Acta Sci. Math., 42 (1980), 299—304

Some general fixed point theorems

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In the late 1960's and early 1970's a large number of fixed point papers were written involving definitions which are generalizations of the original contractive definition attributed to Banach. A classification and comparison of many of these definitions appears in [17]. More recently, several authors have made improvements by recognizing that the contractive definitions need not hold for all points in the space. For example, B. FISHER [7] has proved several fixed point theorems involving contractive definitions which are satisfied only for points x, fy, for x, y in the space. (See also [18].) ĆIRIĆ [3] made the observation that certain contractive definitions imply the boundedness of $O(x)$, the orbit of x, for each x in the space, where $O(x)=\{x, f(x), f^2(x), \ldots\}$. This idea has been utilized by HEGEDŰS [11]. Other authors have used a contractive definition involving a function $\varphi: R_+ \to R_+$, which is nondecreasing and satisfies $\varphi(t)<t$ for each $t>0$, where $R_+=[0, \infty)$. (See, e.g. [1].)

In this paper we establish several fixed point theorems involving hypotheses weak enough to include a number of fixed point theorems as special cases.

Our first result is the following, which is a generalization of Theorem 4.1 of the first author [16].

Let X be a topological space. A function $G: X\to R_+$ is called f-orbitally lower semicontinuous at a point $p\in X$ if, for every $x_0\in X$, $x_{n_k}\to p$ implies $G(p)\leqq \liminf_k G(x_{n_k})$, where $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$, and $\{x_n\}$ is defined by $x_{n+1}=f(x_n)$; i.e., $\{x_n\}=0(x_0)$.

Theorem 1. *Let f be a selfmap of a topological space X, and d a non-negative, real valued function defined on $X\times X$ such that $d(x, y)=d(y, x)$ and $d(x, y)=0$ iff $x=y$. If there exists a point $u\in X$ such that $\lim_n d(f^{n+1}(u), f^n(u))=0$, and, if $\{f^n(u)\}$ has a convergent subsequence with limit $p\in X$, then p is a fixed point of f if and only if $G(x)=d(x, f(x))$ is f-orbitally lower semicontinuous at p.*

Received April 18, 1979 and in revised form February 14, 1980.

*) Supported by the SNU-AID program in 1979.

Proof. Suppose $\{f^{n_k}(u)\}$ converges to a fixed point p of f. Then $0=G(p)\leqq \leqq \liminf_k G(f^{n_k}(u))$.

Conversely, if G is f-orbitally lower semicontinuous at p, then

$$0 = \lim_n d(f^{n+1}(u), f^n(u)) = \liminf_k d(f^{n_k+1}(u), f^{n_k}(u)) \geqq d(p, f(p)),$$

since, for each k sufficiently large, there exists an integer n_k satisfying $n_k \geqq k$.

Theorem 1 includes Theorem 2 of [15].

Corollary 1. *Let X be a metric space, $f: X\to X$, $\varphi: X\to R_+$ such that, there exists a point $u\in X$ with $d(x, f(x))\leqq\varphi(x)-\varphi(f(x))$ for each $x\in O(u)$, and $\bar{O}(u)$ is complete. Then*

(i) $\lim_n f^n(u)=p$ *exists, and*

(ii) *p is a fixed point of f if and only if $G=d(x, f(x))$ is f-orbitally lower semicontinuous at p.*

The proof of Corollary 1 follows from Theorem 1 and the following

Lemma (SIEGEL [20]. *Let $\{x_n\}$ be a sequence in $O(u)$ such that $d(x_n, x_{n+1})\leqq \leqq\varphi(x_n)-\varphi(x_{n+1})$ for all n, φ, u as in Corollary 1. Then $\lim_n x_n$ exists.*

Corollary 1 compares well with CARISTI's Theorem [2] and Theorem 1 of [12] as special cases.

Let $\delta(O(x))$ denote the *diameter of the orbit of* x.

The following result is an extension of HEGEDŰS [11] and Theorem 1 of DANEŠ [4].

Theorem 2. *Let f be a selfmap of a metric space (X, d) satisfying:*

(i) $\delta(O(x))<\infty$ *for each* $x\in X$.

(ii) *There exists $u\in X$ such that $O(u)$ has a cluster point $p\in X$.*

(iii) *There exists a map $\varphi: R_+\to R_+$ which is nondecreasing, continuous from the right and satisfies $\varphi(t)<t$ for each $t>0$ and the inequality,*

$$d(f(x), f^2(y)) \leqq \varphi(\delta(O(x)\cup O(f(y)))) \quad \textit{for each} \quad x, y\in X.$$

Then p is the unique fixed point of f and $\lim_n f^n(u)=p$.

Proof. Define $\varrho_n=\delta(O(f^n(u)))$. From (i), ϱ_n is finite for each n. Since $\varrho_{n+1}\leqq\varrho_n$ for each n, $\{\varrho_n\}$ converges to some number $\varrho\geqq 0$.

For each $j>i\geqq n+1$, from (iii),

$$d(f^i(u), f^j(u)) \leqq \varphi(\delta(O(f^{i-1}(u))\cup O(f^{j-2}(u)))) \leqq \varphi(\delta(O(f^n(u)))) = \varphi(\varrho_n),$$

so that $\varrho_{n+1}\leqq\varphi(\varrho_n)$ for each n. Since φ is continuous from the right, $\varrho\leqq\varphi(\varrho)$, which implies $\varrho=0$. Therefore $\{f^n(u)\}$ is Cauchy, and $f^n(u)\to p$ by (ii).

For each $\varepsilon>0$ there exists an integer N such that $n\geqq N$ implies $d(f^n(u), p)<\varepsilon$.

For any integers $m>0$ and $n>N$, from (iii) it follows

$$d(p, f^m(p)) \leqq d(p, f^{n+1}(u)) + d(f(f^{m-1}(p)), f^2(f^{n-1}(u))) \leqq$$
$$\leqq d(p, f^{n+1}(u)) + \varphi(\delta(O(f^{m-1}(p)) \cup O(f^{n-1}(u)))) \leqq \varepsilon + \varphi(\max\{2\varepsilon, \delta(O(p))+\varepsilon\}).$$

From the Lemma of [11], $\delta(O(p)) = \sup_m d(p, f^m(p))$, so that we have

$$\delta(O(p)) \leqq \varepsilon + \varphi(\max\{2\varepsilon, \delta(O(p))+\varepsilon\}).$$

Since ε is arbitrary, $\delta(O(p)) \leqq \varphi(\delta(O(p)))$, so that $O(p)=0$, which implies $\delta(O(p))=0$. Therefore $p=f(p)$.

Uniqueness follows from (iii).

The next result is an extension of Theorem 2 to 2-metric spaces, and is a generalization of Theorem 1 of [19].

A 2-metric space is a space X in which, for each triple of points a, b, c, there exists a real-valued nonnegative function ϱ satisfying

(1a) for each pair of points $a, b, a \neq b$, of X, there exists a point $c \in X$ such that $\varrho(a, b, c) \neq 0$,

(1b) $\varrho(a, b, c)=0$ when at least two of the points are equal,

(2) $\varrho(a, b, c)=\varrho(a, c, b)=\varrho(b, c, a)$, and

(3) $\varrho(a, b, c) \leqq \varrho(a, b, d)+\varrho(a, d, c)+\varrho(d, b, c)$.

For other properties of 2-metric spaces the reader may consult [5], [6], [8]—[10], and [21]. Fixed point theorems for 2-metric spaces appear in [13], [14], and [19].

For a set $A \subset X$, define $\delta_a(A)=\sup\{\varrho(x, y, a) | x, y \in A\}$. In a 2-metric space a sequence $\{x_n\}$ is called bounded if, for each $a \in X$, $\sup_{m,n} \varrho(x_m, x_n, a) < \infty$, and Cauchy, if, for each $\varepsilon>0$ there exists an integer $N=N(a, \varepsilon)$ such that $\varrho(x_m, x_n, a)<\varepsilon$ for all $m, n>N$. ϱ is always continuous in one coordinate.

Theorem 3. *Let f be a selfmap of a 2-metric space X with the following properties:*

(i) $\delta_a[O(x) \cup O(y)]$ *is finite for each* $x, y, a \in X$.

(ii) *There exists* $u \in X$ *such that* $O(u)$ *has a cluster point* $p \in X$.

(iii) *There exists a map* $\varphi: R_+ \to R_+$ *which is semicontinuous from the right, nondecreasing, and satisfies* $\varphi(t)<t$ *for each* $t>0$.

(iv) f *satisfies* $\varrho(f(x), f^2(y), a) \leqq \varphi[\delta_a(O(x) \cup O(f(y)))]$ *for each* $x, y, a \in X$.

Then p *is the unique fixed point of* f, *and* $\lim_n f^n(u)=p$.

Proof. Let n be an arbitrary integer, i, j integers satisfying $i>j \geqq n$.

$$\varrho(f^i(u), f^j(u), a) = \varrho(f(f^{j-1}(u), f^2(f^{i-2}(u)), a) \leqq$$
$$\leqq \varphi[(\delta_a O(f^{j-1}(u)) \cup O(f^{i-2}(u)))] \leqq \varphi[\delta_a(O(f^{j-1}(u)))] \leqq \delta_a(O(f^{j-1}(u))) < \infty$$

by (iv). Taking the supremum over all $i>j\geqq n$ we obtain

$$\delta_a[O(f^n(u))] \leqq \varphi(\delta_a[O(f^{n-1}(u))]) \leqq \delta_a[O(f^{n-1}(u))]. \tag{4}$$

If we define $\delta_n=\delta_a[O(f^n(u))]$, then $\{\delta_n\}$ is nonincreasing and hence converges to a real number $\delta\geqq 0$. Also, $\delta_{n+1}\leqq\varphi(\delta_n)$. From (iii) it follows that $\delta\leqq\varphi(\delta)$, and hence $\delta=0$.

For each $m>n$,

$$\varrho(f^m(u), f^n(u), a) \leqq \varphi(\delta_a[O(f^{n-1}(u))]) \leqq \delta_a[O(f^{n-1}(u))] = \delta_{n-1} \to 0 \quad \text{as} \quad n \to \infty.$$

Therefore $\{f^n(u)\}$ is Cauchy and, from (ii), converges to p.

It remains to show that p is a fixed point for f. As in the proof of (4) it can be shown that

$$\delta_a[O(f^n(u)\cup f^n(p))] \leqq \varphi(\delta_a[O(f^{n-1}(u)\cup f^{n-1}(p))]),$$

and hence, that

$$\lim_n \delta_a[O(f^n(u)\cup f^n(p))] = 0, \quad \text{for each} \quad a\in X. \tag{5}$$

Using (3),

$$\varrho(p, f^n(p), a) \leqq \varrho(p, f^n(p), f^n(u))+\varrho(p, f^n(u), a)+\varrho(f^n(u), f^n(p), a) \leqq$$
$$\leqq \delta_p[O(f^n(u)\cup f^n(p))]+\varrho(p, f^n(u), a)+\delta_a[O(f^n(u)\cup f^n(p))].$$

Taking the limit as $n\to\infty$, and using (5), we have

$$\lim_n \varrho(p, f^n(p), a) = 0. \tag{6}$$

Now let $\delta_n=\delta_a[O(f^n(p))]$. Again using (3), for any $n>m>0$,

$$\varrho(p, f^m(p), a) \leqq \varrho(p, f^m(p), f^n(p))+\varrho(p, f^n(p), a)+\varrho(f^n(p), f^m(p), a)$$
$$\leqq \varrho(p, f^m(p), f^n(p))+\varrho(p, f^n(p), a)+\delta_1.$$

Taking the limit as $n\to\infty$, and using (5) and (6), one obtains

$$\varrho(p, f^m(p), a) \leqq \delta_1. \tag{7}$$

If, for any $n, \delta_n\neq 0$, then, from (4) and (iii), $\delta_{n+1}\leqq\varphi(\delta_n)<\delta_n$. Also,

$$\delta_n = \max\{\sup_{m>n} \varrho(f^n(p), f^m(p), a), \sup_{m,j>n} \varrho(f^m(p), f^j(p), a)\}.$$

If $\delta_n>0$, then $\sup_{m,j>n} \varrho(f^m(p), f^j(p), a)\leqq\varphi(\delta_n)<\delta_n$, so that

$$\delta_n = \sup_{m>n} \varrho(f^n(p), f^m(p), a). \tag{8}$$

If $\delta_0\neq 0$, then, taking the supremum of (7) for $m>0$, and using (8), yields $\delta_0\leqq\delta_1$. But $\delta_1\leqq\varphi(\delta_0)<\delta_0$, so that $\delta_0<\delta_0$, a contradiction.

Therefore $\delta_0=0$ and p is a fixed point for f.

To establish uniqueness, suppose w is also a fixed point of f. From (iv),

$$\varrho(p, w, a) = \varrho(f(p), f^2(w), a) \leqq \varphi(\delta_a[O(p)\cup O(f(w))]) =$$

$$= \varphi(\delta_a[O(p)\cup O(w)]) = \varphi(\varrho(p, w, a)).$$

From the definition of φ, $\varrho(p, w, a)\neq 0$ yields the contradiction $\varrho(p, w, a) < \varrho(p, w, a)$. Therefore $\varrho(p, w, a)=0$ for all $a\in X$, i.e., $p=w$.

Remark. Wong [22] has noted that, for nondecreasing functions $\varphi\colon R_+ \to R_+$, φ is continuous from the right if and only if φ is upper semicontinuous from the right. It is for this reason that the theorems of this paper have been phrased in terms of φ being continuous from the right.

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