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On a conjecture of Sz.-Nagy and Foiaş

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Sz.-Nagy and Foiaş [10] conjectured that if T is a C_0 contraction with finite multiplicity and $X\in\{T\}'$ then $T|\ker X$ and $(T^*|\ker X^*)^*$ are quasi-similar. Recently, Bercovici [1] and Uchiyama [11] have independently given counterexamples to this conjecture. However, in the present paper we want to establish a weaker form of this conjecture. More specifically, we will show that if T is a $C_0(N)$ contraction and $X\in\{T\}'$ then there exists a $Y\in\{T\}'$ such that $T|\ker X$ and $(T^*|\ker Y^*)^*$ are quasi-similar. Indeed, this follows from the following two main results of Section 1: If T is a $C_0(N)$ contraction, then (1) every invariant subspace for T is of the form $\ker X$ for some $X\in\{T\}'$ (Theorem 1.2) and (2) for every invariant subspace K for T there exists an invariant subspace L for T^* such that $T|K$ is quasi-similar to $(T^*|L)^*$ (Theorem 1.3). In Section 2, we consider the corresponding question for C_{11} contractions. It will be shown that the Sz.-Nagy and Foiaş conjecture holds for completely non-unitary (c.n.u.) C_{11} contractions with finite defect indices. Moreover, result (1) also holds for such operators with bi-invariant subspaces replacing invariant subspaces. ((2) with the same modifications follows from the analogue of the Sz.-Nagy and Foiaş conjecture trivially.) The corresponding results for weak contractions will be considered in Section 3.

In this paper only bounded linear operators on complex, separable Hilbert spaces will be considered. For an operator T, let $\{T\}'$ and $\{T\}''$ denote the commutant and double commutant of T, and let $\operatorname{Lat} T$, $\operatorname{Lat}'' T$ and $\operatorname{Hyperlat} T$ denote the lattices of invariant subspaces, bi-invariant subspaces and hyperinvariant subspaces of T, respectively. If T_1 and T_2 are operators on H_1 and H_2, respectively, $T_1\overset{i}{\prec}T_2$ (resp. $T_1\prec T_2$) denotes that there exists an injection $X\colon H_1\to H_2$ (resp. an injection $X\colon H_1\to H_2$ with dense range, called *quasi-affinity*) which intertwines T_1 and T_2, i. e. $T_1X=XT_2$. T_1 is *quasi-similar* to T_2 ($T_1\sim T_2$) if $T_1\prec T_2$

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and $T_2 \prec T_1$. Readers are referred to Sz.-Nagy and Foiaş [6] for basic definitions and properties of contractions of various classes.

1. $C_0(N)$ contractions. If T is a $C_0(N)$ contraction, then T is quasi-similar to a uniquely determined Jordan operator of the form $S(\varphi_1)\oplus\ldots\oplus S(\varphi_n)$, called the *Jordan model* of T, where $\varphi_1, \ldots, \varphi_n$ are inner functions satisfying $\varphi_j|\varphi_{j-1}$ and $S(\varphi_j)$ denotes the compression of the shift on $H^2\ominus\varphi_j H^2$ for $j=1, 2, \ldots, n$ (cf. [7]). We start with the following lemma.

Lemma 1.1. *Let T be a $C_0(N)$ contraction and let $K\in\operatorname{Lat} T$. Assume that $J=S(\varphi_1)\oplus\ldots\oplus S(\varphi_n)$ and $J_1=S(\psi_1)\oplus\ldots\oplus S(\psi_m)$ are the Jordan models of T and $T|K$, respectively. Then $m\leqq n$ and $\psi_j|\varphi_j$ for $j=1, 2, \ldots, m$.*

Proof. Since $J_1\sim T|K\overset{i}{\prec} T\sim J$, the conclusion follows from [9], Theorem 4.

Theorem 1.2. *Let T be a $C_0(N)$ contraction on H and let K be a subspace of H. Then the following statements are equivalent:*

(1) *$K\in\operatorname{Lat} T$;*
(2) *$K=(\operatorname{Range} X)^-$ for some $X\in\{T\}'$;*
(3) *$K=\ker Y$ for some $Y\in\{T\}'$.*

Proof. It suffices to show (1)⇒(2). Let $J=S(\varphi_1)\oplus\ldots\oplus S(\varphi_n)$ and $J_1=S(\psi_1)\oplus\ldots\oplus S(\psi_m)$ be the Jordan models of T and $T|K$, respectively. Let $V\colon H\to H_1\equiv(H^2\ominus\varphi_1 H^2)\oplus\ldots\oplus(H^2\ominus\varphi_n H^2)$ and $W\colon K_1\equiv(H^2\ominus\psi_1 H^2)\oplus\ldots\oplus(H^2\ominus\psi_m H^2)\to K$ be quasi-affinities intertwining T, J and J_1, $T|K$, respectively. Lemma 1.1 implies that $m\leqq n$ and $\psi_j|\varphi_j$ for $j=1, 2, \ldots, m$; say, $\varphi_j=\psi_j\eta_j$ for each j. Note that $S(\varphi_j)|(\operatorname{Range}\eta_j(S(\varphi_j)))^-\cong S(\psi_j)$ (cf. [9], pp. 315—316). For each j, let Z_j be the operator which implements this unitary equivalence and let $Z\colon H_1\to K_1$ be the operator $Z_1\eta_1(S(\varphi_1))\oplus\ldots\oplus Z_m\eta_m(S(\varphi_m))\oplus\underbrace{0\oplus\ldots\oplus 0}_{n-m}$. Then Z intertwines J, J_1 and has dense range in K_1. Finally, let $X=WZV$. It is obvious that $X\in\{T\}'$ and $K=(\operatorname{Range} X)^-$. This completes the proof.

It is interesting to contrast the preceding theorem with the main result in [14].

Theorem 1.3. *If T is a $C_0(N)$ contraction on H and $K\in\operatorname{Lat} T$, then there exists an $L\in\operatorname{Lat} T^*$ such that $T|K$ is quasi-similar to $(T^*|L)^*$.*

Proof. Let $J=S(\varphi_1)\oplus\ldots\oplus S(\varphi_n)$ and $J_1=S(\psi_1)\oplus\ldots\oplus S(\psi_m)$ be the Jordan models of T and $T|K$, respectively. Then $\tilde{J}\equiv S(\tilde{\varphi}_1)\oplus\ldots\oplus S(\tilde{\varphi}_n)$ is the Jordan model of T^*, where $\tilde{\varphi}_j(z)=\overline{\varphi_j(\bar{z})}$ for each j. Let $V\colon H_1\equiv(H^2\ominus\tilde{\varphi}_1 H^2)\oplus\ldots\oplus(H^2\ominus\tilde{\varphi}_n H^2)\to H$ be the quasi-affinity intertwining $\tilde{J}$ and T^*. From Lemma 1.1 we have $m\leqq n$ and $\psi_j|\varphi_j$ for $j=1, 2, \ldots, m$; say, $\varphi_j=\psi_j\eta_j$ for each j. Then $\tilde{\varphi}_j=\tilde{\psi}_j\tilde{\eta}_j$, which implies that $S(\tilde{\psi}_j)\cong S(\tilde{\varphi}_j)|(\operatorname{Range}\tilde{\eta}_j(S(\tilde{\varphi}_j)))^-$ for $j=1, \ldots, m$.

For each j, let Z_j be the operator which implements this unitary equivalence and let $Z=Z_1\oplus\ldots\oplus Z_m$. Let $\tilde{J}_1=S(\tilde{\psi}_1)\oplus\ldots\oplus S(\tilde{\psi}_m)$ on $\tilde{K}_1\equiv(H^2\ominus\tilde{\psi}_1H^2)\oplus\ldots\oplus(H^2\ominus\tilde{\psi}_mH^2)$ and let $L=(VZ\tilde{K}_1)^-$. Then $L\in\operatorname{Lat}T^*$ and $\tilde{J}_1\prec T^*|L$. It follows that $\tilde{J}_1\sim T^*|L$ and hence $(T^*|L)^*\sim\tilde{J}_1^*\cong S(\psi_1)\oplus\ldots\oplus S(\psi_m)\sim T|K$, completing the proof.

Corollary 1.4. *If T is a $C_0(N)$ contraction and $X\in\{T\}'$, then there exists another $Y\in\{T\}'$ such that $T|\ker X$ is quasi-similar to $(T^*|\ker Y^*)^*$.*

Note that Theorems 1.2 and 1.3 generalize the corresponding results for operators on finite-dimensional spaces proved by HALMOS (cf. [4], Theorems 2 and 3).*)

2. C_{11} contractions. Let T be a c.n.u. C_{11} contraction with finite defect indices defined on $H\equiv\langle H_n^2\oplus\overline{\Delta L_n^2}\rangle\ominus\{\Theta_Tw\oplus\Delta w\colon w\in H_n^2\}$ by $T(f\oplus g)=P(e^{it}f\oplus e^{it}g)$ for $f\oplus g\in H$, where $\Delta=(1-\Theta_T^*\Theta_T)^{1/2}$ and P denotes the (orthogonal) projection onto H. Let U be the operator of multiplication by e^{it} on $(\Delta_*L_n^2)^-$, where $\Delta_*=(1-\Theta_T\Theta_T^*)^{1/2}$, and let $X\colon H\to(\Delta_*L_n^2)^-$ be the quasi-affinity $X(f\oplus g)=-\Delta_*f+\Theta_Tg$ which intertwines T and U (cf. [18], Lemma 3.4). Let δ be an outer scaler multiple of Θ_T and let Ω be a contractive analytic function such that $\Omega\Theta_T=\Theta_T\Omega=\delta 1$. Note that we have $\Omega\Delta_*=\Delta\Omega$.

Define the operator $Y\colon(\Delta_*L_n^2)^-\to H$ by $Yu=P(0\oplus\Omega u)$ for $u\in(\Delta_*L_n^2)^-$. Note that Ωu is in $(\Delta L_n^2)^-$ for any $u\in(\Delta_*L_n^2)^-$. Indeed, if $u\in(\Delta_*L_n^2)^-$, there exists a sequence of vectors $\{u_m\}$ in L_n^2 such that $\Delta_*u_m\to u$ in norm as $m\to\infty$. Since $\Omega\Delta_*u_m=\Delta\Omega u_m\in\Delta L_n^2$ for all m and $\Omega\Delta_*u_m\to\Omega u$, we conclude that $\Omega u\in(\Delta L_n^2)^-$ as asserted.

Lemma 2.1. *Let T, U, X and Y be as above. Then* (1) $YX=\delta(T)$ *and* $XY=\delta(U)$; (2) *Y is a quasi-affinity intertwining U and T.*

Proof. (1) For any $P(0\oplus g)\in H$, we have

$$YX(P(0\oplus g)) = YX((0\oplus g)-(\Theta_Tw\oplus\Delta w)) = Y(-\Delta_*(-\Theta_Tw)+\Theta_T(g-\Delta w)) =$$

$$= Y(\Theta_Tg) = P(0\oplus\Omega\Theta_Tg) = P(0\oplus\delta g) = \delta(T)P(0\oplus g),$$

where $w\in H_n^2$ and in the third equality we used the relation $\Delta_*\Theta_T=\Theta_T\Delta$. Since $\{P(0\oplus g)\colon g\in(\Delta L_n^2)^-\}$ is dense in H (cf. [19], proof of Lemma 2), we conclude that

*) *Editor's Note:* The results in this section are actually true for arbitrary C_0 contractions on (not necessarily separable) Hilbert spacer. We have only to refer to the existence of the Jordan model and the validity of Lemma 1.1 in the general case. These have been proved by H. BERCOVICI (On the Jordan model of C_0 operators. II, *Acta Sci. Math.*, **42** (1980), 43—56). He also proved Theorem 1.2 for arbitrary C_0 contractions.

$YX=\delta(T)$. In a similar fashion, for any $u\in(\Delta_* L^2_n)^-$,

$$XYu = XP(0\oplus\Omega u) = \Theta_T \Omega u = \delta u = \delta(U)u.$$

Hence $XY=\delta(U)$.

(2) For any $u\in(\Delta_* L^2_n)^-$,

$$YUu = Y(e^{it}u) = P(0\oplus\Omega e^{it}u) = TP(0\oplus\Omega u) = TYu.$$

This shows that Y intertwines U and T. That Y is a quasi-affinity follows from (1) and the fact that both $\delta(T)$ and $\delta(U)$ are quasi-affinities (since $\delta\not\equiv 0$ is an outer function; cf. [6], pp. 118 and 121, resp.).

To prove the analogue of the Sz.-Nagy and Foiaş conjecture for C_{11} contractions, we need the next theorem. It implies that for c.n.u. C_{11} contractions with finite defect indices, the lattices of invariant subspaces, bi-invariant subspaces and hyperinvariant subspaces are all preserved under quasi-similarities. Note that the assertions concerning bi-invariant and hyperinvariant subspaces have been obtained before by more elaborate methods (cf. [18] and [15]). By introducing the operator Y we are able to prove all three assertions in one stroke.

Theorem 2.2. *Let T, U, X and Y be as above. Then* Lat $T\cong$ Lat U, Lat$''T\cong$ Lat$''U$ *and* Hyperlat $T\cong$ Hyperlat U. *Moreover, in each case the (lattice) isomorphisms are implemented by the mappings $K\to\overline{XK}$ and $L\to\overline{YL}$ where K is in* Lat T, Lat$''T$ *or* Hyperlat T *and L in* Lat U, Lat$''U$ *or* Hyperlat U, *and $T|K$ is quasi-similar to $U|\overline{XK}$.*

We first prove the following lemma.

Lemma 2.3. *Let U be an absolutely continuous unitary operator and let $L\in$ Lat U. If δ is an outer function, then $\delta(U|L)$ is a quasi-affinity on L.*

Proof. Since $U'=U|L$ is a contraction, we may consider the canonical decomposition $U'=U_1\oplus U_2$ of U', where U_1 and U_2 are the unitary and c.n.u. parts of U', respectively. Then $\delta(U')=\delta(U_1)\oplus\delta(U_2)$. That $\delta\not\equiv 0$ is an outer function implies that both $\delta(U_1)$ and $\delta(U_2)$ are quasi-affinities (cf. [6], pp. 121 and 118, resp.). It follows that $\delta(U')$ is a quasi-affinity.

Proof of Theorem 2.2. Since X and Y intertwine T and U, it is easily seen that $\overline{XK}\in$ Lat U and $\overline{YL}\in$ Lat T for any $K\in$ Lat T and $L\in$ Lat U. Moreover, $(Y\overline{XK})^-=(YXK)^-=(\delta(T)K)^-=(\delta(T|K)K)^-=K$ and $(X\overline{YL})^-=(XYL)^-=(\delta(U)L)^-=(\delta(U|L)L)^-=L$ by Lemmas 2.1 and 2.3. We infer that the mappings $K\to\overline{XK}$ and $L\to\overline{YL}$ implement the (lattice) isomorphisms between Lat T and Lat U and $T|K$ is quasi-similar to $U|\overline{XK}$.

To complete the proof it suffices to show that for any $K\in \mathrm{Lat}'' T$ or $\mathrm{Hyperlat}\, T$, $\overline{XK}\in \mathrm{Lat}'' U$ or $\mathrm{Hyperlat}\, U$ and for any $L\in \mathrm{Lat}'' U$ or $\mathrm{Hyperlat}\, U$, $\overline{YL}\in \mathrm{Lat}'' T$ or $\mathrm{Hyperlat}\, T$, respectively. If $K\in \mathrm{Lat}'' T$, then $\sigma(T|K)\subseteq\sigma(T)$ (cf. [20], Theorem 3), and so $T|K\in C_{11}$. Therefore $T|K$ is quasi-similar to a unitary operator. Since $T|K\prec U|\overline{XK}$, we infer by Lemma 4.1 of [3] that $\overline{XK}$ is a reducing subspace of U, and so $\overline{XK}\in \mathrm{Lat}'' U$. On the other hand, if $L\in \mathrm{Lat}'' U$, then $\overline{YL}=\{P(0\oplus \Omega u)\colon u\in L\}^-$. An operator S in $\{T\}''$ must be of the form $P\begin{bmatrix} A & O\\ B & C\end{bmatrix}$, where A is a bounded analytic function, B and C are bounded measurable functions and C is scalar-valued satisfying $A\Theta_T=\Theta_T A_0$ and $B\Theta_T+C\Delta=\Delta A_0$ for some bounded analytic function A_0 (cf. [19], Lemma 2). Hence $S\big(P(0\oplus\Omega u)\big)=P\begin{bmatrix} A & O\\ B & C\end{bmatrix}P\begin{bmatrix} O\\ \Omega u\end{bmatrix}= =P\begin{bmatrix} O\\ C\Omega u\end{bmatrix}=P\begin{bmatrix} O\\ \Omega C u\end{bmatrix}\in\overline{YL}$ for any $u\in L$, since $L\in \mathrm{Lat}'' U$ and $C\in\{U\}''$. It follows that $S\overline{YL}\subseteq\overline{YL}$, whence $\overline{YL}\in \mathrm{Lat}'' T$.

If $K\in \mathrm{Hyperlat}\, T$, then $\overline{XK}\in \mathrm{Hyperlat}\, U$ by [15], Corollary 1. Now let $L\in \mathrm{Hyperlat}\, U$ and let $S=P\begin{bmatrix} A & 0\\ B & C\end{bmatrix}$ be an operator in $\{T\}'$, where A, B and C are as above except that C may not be scalar-valued (cf. [8]). As before, $\overline{YL}=\{P(0\oplus\Omega u)\colon u\in L\}^-$ and $SP(0\oplus\Omega u)=P\begin{bmatrix} 0\\ C\Omega u\end{bmatrix}$ for any $u\in L$. Note that L, being hyperinvariant for U, is of the form $\chi_F(\Delta_* L_n^2)^-$ for some Borel subset F of the unit circle. Assume that $u=\chi_F\Delta_* v$ for some $v\in L_n^2$. Then $C\Omega u=\chi_F C\Omega\Delta_* v$. Note that $A\Theta_T=\Theta_T A_0$ and $B\Theta_T+C\Delta=\Delta A_0$ imply that $\Omega A=A_0\Omega$ and $B\delta+ +C\Delta\Omega=\Delta A_0\Omega$. Thus

$$C\Omega\Delta_* = C\Delta\Omega = \Delta A_0\Omega - B\delta = \Delta\Omega A - \Omega\Theta_T B = \Omega(\Delta_* A - \Theta_T B)$$

and we have $C\Omega u=\chi_F\Omega(\Delta_* A-\Theta_T B)v$. Note that $\Theta_T B v\in(\Delta_* L_n^2)^-$ and hence $C\Omega u=\Omega w$, where $w=\chi_F(\Delta_* A-\Theta_T B)v\in L$. This shows that $SP(0\oplus\Omega u)\in\overline{YL}$ for any $u=\chi_F\Delta_* v\in L$. Since $\{\chi_F\Delta_* v\colon v\in L_n^2\}$ is dense in L, we conclude that $SP(0\oplus\Omega u)\in\overline{YL}$ for all $u\in L$. Hence $S\overline{YL}\subseteq\overline{YL}$ and $\overline{YL}\in \mathrm{Hyperlat}\, T$, completing the proof.

The next theorem is the analogue of Theorem 1.2 for C_{11} contractions. It should be contrasted with [16], Theorem 3.6.

Theorem 2.4. *Let T be a c.n.u. C_{11} contraction on H with finite defect indices and let $K\subseteq H$ be a subspace. Then the following statements are equivalent:*

(1) $K\in \mathrm{Lat}'' T$;

(2) $K=(\mathrm{Range}\, S)^-$ *for some* $S\in\{T\}'$;

(3) $K=\ker V$ *for some* $V\in\{T\}'$.

Proof. It suffices to show (1)⇒(2). Let U, X and Y be defined as before and $L=\overline{XK}$. Then L, being in $\mathrm{Lat}''\,U$, reduces U. Therefore $L=(W(\Delta_* L_n^2)^-)^-$ for some $W\in\{U\}'$. (W may be taken to be the (orthogonal) projection from $(\Delta_* L_n^2)^-$ onto L.) Let $S=YWX$. Then $S\in\{T\}'$ and $(\mathrm{Range}\,S)^-=(YWXH)^-=$ $=(YW(\Delta_* L_n^2)^-)^-=(YL)^-=(YXK)^-=K$ by Theorem 2.2.

In the remainder of this section we will show that the Sz.-Nagy and Foiaş conjecture holds for C_{11} contractions.

Lemma 2.5. *If T is a normal operator on H with finite multiplicity and $X\in\{T\}'$, then $T|\ker X$ is unitarily equivalent to $(T^*|\ker X^*)^*$.*

Recall that the *multiplicity* of an operator T on H is the least cardinal number of a set of vectors in H which, together with their transforms by $T, T^2, \ldots$, span H.

Proof. Note that $(\mathrm{Range}\,X)^-$ and $(\mathrm{Range}\,X^*)^-$ reduce T and $T|(\mathrm{Range}\,X)^-$ is unitarily equivalent to $T|(\mathrm{Range}\,X^*)^-$ (cf. [3], Lemma 4.1). Let $\mathscr{A}$ be the von Neumann algebra generated by T and I. Since T is normal, $\mathscr{A}$ is abelian, hence finite. On the other hand, T has finite multiplicity implies that $\mathscr{A}'$ is also finite. By [5], Theorem 3, we conclude that $T|\ker X$ is unitarily equivalent to $T|\ker X^*=(T^*|\ker X^*)^*$.

Theorem 2.6. *If T is a c.n.u. C_{11} contraction on H with finite defect indices and $X\in\{T\}'$, then $T|\ker X$ is quasi-similar to $(T^*|\ker X^*)^*$.*

Proof. Let U be the operator of multiplication by e^{it} on $(\Delta_* L_n^2)^-$ and let $Z\colon H\to(\Delta_* L_n^2)^-$ and $Y\colon(\Delta_* L_n^2)^-\to H$ be the quasi-affinities defined in the beginning of Section 2 such that $YZ=\delta(T)$ and $ZY=\delta(U)$, where δ is some outer function. It is easily seen that $ZXY\in\{U\}'$, $(Y(\ker ZXY))^-\subseteqq\ker X$ and $(Z(\ker X))^-\subseteqq$ $\subseteqq\ker ZXY$. From Theorem 2.2 we infer that $\ker ZXY=(ZY(\ker ZXY))^-\subseteqq$ $\subseteqq(Z(\ker X))^-\subseteqq\ker ZXY$. Hence $(Z(\ker X))^-=\ker ZXY$ and $T|\ker X$ is quasi-similar to $U|\ker ZXY$. Note that Z^* and Y^* are also quasi-affinities satisfying $Z^*Y^*=\tilde\delta(T^*)$ and $Y^*Z^*=\tilde\delta(U^*)$ where $\tilde\delta(z)=\overline{\delta(\bar z)}$ is outer. A similar argument as above shows that $T^*|\ker X^*$ is quasi-similar to $U^*|\ker(ZXY)^*$. Lemma 2.5 says that $U|\ker ZXY$ is unitarily equivalent to $U|\ker(ZXY)^*$. We conclude that $T|\ker X$ is quasi-similar to $(T^*|\ker X^*)^*$ as asserted.

3. Weak contractions. In this section we generalize some results in Sections 1 and 2 to weak contractions. The next theorem is the generalization of Theorems 1.2 and 2.4. It should be contrasted with [17], Theorem 3.8 and Corollary 3.9.

Theorem 3.1. *Let T be a c.n.u. weak contraction on H with finite defect indices and let $K\subseteqq H$ be a subspace. Then the following statements are equivalent:*

(1) $K\in\mathrm{Lat}''\,T$;

(2) $K=(\text{Range } S)^{-}$ *for some* $S\in\{T\}'$;
(3) $K=\ker V$ *for some* $V\in\{T\}'$.

Proof. We have only to prove (1)⇒(2). Let $H_0, H_1\subseteq H$ be the invariant subspaces of T on which the C_0 and C_{11} parts of T act, respectively. Since $K\in\text{Lat}'' T$, $T|K$ is also a weak contraction (cf. [18], Theorem 4.1). Hence we may also consider the subspaces $K_0, K_1\subseteq K$ on which the C_0 and C_{11} parts of $T|K$ act. Since $K_0\in\text{Lat}'' T_0$ and $K_1\in\text{Lat}'' T_1$ (cf. [18], Theorem 4.1), $K_0=\overline{S_0 H_0}$ and $K_1=\overline{S_1 H_1}$ for some $S_0\in\{T_0\}'$ and $S_1\in\{T_1\}'$ by Theorems 1.2 and 2.4. We also have $H_0=\overline{V_0 H}$ and $H_1=\overline{V_1 H}$ for some $V_0, V_1\in\{T\}''$ (cf. [17], Theorem 3.1 and [6], p. 334, resp.). Let $S=S_0V_0+S_1V_1$. It is easily seen that $S\in\{T\}'$ and $(\text{Range } S)^{-}=(S_0V_0H+S_1V_1H)^{-}=(S_0H_0+S_1H_1)^{-}=K_0\vee K_1=K$. This completes the proof.

The next theorem (partially) generalizes Theorems 1.3 and 2.6.

Theorem 3.2. *If T is a c.n.u. weak contraction on H with finite defect indices and $K\in\text{Lat}'' T$, then there exists an $L\in\text{Lat}'' T^*$ such that $T|K$ is quasi-similar to $(T^*|L)^*$.*

Proof. As in the proof of the preceding theorem, let $H_0, H_1\subseteq H$ and $K_0, K_1\subseteq K$ be the invariant subspaces for T, $T|K$ such that $T_0\equiv T|H_0$ and $T_1\equiv T|H_1$ are the C_0 and C_{11} parts of T and $T|K_0$ and $T|K_1$ are the C_0 and C_{11} parts of $T|K$, respectively. Since $K_0\in\text{Lat}'' T_0$ and $K_1\in\text{Lat}'' T_1$, there exist $L_0\in\text{Lat}'' T_0^*$ and $L_1\in\text{Lat}'' T_1^*$ such that $T_0|K_0\sim(T_0^*|L_0)^*$ and $T_1|K_1\sim(T_1^*|L_1)^*$ by Theorems 1.3 and 2.6. Let $L'=L_0\oplus L_1\in\text{Lat}'' T_0^*\oplus\text{Lat}'' T_1^*=\text{Lat}''(T_0^*\oplus T_1^*)$ (cf. [2], Prop. 1.3 and Lemma 4.4). By [12], Theorem 1, $T|K\sim(T_0|K_0)\oplus(T_1|K_1)\sim$ $\sim(T_0^*|L_0)^*\oplus(T_1^*)|L_1)^* = ((T_0^*\oplus T_1^*)|L')^*$. Similarly, $T\sim T_0\oplus T_1$, whence $T^*\sim$ $\sim T_0^*\oplus T_1^*$. Note that quasi-similar c.n.u. weak contractions with finite defect indices have isomorphic bi-invariant subspace lattices and the restrictions of the weak contractions to the corresponding bi-invariant subspaces are quasi-similar to each other (cf. [18], Added in proof). We infer that there is an $L\in\text{Lat}'' T^*$ such that $T^*|L\sim(T_0^*\oplus T_1^*)|L'$. Hence $T|K\sim(T^*|L)^*$ as asserted.

We conclude this paper by a simple observation that if T is a weak contraction and $X\in\{T\}'$ then $T|(\text{Range } X)^{-}$ is quasi-similar to $(T^*|(\text{Range } X^*)^{-})^*$. Indeed, this follows from the following

Lemma 3.3. *If T is an operator on H and $X\in\{T\}'$, then $(T^*|(\text{Range } X^*)^{-})^*\prec$ $\prec T|(\text{Range } X)^{-}$.*

Proof. Let X_1 be the operator $X|(\text{Range } X^*)^{-}$ from $(\text{Range } X^*)^{-}$ to $(\text{Range } X)^{-}$. It is routine to check that X_1 is a quasi-affinity intertwining $(T^*|(\text{Range } X^*)^{-})^*$ and $T|(\text{Range } X)^{-}$. We leave the details to the readers.

Corollary 3.4. *If T is a C_0 contraction, a C_{11} contraction or a weak contraction and $X \in \{T\}'$, then $T|(\text{Range } X)^-$ is quasi-similar to $(T^*|(\text{Range } X^*)^-)^*$.*

Proof. This follows from Lemma 3.3 and [13], Lemma 3.

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