*)}\cap D=\sigma(T)\cap D$ and $\sigma(T)\cap D$ consists of isolated points in D. ($\sigma_p(T)$ denotes the point-spectrum of T.) Therefore we can state that a contraction T is p-weak if and only if $\sigma_p(T)$ and $\sigma_p(T^*)$ does not fill the unit disc D and $I-T^*T\in\mathscr{C}_p$.

Definition 3.1 is a generalization of the concept of weak contractions (case $p=1$). Several properties carry over to this case also.

Theorem 3.3. *If T is a p-weak contraction, then so are*
(i) $T_a=(T-aI)(I-\bar{a}T)^{-1}$, *where* $a\in D$;
(ii) T^*;
(iii) $T|\mathfrak{L}$, *where $\mathfrak{L}$ is an invariant subspace for T and* $\sigma(T|\mathfrak{L})\neq D^-$.

Proof. $\mathscr{C}_p$ being a two-sided ideal we see that $I-T_a^*T_a=(1-|a|^2)(I-aT^*)^{-1}\cdot(I-T^*T)(I-\bar{a}T)^{-1}\in\mathscr{C}_p$. Properties (ii) and (iii) follow as in [3], ch. VIII.

Theorem 3.4. *If T is a p-weak contraction then there exists a contractive analytic function $\Theta_0\in H^\infty(\mathscr{L}(\mathfrak{E}))$ coinciding with the characteristic function of T such that $\Theta_0(\lambda)\in I+\mathscr{C}_p(\mathfrak{E})$ for all $\lambda\in D$. All such function is of the form $U\Theta_0$ (regarding unitary equivalence), where U is an arbitrary unitary operator belonging to the class $I+\mathscr{C}_p$. Moreover Θ_0 can be chosen such that for all unitary operator $U\in I+\mathscr{C}_p(\mathfrak{E})$ and $\lambda\in D$ we have*

$$\|I-\Theta(\lambda)\|_p\leqq\|D_T^2\|_p(1-|\lambda|)^{-1}+\|I-U\|_p,$$

where $\Theta=U\Theta_0$.

Proof. Let $a\in D\setminus\sigma(T)$ and $T_a=(T-aI)(I-\bar{a}T)^{-1}$. We have $D_{T_a}^2=\sum_n\mu_n\langle\cdot,\varphi_n\rangle\varphi_n$, $D_{T_a^*}^2=\sum_n\mu_n\langle\cdot,\psi_n\rangle\psi_n$ where $\{\varphi_n\}$, $\{\psi_n\}$ are orthonormal systems (cf. [3], ch. VIII). The operator $U_a\in\mathscr{L}(\mathfrak{D}_{T_a},\mathfrak{D}_{T_a^*})$ defined by $U_a\varphi_n=-\psi_n$ will be unitary and $\Theta_a(\lambda)=U_a^*\Theta_{T_a}(\lambda)\in I+\mathscr{C}_p(\mathfrak{D}_{T_a})$ for all $\lambda\in D$. (Θ_T and Θ_{T_a} are the characteristic functions of T and T_a respectively.)

$$\|I-\Theta_a(\lambda)\|_p\leqq\|(I+U_a^*T_a)|\mathfrak{D}_{T_a}\|_p+\|\lambda U_a^*D_{T_a^*}(I-\lambda T_a^*)^{-1}D_{T_a}\|_p\leqq$$

$$\leqq\|(I+U_a^*T_a)|\mathfrak{D}_{T_a}\|_p+|\lambda|\|D_{T_a^*}\|_{2p}\|D_{T_a}\|_{2p}(1-|\lambda|)^{-1}\leqq\|D_{T_a}^2\|_p(1-|\lambda|)^{-1}.$$

(We have used that $\|(I+U_a^*T_a)|\mathfrak{D}_{T_a}\|_p=(\sum_n(1-(1-\mu_n)^{\frac{1}{2}})^p)^{\frac{1}{p}}\leqq\|D_{T_a}^2\|_p$ and that $\|D_{T_a}\|_{2p}=\|D_{T_a^*}\|_{2p}=(\|D_{T_a}^2\|_p)^{\frac{1}{2}}$.)

There exist unitary operators $U_1\in\mathscr{L}(\mathfrak{D}_T,\mathfrak{D}_{T_a})$, $U_2\in\mathscr{L}(\mathfrak{D}_{T_a^*},\mathfrak{D}_{T^*})$ such that $\Theta_T\left(\frac{\lambda+a}{1+\bar{a}\lambda}\right)=U_2\Theta_{T_a}(\lambda)U_1$. Then for $\Theta_0(\lambda)=U_1^*U_a^*U_2^*\Theta_T(\lambda)$ we have $I-\Theta_0\left(\frac{\lambda+a}{1+\bar{a}\lambda}\right)=$

$=U_1^*(I-\Theta_a(\lambda))U_1$ and $\|I-\Theta_0(\lambda)\|_p \leqq \|D_{T_a}^2\|_p\left(1-\left|\frac{\lambda-a}{1-\bar{a}\lambda}\right|\right)^{-1} \leqq \|D_{T_a}^2\|_p\left(1-\frac{|\lambda|+|a|}{1+|a||\lambda|}\right)^{-1}$.

Since $\|D_{T_a}^2\|_p=\|S^*D_T^2S\|_p \leqq \|S\|^2\|D_T^2\|_p$ where $\|S\|=\|(1-|a|^2)^{\frac{1}{2}}(I-\bar{a}T)^{-1}\| \leqq$ $\leqq \left(\frac{1+|a|}{1-|a|}\right)^{\frac{1}{2}}$ we infer that $\|I-\Theta_0(\lambda)\|_p \leqq \left(\frac{1+|a|}{1-|a|}\right)^2\|D_T^2\|_p(1-|\lambda|)^{-1}$. By Remark 3.2 $a\in D\setminus\sigma(T)$ can be chosen arbitrary small, therefore

$$\|I-\Theta_0(\lambda)\|_p \leqq \|D_T^2\|_p(1-|\lambda|)^{-1}.$$

Because $\|I-U\Theta_0(\lambda)\|_p \leqq \|I-\Theta_0(\lambda)\|_p+\|I-U\|_p\|\Theta_0(\lambda)\| \leqq \|I-\Theta_0(\lambda)\|_p+\|I-U\|_p$ if $U\in I+\mathscr{C}_p(\mathfrak{D}_T)$ is a unitary operator, the Lemma follows.

The next converse of Theorem 3.4 is true.

Theorem 3.5. *Let $\Theta\in H^\infty(\mathscr{L}(\mathfrak{E},\mathfrak{E}_*))$ be a purely contractive analytic function. Let us assume that there exists a $\lambda_0\in D$ such that $\Theta(\lambda_0)$ is invertible and there exist a $\lambda_1\in D$ and an unitary operator $U\in\mathscr{L}(\mathfrak{E},\mathfrak{E}_*)$ such that $U^*\Theta(\lambda_1)\in I+\mathscr{C}_p(\mathfrak{E})$. Then $S(\Theta)$ is a p-weak contraction.*

Proof. The characteristic function of $S=S(\Theta)$, Θ_S coincides with Θ. (Cf. [3], VI. 3.) So there exists a unitary operator $U_1\in\mathscr{L}(\mathfrak{D}_S,\mathfrak{D}_{S^*})$ such that $U_1^*\Theta_S(\lambda_1)\in I+\mathscr{C}_p(\mathfrak{D}_S)$. Let us denote $S_1=S_{\lambda_1}=(S-\lambda_1 I)(I-\bar{\lambda}_1 S)^{-1}$. Since the characteristic function of S_1, $\Theta_1(\lambda)$ coincides with $\Theta_S\left(\frac{\lambda+\lambda_1}{1+\bar{\lambda}_1\lambda}\right)$ (cf. [3], ch. VI), there exists a unitary operator $U_2\in\mathscr{L}(\mathfrak{D}_{S_1},\mathfrak{D}_{S_1^*})$ such that $U_2^*\Theta_1(0)=$ $=-U_2^*S_1|\mathfrak{D}_{S_1}\in I+\mathscr{C}_p(\mathfrak{D}_{S_1})$. Therefore $(I-S_1^*S_1)|\mathfrak{D}_{S_1}=I-(S_1^*(-U_2))(-U_2^*S_1)|\mathfrak{D}_{S_1}\in$ $\in\mathscr{C}_p(\mathfrak{D}_{S_1})$ and so $I-S_1^*S_1\in\mathscr{C}_p$. $\Theta_1\left(\frac{\lambda_0-\lambda_1}{1-\bar{\lambda}_1\lambda_0}\right)$ being invertible we see that $\sigma(S_1)\neq D^-$ (cf. [3], VI). Therefore S_1 is a p-weak contraction and by Theorem 3.3 so is $S=(S_1)_{-\lambda_1}$.

Remark 3.6. Regarding Remark 3.2 we see that Theorem 3.5 remains valid if instead of the existence of $\lambda_0\in D$ such that $\Theta(\lambda_0)$ is invertible we assume that there exist $\lambda_0', \lambda_0''\in D$ such that $\Theta(\lambda_0')$ and $\Theta^\sim(\lambda_0'')$ are injections. (Cf. [3], VI. 4.)

Corollary 3.7. *Let $\Theta\in H^\infty(\mathscr{L}(\mathfrak{E}))$ be a puraly contractive analytic function. Let us assume that there exists a $\lambda_1\in D$ for which $\Theta(\lambda_1)\in I+\mathscr{C}_p(\mathfrak{E})$ and there exist $\lambda_0', \lambda_0''\in D$ such that $\Theta(\lambda_0')$ and $\Theta^\sim(\lambda_0'')$ are injections. Then $\Theta(\lambda)\in I+\mathscr{C}_p(\mathfrak{E})$ for all $\lambda\in D$ and $\Theta(\lambda)$ is invertible except isolated points in D.*

Proof. By Remark 3.6 $S(\Theta)$ is a p-weak contraction. So by Theorem 3.4 there exists a unitary operator $U\in\mathscr{L}(\mathfrak{E})$ such that $U\Theta(\lambda)\in I+\mathscr{C}_p(\mathfrak{E})$ for all $\lambda\in D$. In particular $U\Theta(\lambda_1)\in I+\mathscr{C}_p(\mathfrak{E})$. Since we have also $\Theta(\lambda_1)\in I+\mathscr{C}_p(\mathfrak{E})$ we infer that $U\in I+\mathscr{C}_p(\mathfrak{E})$ and so that $\Theta(\lambda)\in I+\mathscr{C}_p(\mathfrak{E})$ for all $\lambda\in D$. The other part of the theorem follows by Remark 3.2.

Corollary 3.8. *Let $\{m_n\}$ be an arbitrary sequence of non-constant inner functions.*

(i) *If there exists a $\lambda_1\in D$ such that $\sum_n |1-m_n(\lambda_1)|^p<\infty$, then for all $\lambda\in D$ we have $\sum_n |1-m_n(\lambda)|^p<\infty$.*

(ii) *If there exists a $\lambda_1\in D$ such that $\sum_n (1-|m_n(\lambda_1)|)^p<\infty$, then for all $\lambda\in D$ we have $\sum_n (1-|m_n(\lambda)|)^p<\infty$.*

Proof. (i) is an immediate consequence of the Corollary 3.7.

If $S=\bigoplus_n S(m_n)$ then $I-SS^*=\sum_n (1-|m_n(0)|^2)\langle\cdot,\varphi_n\rangle\varphi_n$ where $\{\varphi_n\}$ is an orthonormal system. So (ii) follows by Theorem 3.3 (i).

Now we regard some important corollaries of Theorem 3.4.

Corollary 3.9. *Let $p\geqq 1$ be an integer. If Θ is the contractive analytic function occurring in Theorem 3.4, then for all $\lambda\in D$ we have*

(i) $\|\Theta(\lambda)^{\overset{(p)}{\mathrm{Ad}}}\| \leqq D_p \exp\left[C_p(\|D_T^2\|_p(1-|\lambda|)^{-1}+\|I-U\|_p)^{\gamma_p}\right]$;

(ii) $|\overset{(p)}{\det}\,\Theta(\lambda)| \leqq \exp\left[C_p^*(\|D_T^2\|_p(1-|\lambda|)^{-1}+\|I-U\|_p)^p\right]$,

where C_p, D_p, γ_p are the constants from Theorem 2.10 and C_p^* is the constant from Lemma 2.3.

Proof. (i) is an immediate consequence of Theorem 3.4 and Theorem 2.10.

Let $A=I-X$ where $X\in\mathscr{C}_p$. Let us denote by $\{\lambda_n\}$, $\{s_n\}$ the characteristic values with algebraic multiplicities of X and $|X|$ respectively. Then by Lemma 2.3 and [2], ch. II. 3.1 we infer

$$|\overset{(p)}{\det} A| = \left|\prod_n \left[(1-\lambda_n)\exp\left(\lambda_n+\frac{1}{2}\lambda_n^2+\ldots+\frac{1}{p-1}\lambda_n^{p-1}\right)\right]\right| \leqq$$
$$\leqq \exp\left[C_p^*\sum_n |\lambda_n|^p\right] \leqq \exp\left[C_p^*\sum_n s_n^p\right] = \exp\left[C_p^*\|I-A\|_p^p\right].$$

(ii) follows from this relation and Theorem 3.4.

Theorem 3.10. *If T is a p-weak contraction ($p\geqq 1$ integer) and Θ is a contractive analytic function coinciding with the characteristic function of T such that $\Theta(\lambda)\in I+\mathscr{C}_p$ for all $\lambda\in D$, then $\Theta^{\overset{(p)}{\mathrm{Ad}}}$ and $\overset{(p)}{\det}\,\Theta$ are analytic functions on D.*

Proof. Let $\{P_n\}$ be a sequence of orthogonal projections of finite rank which converges strongly to the identity operator. Then $\Theta_n=P_n\Theta P_n+(I-P_n)\,\overset{(p)}{\det}\,(P_n\Theta P_n)$

is a contractive analytic function for every n, and $\{\|\Theta(\lambda)-\Theta_n(\lambda)\|_p\}_n$ converges to 0 for all $\lambda\in D$. Therefore $\{\overset{(p)}{\det}\,\Theta_n\}_n$ converges to $\overset{(p)}{\det}\,\Theta$, and by Theorem 3.4 and Corollary 3.9 this sequence is uniformly bounded in every compact subset of D. Now analycity of $\overset{(p)}{\det}\,\Theta$ follows by Vitali's theorem.

Regarding Theorem 2.10 (ii) analycity of $\overset{(p)}{\Theta^{\mathrm{Ad}}}$ follows similarly.

Definition 3.11. The $\mathscr{L}(\mathfrak{E},\mathfrak{E}_*)$-valued analytiy function Θ is said to have the *general scalar multiple* $\delta(\lambda)$, if $\delta(\lambda)$ is a scalar valued analytic function, $\delta(\lambda)\not\equiv 0$, and there exists an $\mathscr{L}(\mathfrak{E}_*,\mathfrak{E})$-valued analytic function Ω such that for all $\lambda\in D$ we have

$$\Omega(\lambda)\Theta(\lambda)=\delta(\lambda)I_{\mathfrak{E}},\quad \Theta(\lambda)\Omega(\lambda)=\delta(\lambda)I_{\mathfrak{E}_*}.$$

Theorem 3.12. *If T is a p-weak contraction ($p\geqq 1$ is real), then its characteristic function Θ_T has a general scalar multiple. Particularly $\overset{(q)}{\det}\,\Theta$ will be a general scalar multiple, where $q\geqq p$ is an arbitrary integer and Θ is a function coinciding with Θ_T and such that $\Theta(\lambda)\in I+\mathscr{C}_q$ for all $\lambda\in D$.*

Proof. Theorem follows by Theorems 3.4, 3.10 and 2.10 (i).

Remark 3.13. If $p>1$ real and T is a p-weak contraction, then Θ_T does not have generally a scalar multiple. Even it may happen that there is not a general scalar multiple belonging to some Hardy-class H^q ($q>0$). For example let $\{a_n\}$ be a sequence of complex numbers such that $0<|a_n|<1$, $\sum_n(1-|a_n|)^p<\infty$ for all $p>1$, and $\sum_n(1-|a_n|)=\infty$. Let us denote

$$m_n(\lambda)=\frac{|a_n|}{a_n}\,\frac{a_n-\lambda}{1-\bar{a}_n\lambda}\quad\text{and}\quad T=\bigoplus_n S(m_n).$$

Then T is a p-weak contraction for all $p>1$, but Θ_T does not have a general scalar multiple belonging to some class H^q. (Cf. [6] Theorem 2.3 and [3], ch. VI.)

4. The spectrum of a p-weak contraction

Let T be a p-weak contraction ($p\geqq 1$ integer). By Theorem 3.4 there exists an analytic function Θ coinciding with Θ_T such that $\Theta(\lambda)\in I+\mathscr{C}_p$ for all $\lambda\in D$. We can define $\overset{(p)}{\det}\,\Theta$ which will be an analytic function on D. Because $\overset{(p)}{\det}\,\Theta(\lambda)=0$ if and only if $\Theta(\lambda)$ is not invertible, we infer that $\sigma(T)\cap D$ coincides with the set of zeros of $\overset{(p)}{\det}\,\Theta$. (Cf. [3], VI. 4.) (For the unitary part T_u of T we have

$\sigma(T_u)\cap D=\emptyset$.) We can estimate the growth of $|\overset{(p)}{\det}\,\Theta(\lambda)|$ by Corollary 3.9. On the other hand there is a connection between the growth of absolute value and distribution of zeros of a function analytic on D. So we can get information about the distribution of points of $\sigma(T)\cap D$.

Definition 4.1. If $\alpha=\{\alpha_n\}$ is a sequence in D then $\tau_\alpha:=\inf\{x>0 \mid \sum_n (1-|\alpha_n|)^x<<\infty\}$. If $\beta=\{\beta_n\}$ is an arbitrary sequence of non-zero numbers in $\mathbf{C}$ then

$$\tau'_\beta:=\inf\left\{x>0 \,\middle|\, \sum_n \left(\frac{1}{|\beta_n|}\right)^x<\infty\right\}.$$

Definition 4.2. If f is an analytic function on D, then ϱ_f denotes the infimum of positive μ's which satisfy that there exists an $r_\mu<1$ such that for all $r_\mu<r<1$ we have $M_f(r)=\max\{|f(\lambda)| \mid |\lambda|=r\}\leqq\exp[(1-r)^{-\mu}]$. We call ϱ_f the *order of the function* f.

Lemma 4.3. *If f is an analytic function on D and $\alpha=\{\alpha_n\}$ is the sequence of its zeros, taking them with multiplicities, then*

$$\tau_\alpha\leqq\varrho_f+1.$$

Proof. Let us denote $\beta=\{\beta_n=(1-|\alpha_n|)^{-1}\}$. For $\tau_\alpha=\tau'_\beta$ and $\tau'_\beta=\overline{\lim_{r\to\infty}}\dfrac{\ln\nu_\beta(r)}{\ln r}$, where $\nu_\beta(r)$ denotes the number of β_n's having absolute value less than r, (cf. [7], V. § 15), we have

$$\tau_\alpha=\overline{\lim_{r\to 1-0}}\frac{\ln\nu_\beta((1-r)^{-1})}{-\ln(1-r)}=\overline{\lim_{r\to 1-0}}\frac{\ln\nu_\alpha(r)}{-\ln(1-r)}=\overline{\lim_{r\to 1-0}}\frac{\ln\nu_\alpha(1-e(1-r))}{-1-\ln(1-r)}$$

If $f(0)\neq 0$ and $0<r<1$, then

$$\int_0^r\frac{\nu_\alpha(t)}{t}\,dt=\int_1^{(1-r)^{-1}}\frac{\nu_\beta(u)\,du}{u(u-1)}\geqq\frac{1-r}{r}\int_{(e(1-r))^{-1}}^{e(e(1-r)^{-1}}\frac{\nu_\beta(u)\,du}{u}\geqq$$

$$\geqq\frac{1-r}{r}\nu_\beta((e(1-r))^{-1})=\frac{1-r}{r}\nu_\alpha(1-e(1-r)).$$

Moreover if $|f(0)|=1$, then $\displaystyle\int_0^r\frac{\nu_\alpha(t)\,dt}{t}\leqq\ln M_f(r)$ (cf. [7], V. § 15), so in this case $\nu_\alpha(1-e(1-r))\leqq\dfrac{r}{1-r}\ln M_f(r)$. We infer that if f is an arbitrary function having 0 as a zero with multiplicity n, and $\dfrac{3}{4}<r<1$ then

$$\nu_\alpha(1-e(1-r))\leqq\frac{r}{1-r}\ln M_f(r)+\frac{r}{1-r}\ln\frac{n!}{2^{-n}|f^{(n)}(0)|}+n.$$

Therefore

$$\tau_\alpha \leqq 1 + \varlimsup_{r\to 1-0} \frac{\ln\ln M_f(r)}{-\ln(1-r)}.$$

It is easily seen that

$$\varrho_f = \varlimsup_{r\to 1-0} \frac{\ln\ln M_f(r)}{-\ln(1-r)},$$

so the Lemma is proved.

Theorem 4.4. *If T is a p-weak contraction ($p \geqq 1$ integer) and $\sigma(T)\cap D = \{\lambda_n\} = \lambda$, then $\tau_\lambda \leqq p+1$.*

Proof. The Theorem follows by Corollary 3.9 and Lemma 4.3.

This estimation is not exact in the case $p=1$. Indeed then $\det \Theta \in H^\infty$ and so $\sum_n (1-|\lambda_n|) < \infty$. There is a question whether it is exact in the case $p \geqq 2$. In Theorem 3.12 we verified the existence of a general scalar multiple of order p of Θ_T if T is a p-weak contraction. It remains a question whether there is a general scalar multiple of order $(p-1)$ of Θ_T for arbitrary p.

We give exact estimation for a special class of operators.

Lemma 4.5. *Let $\{m_i\}_i$ be a sequence of inner functions. Let us denote $\{\alpha_n^{(i)}\}_n$ the zeros of m_i with multiplicities. If there exists a $\lambda_1 \in D$ such that $\sum_i (1-|m_i(\lambda_1)|)^p < \infty$ ($p \geqq 1$ real) then $\sum_i \sum_n (1-|\alpha_n^{(i)}|)^p < \infty$.*

Proof. By Corollary 3.8 we know $\sum_i (1-|m_i(0)|)^p < \infty$. Regarding the factorization of m_i into the product of a Blaschke product and an inner function nonvanishing on D, we see that $\sum_i (1-\prod_n |\alpha_n^{(i)}|)^p \leqq \sum_i (1-|m_i(0)|)^p < \infty$. There exists a $\delta > 0$ such that for every $\frac{1}{2} \leqq x \leqq 1$ we have $1-x \leqq -\ln x \leqq \delta(1-x)$. We may assume that $\frac{1}{2} \leqq \prod_n |\alpha_n^{(i)}| \leqq 1$ for every i. So for arbitrary i

$$0 \leqq \sum_n (1-|\alpha_n^{(i)}|) \leqq -\sum_n \ln|\alpha_n^{(i)}| = -\ln\big(\prod_n |\alpha_n^{(i)}|\big) \leqq \delta\big(1-\prod_n |\alpha_n^{(i)}|\big).$$

Therefore $\sum_{i,n} (1-|\alpha_n^{(i)}|)^p \leqq \sum_i \big(\sum_n (1-|\alpha_n^{(i)}|)\big)^p \leqq \delta^p \sum_i \big(1-\prod_n |\alpha_n^{(i)}|\big)^p < \infty$. The Lemma is proved.

Theorem 4.6. *Let $\{m_i\}_i$ be an arbitrary sequence of inner functions, and $S = \bigoplus_i S(m_i)$. If S is a p-weak contraction ($p \geqq 1$ real) and $\sigma(S)\cap D = \{\lambda_n\}_n$, then $\sum_n (1-|\lambda_n|)^p < \infty$. (It is easily seen that this is an exact estimation.)*

Proof. Since S is a p-weak contraction, we have $\sum_i (1-|m_i(0)|)^p < \infty$. By Lemma 4.5 we infer $\sum_{i,n} (1-|\alpha_n^{(i)}|)^p < \infty$, where $\{\alpha_n^{(i)}\}_n$ is the sequence of zeros of

m_i without multiplicities. For Θ_S coincides with $\begin{pmatrix} m_1 & & & 0 \\ & m_2 & & \\ & & m_3 & \\ 0 & & & \ddots \end{pmatrix}$, so $\sigma_p(S)\cap D=$ $=\overline{\sigma_p(S^*)}\cap D=\{\alpha_n^{(i)}\}_{n,i}\neq D$. (Cf. [3], VI. 4.) Then by Remark 3.2 we see that $\sigma(S)\cap$ $\cap D=\sigma_p(S)\cap D$, therefore $\sigma(S)\cap D=\{\alpha_n^{(i)}\}_{n,i}$ and the Theorem follows.

5. The spectra of contractions of class $C_{\cdot 1}$

Lemma 5.1. *If $\Theta\in H^\infty(\mathcal{L}(\mathfrak{E},\mathfrak{E}_*))$ is an outer function, then for all $0<r<1$ $\Theta_r(\lambda)=\Theta(r\lambda)$ will be also an outer function.*

Proof. Let us denote $U_r\in\mathcal{L}(H^2(\mathfrak{E}))$ the operator defined by $(U_ru)(\lambda)=u(r\lambda)$, $u\in H^2(\mathfrak{E})$. Regarding the decomposition $H^2(\mathfrak{E})=\bigoplus_{n=0}^{\infty} U_+^n\mathfrak{E}$, where U_+ is the operator of multiplication by λ, U_r has the form $U_r=\bigoplus_{n=0}^{\infty} r^n I_{U_+^n\mathfrak{E}}$. So U_r is a quasiaffinity. Let $U_{*r}\in\mathcal{L}(H^2(\mathfrak{E}_*))$ be defined similarly. We can easily see that $\Theta_rU_r=U_{*r}\Theta$. Therefore $(\Theta_rH^2(\mathfrak{E}))^-=(\Theta_rU_rH^2(\mathfrak{E}))^-=(U_{*r}\Theta H^2(\mathfrak{E}))^-=(\Theta H^2(\mathfrak{E}))^-=$ $=H^2(\mathfrak{E}_*)$. That is Θ_r is outer and the Lemma is proved.

Remark 5.2. The converse of Lemma 5.1 is not true. Namely, there exists a 2-weak contraction T of class C_{01} (cf. [8]). (For the definition of the class C_{01} see [3], II. 4.) Θ_T is an outer function (cf. [3], VI. 3) which by Theorem 3.12 has a general scalar multiple. Then by Lemma 5.1, for all $0<r<1$ $\Theta_T(r\lambda)$ will be outer, and it is easily seen that it has a scalar multiple. So $\Theta_T^{\sim}(r\lambda)$ is also an outer function for all $0<r<1$ (cf. [3], V. 6), but $\Theta_T^{\sim}(\lambda)$ is not outer for $T\in C_{0\cdot}$.

From the above Lemma, using Theorem 6.2 of [3] ch. V, we infer.

Theorem 5.3. *If the outer function Θ has a general scalar multiple then $\Theta(\lambda)$ is boundedly invertible for all $\lambda\in D$.*

By this Theorem we get the next:

Corollary 5.4. *Let T be a c.n.u. contraction of class $C_{\cdot 1}$ whose characteristic function admits a general scalar multiple. Particularly this is the case if T is a c.n.u. p-weak contraction of class $C_{\cdot 1}$. Then the spectrum $\sigma(T)$ is situated on the circle C.*

Remark 5.5. It would be interesting to know whether Proposition 4.4 of [3], ch. VI remains true replacing scalar multiple by general scalar multiple. The answer depends on the validity of Proposition 6.7 of [3], ch. V in this general situation.

6. p-weak contractions of class C_0

The next theorem is a generalization of [1], Proposition 4.3.

Theorem 6.1. *Let T be a C_0 operator and let S be its Jordan-model. If S is a p-weak contraction ($p \geqq 1$ real), then T is also a p-weak contraction.*

Proof. Regarding the proof of [1], Proposition 4.3 Theorem follows from the next Lemma.

Lemma 6.2. *Let $\{a_k\}_k$, $\{b_k\}_k$ be increasing sequences such that $0<a_k \leqq 1$, $0<b_k \leqq 1$ for every k and $\prod_{k=1}^{n} a_k \leqq \prod_{k=1}^{n} b_k$ for every n. If $\sum_{k=1}^{\infty}(1-a_k)^p<\infty$, where $p \geqq 1$ real, then $\sum_{k=1}^{\infty}(1-b_k)^p<\infty$.*

Proof. From the assumption it follows that $\sum_{k=1}^{n} \log \frac{1}{a_k} \geqq \sum_{k=1}^{n} \log \frac{1}{b_k}$ for arbitrary n. Since $\left\{\log \frac{1}{a_k}\right\}_k$ and $\left\{\log \frac{1}{b_k}\right\}_k$ are decreasing sequences we infer by Lemma 3.4 of [2], ch. II that $\sum_{k=1}^{n} \frac{1}{a_k} \geqq \sum_{k=1}^{n} \frac{1}{b_k}$, that is $\sum_{k=1}^{n} \frac{1-a_k}{a_k} \geqq \sum_{k=1}^{n} \frac{1-b_k}{b_k}$ for every n. $\left\{\frac{1-a_k}{a_k}\right\}_k$ and $\left\{\frac{1-b_k}{b_k}\right\}_k$ being also decreasing sequences we can employ again the above Lemma. So we get $\sum_{k=1}^{\infty}\left(\frac{1-a_k}{a_k}\right)^p \geqq \sum_{k=1}^{\infty}\left(\frac{1-b_k}{b_k}\right)^p$. It follows from the assumption $\sum_{k=1}^{\infty}(1-a_k)^p<\infty$ that $\lim_{k\to\infty} a_k=1$, so $\sum_{k=1}^{\infty}\left(\frac{1-a_k}{a_k}\right)^p<\infty$. Therefore $\sum_{k=1}^{\infty}(1-b_k)^p \leqq \sum_{k=1}^{\infty}\left(\frac{1-b_k}{b_k}\right)^p \leqq \sum_{k=1}^{\infty}\left(\frac{1-a_k}{a_k}\right)^p<\infty$. The Lemma is proved.

Remark 6.3. With a slight modification of the example of [1], Remark 4.4 we can show that the converse of Theorem 6.1 is in general false. Namely let μ be a finite non-negative measure on $[0, 2\pi]$, singular with respect to Lebesgue measure and without atoms. Let us assume that $\mu([0, 2\pi])=1$. For every n there exists a decomposition of $[0, 2\pi]$ into disjoint intervals $i_1^{(n)}, \ldots, i_{2^n}^{(n)}$ such that $\mu(i_k^{(n)})=2^{-n}$ for $k=1, \ldots, 2^n$. Let $m_{k,n}(\lambda)$ and $m(\lambda)$ be inner functions defined by

$$m_{k,n}(\lambda)=\exp\left[-\int_{i_k^{(n)}} \frac{e^{it}+\lambda}{e^{it}-\lambda}\, d\mu(t)\right]$$

$(n=1, 2, \ldots;\ k=1, \ldots, 2^n)$, and

$$m(\lambda)=\exp\left[-\int_{0}^{2\pi} \frac{e^{it}+\lambda}{e^{it}-\lambda}\, d\mu(t)\right].$$

Let T, S be the operators $T=\bigoplus_{n=1}^{\infty}\left(\bigoplus_{k=1}^{2^n} S(m_{k,n})\right)$ and $S=S(m)\oplus S(m)\oplus\dots$. Then S is the Jordan-model of T (cf. [9]). Since $\sum_{n=1}^{\infty}\sum_{k=1}^{2^n}(1-|m_{k,n}(0)|^2)^p=$
$=\sum_{n=1}^{\infty}2^n\left(1-\exp\left(-\frac{2}{2^n}\right)\right)^p\le 2^p\sum_{n=1}^{\infty}\left(\frac{2}{2^p}\right)^n<\infty$ if $p>1$, it follows that T is a p-weak contraction for all real number $p>1$. On the other hand, $I-SS^*$ is not compact.

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