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Uniform lattices

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In this paper we shall give a method of embedding a lattice into a uniform lattice. We shall use the notation and the terminology of [1] and [2]. Let us recall some of this terminology first.

If $L, \wedge, \vee$ is any lattice, and $e \in L$, then we denote the principal ideal generated by e in L by eL. If e and f are any elements of L such that $\alpha: eL \to fL$ is an isomorphism of eL onto fL, then we shall call α a *partial isomorphism* of L. The set of partial isomorphisms of L forms an inverse subsemigroup T_L of the inverse semigroup $\mathscr{I}_L$ of one-to-one partial transformations of L; T_L will be called the *Munn semigroup* of L [3]. We define an equivalence relation $\mathscr{U}_L$ on L by

$$\mathscr{U}_L = \{(e, f) \in L \times L \mid eL \cong fL\}.$$

The lattice L will be called *uniform* if $\mathscr{U}_L = L \times L$. It can be shown that L is uniform if and only if $L, \wedge$ is the semilattice of idempotents of some bisimple inverse semigroup [3].

If L is any lattice, then the automorphism group of L will be denoted by $\mathrm{Aut}\,(L)$, the endomorphism semigroup of L will be denoted by $\mathrm{End}\,(L)$, and the lattice of congruences of L will be denoted by $\theta(L)$.

We now proceed with our construction. Let $L, \wedge, \vee$ be a lattice. Let Z^+ denote the set of positive integers. For any $e \in L$ and any $i \in Z^+$ let $X_e^{(i)}$ be a set and

$$\varkappa_e^{(i)}: eL \to X_e^{(i)}$$

a one-to-one mapping of eL onto $X_e^{(i)}$. We shall thereby suppose that $X_f^{(j)} \cap X_e^{(i)} = \square$ if $i \neq j$ or $e \neq f$, and that $\big(\bigcup_{e \in L} \big(\bigcup_{i \in Z^+} X_e^{(i)}\big)\big) \cap L = \square$. Let us put $X_e = \bigcup_{i \in Z^+} X_e^{(i)}$ for all $e \in L$, and let $X = \bigcup_{e \in L} X_e$. If Y is a subset of X, then we shall put

$$Y_e = Y \cap X_e, \quad Y_e^{(i)} = Y \cap X_e^{(i)}$$

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for all $e \in L$ and all $i \in Z^+$. Let $\mathscr{A}$ be a set, the elements of which are the subsets Y of X which satisfy the following conditions:

(i) there is only a finite number of pairs $(e, i) \in L \times Z^+$ for which $Y_e^{(i)} \neq X_e^{(i)}$,

(ii) for every $(e, i) \in L \times Z^+$, either $Y_e^{(i)} = \square$ or $Y_e^{(i)} \varkappa_e^{(i)-1}$ is of the form gL for some $g \in eL$.

Remark that $X \in \mathscr{A}$. Clearly $\mathscr{A}$ is a subset of the power set $P(X)$.

Let $\mathscr{B}$ be the subset of $P(X \cup L)$ which is defined by

$$\mathscr{B} = \{eL \cup Y \mid e \in L,\ Y \in \mathscr{A}\} \cup \mathscr{A}.$$

$\mathscr{B}, \subseteqq$ is a partially ordered set. It is easy to check that $\mathscr{B}, \subseteqq$ is in fact a lattice. Let us for instance compute the l.u.b. and the g.l.b. of $eL \cup V$ and $fL \cup W$, $e, f \in L$, $V, W \in \mathscr{A}$, in $\mathscr{B}$. It is obvious that

$$\text{g.l.b.}\,(eL \cup V,\ fL \cup W) = (eL \cup V) \cap (fL \cup W) = (e \wedge f)L \cup (V \cap W)$$

since $\mathscr{B}$ is closed for taking intersections. Let us now define an element U of $\mathscr{A}$ in the following way: for every $(e, i) \in L \times Z^+$ we take

$$\begin{aligned} U_e^{(i)} &= \square && \text{if } V_e^{(i)} = W_e^{(i)} = \square, \\ U_e^{(i)} &= W_e^{(i)} && \text{if } V_e^{(i)} = \square, \\ U_e^{(i)} &= V_e^{(i)} && \text{if } W_e^{(i)} = \square, \end{aligned}$$

and in case $V_e^{(i)} = (vL)\varkappa_e^{(i)}$, $W_e^{(i)} = (wL)\varkappa_e^{(i)}$, take

$$U_e^{(i)} = ((v \vee w)L)\varkappa_e^{(i)}.$$

Then

$$\text{l.u.b.}\,(eL \cup V,\ fL \cup W) = (e \vee f)L \cup U.$$

From this it follows that the mapping

$$\varphi\colon L \to \mathscr{B}, \quad e \to eL \cup X$$

embeds L isomorphically as a dual ideal in $\mathscr{B}$. It is therefore possible to conceive a lattice L_1 which contains L as a dual ideal, and an isomorphism $\varphi_1\colon L_1 \to \mathscr{B}$ of L_1 onto $\mathscr{B}$ which extends the isomorphism φ of L into $\mathscr{B}$. We shall investigate the embedding of L into L_1 in several lemmas.

Lemma 1. $L \times L \subseteqq \mathscr{U}_{L_1}$.

Proof. Let us consider any element e of L. Any element in the principal ideal of $eL \cup X$ in $\mathscr{B}$ is of the form $gL \cup Y$ or of the form Y, where $g \in eL$ and $Y \in \mathscr{A}$. Let φ_e be the mapping of the principal ideal of $eL \cup X$ in $\mathscr{B}$ onto the principal ideal of X in $\mathscr{B}$ which is defined by

$$(gL \cup Y)\varphi_e = (gL)\varkappa_e^{(1)} \cup \Big(\bigcup_{i \in Z^+} Y_e^{(i)} \varkappa_e^{(i)-1} \varkappa_e^{(i+1)}\Big) \cup (Y \setminus Y_e)$$

and

$$Y\varphi_e = \Big(\bigcup_{i\in Z^+} Y_e^{(i)}\varkappa_e^{(i)-1}\varkappa_e^{(i+1)}\Big)\cup(Y\setminus Y_e).$$

It is easy to verify that φ_e is a partial isomorphism of $\mathscr{B}$. Thus $(X, eL\cup X)\in\mathscr{U}_{\mathscr{B}}$ for all $e\in L$. From this it follows that $(eL\cup X, fL\cup X)\in\mathscr{U}_{\mathscr{B}}$ for all $e, f\in L$. Hence, $(e,f)\in\mathscr{U}_{L_1}$, for all $e,f\in L$, and so $L\times L\subseteqq\mathscr{U}_{L_1}$.

Lemma 2. *Every partial isomorphism α of L can be extended to a partial isomorphism $\alpha^{(1)}$ of L_1 in such a way that the mapping*

$$\psi_1\colon T_L\to T_{L_1},\quad \alpha\to\alpha^{(1)}$$

is an isomorphism of T_L into T_{L_1}.

Proof. Let $\alpha\colon eL\to fL$ be any partial isomorphism of L, and let us define the partial isomorphism $\bar\alpha$ of the principal ideal of $eL\cup X$ in $\mathscr{B}$ onto the principal ideal of $fL\cup X$ in $\mathscr{B}$ by

$$(gL\cup Y)\bar\alpha = (g\alpha)L\cup Y,\quad Y\bar\alpha = Y,\quad g\in eL,\quad Y\in\mathscr{A}.$$

Let $\alpha^{(1)}=\varphi_1\bar\alpha\varphi_1^{-1}$. Clearly $\alpha^{(1)}$ is a partial isomorphism of L_1 which maps eL_1 isomorphically onto fL_1, and the restriction of $\alpha^{(1)}$ to L is precisely α. Let us now consider the mapping $\psi_1\colon T_L\to T_{L_1}$, $\alpha\to\alpha^{(1)}$. We have

$$(\alpha\beta)\psi_1 = \varphi_1\overline{\alpha\beta}\varphi_1^{-1} = \varphi_1\bar\alpha\bar\beta\varphi_1^{-1} = (\varphi_1\bar\alpha\varphi_1^{-1})(\varphi_1\bar\beta\varphi_1^{-1}) = (\alpha\psi_1)(\beta\psi_1).$$

Since ψ_1 is clearly injective it follows that ψ_1 is an isomorphism of T_L into T_{L_1}.

Lemma 3. *Every endomorphism γ of L can be extended to an endomorphism $\gamma^{(1)}$ of L_1, in such a way that the mapping*

$$\xi_1\colon \mathrm{End}(L)\to\mathrm{End}(L_1),\quad \gamma\to\gamma^{(1)}$$

is an isomorphism of End (L) *into* End (L_1).

Proof. Let γ be any element of End (L), and let us define the endomorphism $\bar\gamma$ of $\mathscr{B}$ by

$$(eL\cup Y)\bar\gamma = (e\gamma)L\cup Y,\ Y\bar\gamma = Y,\quad e\in L,\ Y\in\mathscr{A}.$$

Let $\gamma^{(1)}=\varphi_1\bar\gamma\varphi_1^{-1}$. Then $\gamma^{(1)}\in\mathrm{End}(L_1)$, and the restriction of $\gamma^{(1)}$ to L is precisely γ. The mapping $\xi_1\colon\mathrm{End}(L)\to\mathrm{End}(L_1)$, $\gamma\to\gamma^{(1)}$ is clearly injective, and for every $\gamma,\delta\in\mathrm{End}(L)$ we have

$$(\gamma\delta)\xi_1 = \varphi_1\overline{\gamma\delta}\varphi_1^{-1} = \varphi_1\bar\gamma\bar\delta\varphi_1^{-2} = (\varphi_1\bar\gamma\varphi_1^{-1})(\varphi_1\bar\delta\varphi_1^{-1}) = (\gamma\xi_1)(\delta\xi_1)$$

Thus ξ_1 is an isomorphism of End (L) into End (L_1).

Lemma 4. *Every automorphism* γ *of* L *can be extended to an automorphism* $\gamma^{(1)}$ *of* L_1, *and the mapping*

$$\xi_1|\mathrm{Aut}(L)\colon \mathrm{Aut}(L) \to \mathrm{Aut}(L_1), \quad \gamma \to \gamma^{(1)}$$

is an isomorphism of $\mathrm{Aut}(L)$ *into* $\mathrm{Aut}(L_1)$.

Proof. Immediate from the definition of ξ_1 in the proof of Lemma 3.

From Lemma 4 it follows that the mapping ξ_1 embeds End (L) isomorphically as a submonoid of End (L_1).

Lemma 5. *Every congruence* ϱ *on* L *is the restriction to* L *of some congruence* $\varrho^{(1)}$ *on* L_1, *where the mapping*

$$\zeta_1\colon \theta(L) \to \theta(L_1), \quad \varrho \to \varrho^{(1)}$$

is a lattice isomorphism of $\theta(L)$ *onto a closed sublattice of* $\theta(L_1)$.

Proof. If ϱ is any congruence on L, then we define the relation $\bar{\varrho}$ on $\mathscr{B}$ by

$$\bar{\varrho} = \{(eL\cup Y, fL\cup Y)|e, f\in L,\ e\varrho f,\ Y\in\mathscr{A}\}\cup\{(Y, Y)|Y\in\mathscr{A}\}.$$

Let $\varrho^{(1)}=\varphi_1\bar{\varrho}\varphi_1^{-1}$. It can be checked easily that $\bar{\varrho}$ and $\varrho^{(1)}$ are congruences on $\mathscr{B}$ and on L_1 respectively, and that ϱ is the restriction of $\varrho^{(1)}$ to L. Let us now consider the injective mapping $\zeta_1\colon \theta(L)\to\theta(L_1)$, $\varrho\to\varrho^{(1)}$. Let $\{\varrho_i|i\in I\}$ be any subset of $\theta(L)$. Clearly

$$\Big(\bigcap_{i\in I}\varrho_i\Big)\zeta_1 = \varphi_1\overline{\Big(\bigcap_{i\in I}\varrho_i\Big)}\varphi_1^{-1} = \varphi_1\Big(\bigcap_{i\in I}\bar{\varrho}_i\Big)\varphi_i^{-1} = \bigcap_{i\in I}\varphi_1\bar{\varrho}_i\varphi_1^{-1} = \bigcap_{i\in I}(\varrho_i\zeta_1).$$

Let A and B any elements of $\mathscr{B}$ such that

$$A\Big(\bigvee_{i\in I}\bar{\varrho}_i\Big)B.$$

Then there exist elements $A=A_0, \dots, A_j, A_{j+1}, \dots, A_k=B$ such that for every $j\in\{0, \dots, k-1\}$, $A_j\bar{\varrho}_jA_{j+1}$ for some $\varrho_j\in\{\varrho_i|i\in I\}$. If A is of the form $A=Y$, $Y\in\mathscr{A}$, then $A=A_0=A_1=\dots=A_k=B$. If A is of the form $eL\cup Y$, $e\in L$, $Y\in\mathscr{A}$, then A_j is of the form $e_jL\cup Y$ for all $j\in\{0, \dots, k\}$, and for all $j\in\{0, \dots, k-1\}$, $e_j\varrho_je_{j+1}$; thus B is then of the form $fL\cup Y$, where $e\,\big(\bigvee_{i\in I}\varrho_i\big)\,f$ in L. We conclude that $\bigvee_{i\in I}\bar{\varrho}_i\subseteq \overline{\bigvee_{i\in I}\varrho_i}$. Similarly we can show that $\overline{\bigvee_{i\in I}\varrho_i}\subseteq\bigvee_{i\in I}\bar{\varrho}_i$. We conclude that $\overline{\bigvee_{i\in I}\varrho_i}=\bigvee_{i\in I}\bar{\varrho}_i$, and so

$$\Big(\bigvee_{i\in I}\varrho_i\Big)\zeta_1 = \varphi_1\overline{\Big(\bigvee_{i\in I}\varrho_i\Big)}\varphi_1^{-1} = \varphi_1\Big(\bigvee_{i\in I}\bar{\varrho}_i\Big)\varphi_1^{-1} = \bigvee_{i\in I}(\varphi_1\bar{\varrho}_i\varphi_1^{-1}) = \bigvee_{i\in I}(\varrho_i)\zeta_1.$$

Therefore ζ_1 is a lattice isomorphism of $\theta(L)$ onto a closed sublattice of $\theta(L_1)$.

We are now in the position to prove our main theorem.

Theorem. *Every lattice* L *can be isomorphically embedded as a dual ideal into a uniform lattice* L' *in such a way that*

(i) *every partial isomorphism* α *of* L *can be extended to a partial isomorphism* α' *of* L' *such that the mapping*

$$\psi: T_L \to T_{L'}, \quad \alpha \to \alpha'$$

is an isomorphism of T_L *into* $T_{L'}$,

(ii) *every endomorphism [automorphism]* γ *of* L *can be extended to an endomorphism [automorphism]* γ' *of* L' *such that the mapping*

$$\xi: \mathrm{End}(L) \to \mathrm{End}(L'), \quad \gamma \to \gamma'$$

is an isomorphism of $\mathrm{End}(L)$ *into* $\mathrm{End}(L')$ *which induces an isomorphism of* $\mathrm{Aut}(L)$ *into* $\mathrm{Aut}(L')$,

(iii) *every congruence* ϱ *on* L *is the restriction to* L *of some congruence* ϱ' *on* L' *where the mapping*

$$\zeta: \theta(L) \to \theta(L'), \quad \varrho \to \varrho'$$

is a lattice isomorphism of $\theta(L)$ *onto a closed sublattice of* $\theta(L')$.

Proof. Let us consider the sequence of lattices

$$L = L_0, L_1, \ldots, L_j, L_{j+1}, \ldots$$

where for every $j \in N$, L_{j+1} is a lattice which contains L_j as a dual ideal and where L_{j+1} is constructed from L_j in the same way as L_1 is constructed from $L=L_0$. Then $L' = \bigcup_{j=0}^{\infty} L_j$ is a lattice which contains each $L_j, j \in Z^+$ as a dual ideal; in particular L is a dual ideal of L'.

Let $\alpha^{(j)}$ be any partial isomorphism of L_j for some $j \in N$. Let us consider the sequence of partial isomorphisms

$$\alpha^{(j)}, \alpha^{(j+1)}, \ldots, \alpha^{(j+k)}, \alpha^{(j+k+1)}, \ldots$$

where for every $k \in N$, $\alpha^{(j+k+1)}$ is a partial isomorphism of L_{j+k+1} which extends the partial isomorphism $\alpha^{(j+k)}$ of L_{j+k} in the way prescribed by the proof of Lemma 2. Therefore $\bigcup_{k \in N} \alpha^{(j+k)} = \alpha'$ is a partial isomorphism of L' which extends $\alpha^{(j)}$.

Let us now consider any two elements $x, y \in L'$. There exists a $j \in Z^+$ such that $x, y \in L_{j-1}$. By Lemma 1 we know that there exists a partial isomorphism $\alpha^{(j)}$ of L_j which maps xL_j isomorphically onto yL_j. Let $\alpha' = \bigcup_{k \in N} \alpha^{(j+k)}$ be the partial isomorphism of L' which is obtained from $\alpha^{(j)}$ in the way described above. The partial isomorphism α' maps xL' isomorphically onto yL', and therefore $(x, y) \in \mathscr{U}_{L'}$. We conclude that L' is uniform.

If $\alpha=\alpha^{(0)}$ is any partial isomorphism of L, then $\alpha'=\bigcup_{j\in N}\alpha^{(j)}$ is a partial isomorphism of L' which extends α. Let us investigate the mapping $\psi\colon T_L\to T_{L'}$, $\alpha\to\alpha'$. If $\beta=\beta^{(0)}$ is any other partial isomorphism of L, then $\beta'=\bigcup_{j\in N}\beta^{(j)}=\beta\psi\in T_{L'}$, and it follows from Lemma 2 that for all $j\in N$, $\alpha^{(j)}\beta^{(j)}=(\alpha\beta)^{(j)}$. From this it follows that $\alpha'\beta'=(\alpha\beta)'$, and so ψ is an isomorphism of T_L into $T_{L'}$. We conclude that (i) is satisfied. Using Lemma 3 and Lemma 4 we can introduce an injective mapping $\xi\colon \mathrm{End}\,(L)\to\mathrm{End}\,(L')$, $\gamma\to\gamma'$ which satisfies (ii): the proof thereof proceeds along the same lines as for the foregoing case.

Let $\varrho=\varrho^{(0)}$ be any congruence on L, and let us consider the sequence of congruences

$$\varrho=\varrho^{(0)},\ \varrho^{(1)},\ \dots,\ \varrho^{(j)},\ \varrho^{(j+1)},\ \dots$$

where for every $j\in N$, $\varrho^{(j+1)}$ is a congruence on L_{j+1} which is constructed from $\varrho^{(j)}$ in the way prescribed by the proof of Lemma 5. It should be clear that for all $i,j\in N$, $i\leqq j$, we have $\varrho^{(j)}\cap L_i\times L_i=\varrho^{(i)}$. Furthermore $\varrho'=\bigcup_{j\in N}\varrho^{(j)}$ is a congruence on L', and the restriction of ϱ' to L is precisely ϱ. Let us investigate the injective mapping $\zeta\colon\theta(L)\to\theta(L')$, $\varrho\to\varrho'$. Let $\{\varrho_i|i\in I\}$ be any subset of $\theta(L)$. Clearly by Lemma 5 we have

$$(\bigcap_{i\in I}\varrho_i)\zeta=(\bigcap_{i\in I}\varrho_i)'=\bigcup_{j\in N}(\bigcap_{i\in I}\varrho_i)^{(j)}=\bigcup_{j\in N}(\bigcap_{i\in I}\varrho_i^{(j)})=$$
$$=\bigcap_{i\in I}(\bigcup_{j\in N}\varrho_i^{(j)})=\bigcap_{i\in I}\varrho_i'=\bigcap_{i\in I}(\varrho_i)\zeta.$$

Let us consider $(\bigvee_{i\in I}\varrho_i)\zeta=(\bigvee_{i\in I}\varrho_i)'$, and let us suppose that x and y are any elements of L' such that $x\,(\bigvee_{i\in I}\varrho_i)'\,y$. There exists a $j\in N$ such that $x,y\in L_j$. Since the restriction of $(\bigvee_{i\in I}\varrho_i)'$ to L_j is precisely $(\bigvee_{i\in I}\varrho_i)^{(j)}$, and since by Lemma 5 $(\bigvee_{i\in I}\varrho_i)^{(j)}=\bigvee_{i\in I}\varrho_i^{(j)}$, we must have $x\,(\bigvee_{i\in I}\varrho_i^{(j)})\,y$. From $(\bigvee_{i\in I}\varrho_i^{(j)})\subseteq(\bigvee_{i\in I}\varrho_i')$ it then follows that $x(\bigvee_{i\in I}\varrho_i')\,y$. We conclude that $(\bigvee_{i\in I}\varrho_i)'\subseteq(\bigvee_{i\in I}\varrho_i')$. Let us conversely suppose that x and y are elements of L' such that $x\,(\bigvee_{i\in I}\varrho_i')\,y$. Then there exist elements $x=x_0, x_1, \dots$ $\dots, x_k=y$ in L' such that for every $j\in\{0,\dots,k-1\}$, $x_j\varrho_j'x_{j+1}$, $\varrho_j\in\{\varrho_i|i\in I\}$. There exists some $n\in N$ such that $\{x_0,\dots,x_k\}\subseteq L_n$, and then $x_j\varrho_j^{(n)}x_{j+1}$ for every $j\in\{0,\dots,k-1\}$. Therefore $x\,(\bigvee_{i\in I}\varrho_i^{(n)})\,y$, and by Lemma 5 we have $(\bigvee_{i\in I}\varrho_i^{(n)})=(\bigvee_{i\in I}\varrho_i)^{(n)}$. Clearly $(\bigvee_{i\in I}\varrho_i)^{(n)}\subseteq(\bigvee_{i\in I}\varrho_i)'$, and so $x\,(\bigvee_{i\in I}\varrho_i)'\,y$. We conclude that $(\bigvee_{i\in I}\varrho_i)\zeta=(\bigvee_{i\in I}\varrho_i)'=\bigvee_{i\in I}\varrho_i'=\bigvee_{i\in I}(\varrho_i)\zeta$. Thus the mapping ζ satisfies (iii). This concludes the proof of the theorem.

Remark. The concepts "partial isomorphism", "Munn semigroup", "uniform" were originally introduced for semilattices. The results of this paper remain valid if we deal with semilattices only; if we do so several simplifications in our construction may be conceived. Anyhow, our main theorem still holds if L and L' are semilattices; L is then embedded as a dual ideal in the uniform semilattice L' in such a way that (i), (ii) and (iii) are satisfied. That every semilattice can be embedded as a subsemilattice in a uniform semilattice also follows from Reilly's results in [4].

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