Acta Sci. Math., 43 (1981), 5—13

Reflexive and hyper-reflexive operators of class C_0

H. BERCOVICI, C. FOIAŞ, B. SZ.-NAGY

Dedicated to P. R. Halmos on his 65th birthday

The Jordan model of a finite matrix was used for the first time in the study of reflexive operators (on finite dimensional spaces) by DEDDENS and FILLMORE [5]. Their result was extended in [1] to the class of algebraic operators on Hilbert space, using the quasi-similar Jordan model (in fact in [1] the notion of para-reflexivity is studied, but one can easily see that reflexivity and para-reflexivity are equivalent for algebraic operators). The possibility of extending these results to the entire class C_0 was then indicated in [6] for the separable case and [2] (where a sketch of proof is done) for the nonseparable case. It appeared that the reflexivity of an operator of class C_0 is equivalent to the reflexivity of a single "Jordan block" $S(m)$ (cf. § 1 below for the precise statement).

In this note we give a simplified version of the proofs of [6] and [2]. We further study the related notion of hyper-reflexivity (stronger than reflexivity for the class C_0) and prove an analogous characterization of hyper-reflexive operators of class C_0.

1. Notations and results

We shall denote by $\mathfrak{H}$ a complex Hilbert space and by $\mathscr{B}(\mathfrak{H})$ the algebra of linear and bounded operators acting on $\mathfrak{H}$. For an algebra $\mathscr{A}\subset\mathscr{B}(\mathfrak{H})$, $\operatorname{Lat}\mathscr{A}$ will stand for the set of closed linear subspaces $\mathfrak{M}\subset\mathfrak{H}$ invariant with respect to all elements of $\mathscr{A}$: $X\mathfrak{M}\subset\mathfrak{M}$, $X\in\mathscr{A}$. For a family $\mathscr{L}$ of closed linear subspaces of $\mathfrak{H}$, $\operatorname{Alg}\mathscr{L}$ will denote the algebra of operators $X\in\mathscr{B}(\mathfrak{H})$ for which $X\mathfrak{M}\subset\mathfrak{M}$ whenever $\mathfrak{M}\in\mathscr{L}$. The algebra $\mathscr{A}\subset\mathscr{B}(\mathfrak{H})$ is called *reflexive* if $\mathscr{A}=\operatorname{Alg}\operatorname{Lat}\mathscr{A}$. An operator $T\in\mathscr{B}(\mathfrak{H})$ is *reflexive* if the weakly closed algebra $\mathscr{A}_T$ generated by T and $I_{\mathfrak{H}}$ is a reflexive algebra. An operator $T\in\mathscr{B}(\mathfrak{H})$ will be called *hyper-reflexive* if its commutant $\{T\}'=(\mathscr{A}_T)'$ is a reflexive algebra.

Received September 26, 1980.

Recall that a completely nonunitary contraction $T\in\mathscr{B}(\mathfrak{H})$ is an operator of class C_0 if $u(T)=0$ for some $u\in H^\infty$, $u\neq 0$ (cf. [10], ch. V). The simplest operators of class C_0 are the "Jordan blocks" $S(m)$, with $m\in H^\infty$ an inner function, defined by

$$S(m)u = P_{\mathfrak{H}(m)}(z\,u(z)),\ u\in\mathfrak{H}(m) = H^2\ominus mH^2. \tag{1.1}$$

By the results of [11], [4] and [3], every operator T of class C_0 is quasi-similar to a unique Jordan operator, that is to an operator of the form

$$S = \bigoplus_\alpha S(m_\alpha) \tag{1.2}$$

where the values of α are ordinal numbers and the inner functions m_α are subject to the conditions

$$m_\alpha = 1 \quad \text{for some} \quad \alpha \geqq 0; \tag{1.3}$$

$$m_\alpha \text{ divides } m_\beta \text{ whenever } \alpha \geqq \beta; \tag{1.4}$$

$$m_\alpha = m_\beta \quad \text{whenever} \quad \operatorname{card}(\alpha) = \operatorname{card}(\beta). \tag{1.5}$$

Let us note that m_0 coincides with the minimal function m_T of T. The operators quasi-similar to some $S(m)$ are precisely the cyclic operators of class C_0 (multiplicity-free operators). For multiplicity-free T it follows from [12] that $\operatorname{Lat} T=\operatorname{Lat}\{T\}'$ and $\mathscr{A}_T=(\mathscr{A}_T)'$ so for such operators reflexivity and hyper-reflexivity are equivalent.

We are now able to state the main results of this note.

Theorem A. *An operator T of class C_0 with Jordan model $S=\bigoplus_\alpha S(m_\alpha)$ is reflexive if and only if $S(m_0/m_1)$ is reflexive.*

Theorem B. *Let T and S be as in Theorem* A. *Then T is hyper-reflexive if and only if $S(m_0)$ is reflexive.*

Recently P. Y. Wu [15] published a proof of Theorem A for the particular case of operators of class C_0 with finite defect indices.

2. Preliminary results

The following theorem plays an important role in the study of reflexive operators of class C_0 (cf. [13] and [14] for the proof).

Theorem 2.1. *For every operator T of class C_0 we have*

$$\mathscr{A}_T = \{T\}'' = \{T\}'\cap \operatorname{Alg}\operatorname{Lat} T. \tag{2.1}$$

Corollary 2.2. *An operator T of class C_0 is reflexive if and only if* Alg Lat $T \subset \subset \{T\}'$. — Obvious from relation (2.1).

Corollary 2.3. *Let $T \in \mathcal{B}(\mathfrak{H})$ be an operator of class C_0 and let $\mathfrak{M}_j \in$* Lat T *$(j \in J)$ be such that $T|\mathfrak{M}_j$ is reflexive for each j. If $\mathfrak{H} = \bigvee_{j \in J} \mathfrak{M}_j$ then T is reflexive.*

Proof. It follows from Corollary 2.2 that it is enough to show that every $X \in$ Alg Lat T commutes with T. But it is obvious that for $X \in$ Alg Lat T we have $X|\mathfrak{M}_j \in$ Alg Lat $(T|\mathfrak{M}_j)$ so that $X|\mathfrak{M}_j \in \{T|\mathfrak{M}_j\}'$ by the hypothesis. Therefore,

$$\ker(XT - TX) \supset \bigvee_{j \in J} \mathfrak{M}_j = \mathfrak{H}, \quad \text{that is} \quad X \in \{T\}'.$$

Corollary 2.4. *Let $T \in \mathcal{B}(\mathfrak{H})$ be a reflexive operator of class C_0. For every $X \in \mathcal{A}_T$ the operator $T|(X\mathfrak{H})^-$ is reflexive.*

Proof. Let us take $Y \in$ Alg Lat $(T|(X\mathfrak{H})^-)$. Since $X \in$ Alg Lat T we infer $YX \in$ Alg Lat T and therefore $YX \in \{T\}'$, by the reflexivity of T and Corollary 2.2. As X and T commute, we have $YT \cdot X = YX \cdot T = TY \cdot X$ such that $Y \in \{T|(X\mathfrak{H})^-\}'$ and the conclusion follows again by Corollary 2.2.

We shall introduce now an auxiliary property.

Definition 2.5. A completely nonunitary contraction T has *property* $(*)$ if for any quasi-affinity $X \in \{T\}'$ there exists a quasi-affinity $Y \in \{T\}'$ such that

$$XY = YX = u(T) \quad \text{for some } u \in H^\infty \tag{2.3}$$

for some $u \in H^\infty$.

Lemma 2.6. *Let T and T' be two quasi-similar completely nonunitary contractions. If T has property $(*)$ then T' does also. Moreover, if T has property $(*)$ then there exist quasi-affinities A, B such that $T'B = BT$, $TA = AT'$ and*

$$AB = u(T), \quad BA = u(T') \quad \textit{for some} \quad u \in H^\infty. \tag{2.4}$$

Proof. Let us assume that T has property $(*)$ and A, B' are two quasi-affinities such that $T'B' = B'T$ and $TA = AT'$. For any quasi-affinity $X \in \{T'\}'$ we have $AXB' \in \{T\}'$ so that, by the assumption, we have $AXB' \cdot Y' = Y' \cdot AXB' = u(T)$ for some quasi-affinity $Y' \in \{T\}'$ and $u \in H^\infty$. We obviously have

$$A(X \cdot B'Y'A - u(T')) = AXB'Y' \cdot A - Au(T') = u(T)A - Au(T') = 0,$$

$$(B'Y'A \cdot X - u(T'))B' = B' \cdot Y'AXB' - u(T')B' = B'u(T) - u(T')B' = 0$$

so that $X \cdot B'Y'A = u(T')$ by the injectivity of A, and $B'Y'A \cdot X = u(T')$ by the quasi-surjectivity of B'. So we have $XY = YX = u(T')$ for $Y = B'Y'A$ and therefore T' has property $(*)$. For the last assertion of the Lemma it is enough to set $B = B'Y'$ where Y' is obtained from the preceding proof for $X = I$. The Lemma follows.

Lemma 2.7. *Every Jordan operator of the form* $S=S(m_0)\oplus S(m_1)$ *has property* $(*)$.

Proof. Let $X\in\{S\}'$ be a quasi-affinity. By the Lifting Theorem ([10], sec. II. 2.3) there exists a 2×2 matrix $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with entries in H^∞, such that

(2.5) $P_{\mathfrak{H}}A(I-P_{\mathfrak{H}})=0$ on $H^2\oplus H^2$, and $X=P_{\mathfrak{H}}A$ on $\mathfrak{H}=\mathfrak{H}(m_0)\oplus\mathfrak{H}(m_1)$.

Let us remark that

$$a\wedge b\wedge m_0=1. \tag{2.6}$$

Indeed, if $q=a\wedge b\wedge m_0\neq 1$ it follows that $\hat{q}=(1-\overline{q(0)}q)\oplus 0$ is a non-zero vector in $\mathfrak{H}$ such that for every vector of the form Xh $(h=h_0\oplus h_1\in\mathfrak{H})$ we have

$$(Xh,\hat{q})=(P_{\mathfrak{H}}Ah,\hat{q})=(Ah,\hat{q})=\int\left((a/q)h_0+(b/q)h_1\right)\left(q-q(0)\right)=0,$$

and this is impossible since X has dense range. Moreover, we have

$$\det A\wedge m_1=1. \tag{2.7}$$

Indeed, let us set $p=\det A\wedge m_1$ and denote $h=-b(m_1/p)\oplus a(m_1/p)$. Then we have, by (2.5),

$$XP_{\mathfrak{H}}h=P_{\mathfrak{H}}AP_{\mathfrak{H}}h=P_{\mathfrak{H}}Ah=P_{\mathfrak{H}}\left(0\oplus m_1\cdot(\det A)/p\right)=0$$

and therefore $P_{\mathfrak{H}}h=0$ by the injectivity of X. Hence, $h\in m_0H^2\oplus m_1H^2$, which implies that p divides b and a; taking account of the definition ot p we infer that p divides $a\wedge b\wedge m_1\wedge\det A$ also. Then (2.6) forces p to equal 1, concluding the proof of (2.7). From (2.6—7) it obviously follows that

$$\det A\wedge m_1 a\wedge m_1 b\wedge m_0=1$$

so that [7] (cf. also [9]) implies the existence of $c', d', e'\in H^\infty$ (even constans) such that $(\det A+m_1(ad'-be')+m_0e')\wedge m_0=1$ or, equivalently,

$$(\det A+m_1(ad'-bc'))\wedge m_0=1. \tag{2.8}$$

Let us remark now that the matrix $A'=\begin{bmatrix} a & b \\ c+m_1c' & d+m_1d' \end{bmatrix}$ satisfies the relations analogous to (2.5) and moreover $\det A'\wedge m_0=1$ by (2.8). Let us define $Yh=P_{\mathfrak{H}}Bh$ for $h\in\mathfrak{H}=\mathfrak{H}(m_0)\oplus\mathfrak{H}(m_1)$, where $B=\begin{bmatrix} d+m_1d' & -b \\ -c-m_1c' & a \end{bmatrix}$. It follows by direct computation that $Y\in\{S\}'$ and $XY=YX=u(S)$ with $u=\det A'$. Now $u(S)$ is a quasi-affinity because $u\wedge m_0=1$ (cf. [10], Prop. III. 4.7b) and therefore Y is also a quasi-affinity. The Lemma follows.

Remark 2.8. Lemma 2.7 also applies to operators of the form $S=S(m)$ (take $m_1=1$). By the celebrated theorem of Sarason [8] we have then, in fact, $X=u(S)$ with some $u\in H^\infty$, for *every* $X\in\{S\}'$.

3. Reflexive operators

The role of property $(*)$ in the study of reflexive operators is underlined by the following result.

Lemma 3.1. *Let T and T' be two quasi-similar operators of class C_0 having property $(*)$. Then T is reflexive if and only if T' is reflexive.*

Proof. By Lemma 2.6 there exist quasi-affinities A, B such that $T'B=BT$, $TA=AT'$ and $AB=u(T)$, $BA=u(T')$ for some $u\in H^\infty$. Assume T is reflexive. For any $X\in \operatorname{Alg}\operatorname{Lat} T'$ and $\mathfrak{M}\in \operatorname{Lat} T$ we have $AXB\mathfrak{M}\subset A(B\mathfrak{M})^-\subset(AB\mathfrak{M})^-=$ $=(u(T)\mathfrak{M})^-\subset\mathfrak{M}$ because $(B\mathfrak{M})^-\in \operatorname{Lat} T'$ and $u(T)\in \operatorname{Alg}\operatorname{Lat} T$. By the reflexivity of T we have $AXB\in\{T\}'$ and from the relations

$$A\cdot XT'\cdot B = AXB\cdot T = T\cdot AXB = A\cdot T'X\cdot B$$

it follows that $X\in\{T'\}'$. The reflexivity of T' follows then by Corollary 2.2, and Lemma 3.1 is proved.

For easier reference, let us formulate the following:

Lemma 3.2. *For two ('comparable') inner functions, say p and q, the operator V_{pq}: $\mathfrak{H}(p)\to\mathfrak{H}(q)$, defined by*

(3.1)
(3.2)
$$V_{pq}h=\begin{cases} P_{\mathfrak{H}(q)}h & \text{if } q \text{ divides } p \\ (q/p)h & \text{if } p \text{ divides } q\end{cases} \quad (h\in\mathfrak{H}(p)),$$

intertwines $S(p)$ and $S(q)$.

Proof. If q divides p, we have for $h\in\mathfrak{H}(p)$, using (1.1) and (3.1),

$$(S(q)V_{pq}-V_{pq}S(p))h = P_{\mathfrak{H}(q)}\{zP_{\mathfrak{H}(q)}h-P_{\mathfrak{H}(p)}zh\}=0$$

because $zP_{\mathfrak{H}(q)}h=z(h+qw)=zh+qw'$, $P_{\mathfrak{H}(p)}zh=zh+pw'=zh+qw''$ with some $w, w', w''\in H^2$, and hence $\{...\}\in qH^2$.

If, conversely, p divides q, then we use the relation $P_{\mathfrak{H}(m)}u=u-m[\bar{m}u]_+$, valid for any inner m and for any $u\in H^2$, $[...]_+$ denoting here the natural projection $L^2\to H^2$. We get by (1.1) and (3.2)

$$(S(q)V_{p,q}-V_{p,q}S(p))h = P_{\mathfrak{H}(q)}z\frac{q}{p}h-\frac{q}{p}P_{\mathfrak{H}(p)}(zh)=$$

$$=\left(z\frac{q}{p}h-q\left[\bar{q}z\frac{q}{p}h\right]_+\right)-\frac{q}{p}(zh-p[\bar{p}zh]_+)=0$$

because $\bar{q}q=1, \dfrac{1}{p}=\bar{p}$ on the circle $\{z\colon |z|=1\}$.

Lemma 3.3. *Let* $S=S(m_0)\oplus S(m_1)$ *be a Jordan operator. Then for every* $X\in \operatorname{Alg\,Lat} S$ *there exists* $Y\in\mathcal{A}_S$ *such that* $X-Y=Z\oplus 0$ *with some operator* Z *on* $\mathfrak{H}(m_0)$ *and the zero operator on* $\mathfrak{H}(m_1)$.

Proof. The subspaces $\mathfrak{H}(m_0)\oplus\{0\}$ and $\{0\}\oplus\mathfrak{H}(m_1)$ are invariant for S so the assumption $X\in\operatorname{Alg\,Lat} S$ implies

$$X=X_0\oplus X_1,\quad X_j\in\operatorname{Alg\,Lat} S(m_j)\quad (j=1,2).$$

Consider the (obviously isometric) operator $V=V_{m_0,m_1}$ defined by (3.2), and the subspaces

$$\{Vh\oplus h\colon h\in\mathfrak{H}\{m_1)\}\quad\text{and}\quad\{VS(m_1)h\oplus h\colon h\in\mathfrak{H}(m_1)\}.$$

By Lemma 3.2, both are invariant for S, and hence for X also. So we infer

$$X_0Vh=VX_1h\quad\text{and}\quad X_0VS(m_1)h=VS(m_1)X_1h\quad\text{for}\quad h\in\mathfrak{H}\{m_1).$$

Apply the first equation for $S(m_1)h$ in place of h and compare the results to obtain $VX_1S(m_1)h=VS(m_1)X_1h$ for all $h\in\mathfrak{H}(m_1)$. Hence, $X_1S(m_1)=S(m_1)X_1$. By a well-known theorem of SARASON [8] this implies that $X_1=u(S(m_1))$ for some $u\in H^\infty$. Hence, $Y=u(S)=u(S(m_0))\oplus u(S(m_1))$ has the property we needed.

Lemma 3.4. *Let* $S=S(m_0)\oplus S(m_1)$ *be a Jordan operator and let* Z *be an operator on* $\mathfrak{H}(m_0)$ *such that* $Z\oplus 0\in\operatorname{Alg\,Lat} S$. *Then*

$$Z(qH^2\ominus m_0H^2)\subset qm_1H^2\ominus m_0H^2 \tag{3.3}$$

for every inner divisor q *of* m_0/m_1.

Proof. As m_1 is a divisor of m_0/q, which, in turn, is a divisor of m_0, we can consider the operators $V_0=V_{m_0/q,m_0}$ and $V_1=V_{m_0/q,m_1}$ defined by (3.2) and (3.1), respectively, and observe that $\{V_0h\oplus V_1h\colon h\in\mathfrak{H}(m_0/q)\}$ is a subspace invariant for S (closure follows from the fact that V_0 is an isometry, namely multiplication by the inner function q). Then it is invariant for $Z\oplus 0$ also. Hence we infer that for every $h\in\mathfrak{H}(m_0/q)$ there exists $h'\in\mathfrak{H}(m_0/q)$ such that $ZV_0h=V_0h'$ and $0=V_1h'$. As $V_1h'=P_{\mathfrak{H}(m_1)}h'$ by (3.1), we must have $h'\in\left(H^2\ominus\frac{m_0}{q}H^2\right)\ominus(H^2\ominus m_1H^2)$ i.e. $h'\in m_1H^2\ominus\frac{m_0}{q}H^2$. We conclude that $Zq\mathfrak{H}\left(\frac{m_0}{q}\right)\subset q\left(m_1H^2\ominus\frac{m_0}{q}H^2\right)$, and this obviously implies (3.3).

Remark. In the particular cases $q=1$ and $q=\dfrac{m_0}{m_1}$ (3.3) implies

$$\operatorname{ran} Z\subset m_1H^2\ominus m_0H^2\quad\text{and}\quad\ker Z\supset(m_0/m_1)H^2\ominus m_0H^2. \tag{3.4}$$

In the proof of the following result we shall use the *unitary* operator

$$R\colon m_1H^2\ominus m_0H^2 \to \mathfrak{H}(m_0/m_1) \quad \text{defined by} \quad Rh = h/m_1, \tag{3.5}$$

which satisfies the relation

$$RS(m_0)|(m_1H^2\ominus m_0H^2) = S(m_0/m_1)R = P_{\mathfrak{H}(m_0/m_1)}S(m_0)R. \tag{3.6}$$

Proposition 3.5. *The Jordan operator* $S=S(m_0)\oplus S(m_1)$ *is reflexive whenever* $S(m_0/m_1)$ *is reflexive.*

Proof. By Lemmas 3.3, 3.4, and Corollary 2.2 it suffices to show that every operator $Z\in\operatorname{Alg\,Lat} S(m_0)$ satisfying (3.3) commutes with $S(m_0)$. We claim that for such a Z we have $RZ|\mathfrak{H}(m_0/m_1)\in\operatorname{Alg\,Lat} S(m_0/m_1)$. Indeed, the general form of the subspaces in $\operatorname{Lat} S(m_0/m_1)$ is $qH^2\ominus(m_0/m_1)H^2$ for q a divisor of m_0/m_1. By (3.3—4) we have $RZ(qH^2\ominus(m_0/m_1)H^2)\subset RZ(qH^2\ominus m_0H^2)\subset R(qm_1H^2\ominus m_0H^2)=$ $=qH^2\ominus(m_0/m_1)H^2$. The reflexivity of $S(m_0/m_1)$ implies $RZ|\mathfrak{H}(m_0/m_1)\in\{S(m_0/m_1)\}'$. Therefore,

$$R(ZS(m_0)-S(m_0)Z)|\mathfrak{H}(m_0/m_1) = ((RZ)S(m_0)-RS(m_0)Z)|\mathfrak{H}(m_0/m_1) =$$

$$= ((RZ)P_{\mathfrak{H}(m_0/m_1)}S(m_0)-S(m_0/m_1)RZ)|\mathfrak{H}(m_0/m_1) = 0$$

so that Z commutes with $S(m_0)$ on $\mathfrak{H}(m_0/m_1)$. Because by (3.4) we have $ZS(m_0)=$ $=S(m_0)Z=0$ on $(m_0/m_1)H^2\ominus m_0H^2$ it follows that $Z\in\{S(m_0)\}'$. The Proposition is proved.

Proof of Theorem A. Let $T\in\mathscr{B}(\mathfrak{H})$ be of class C_0, with Jordan model $S=\bigoplus_\alpha S(m_\alpha)$ on $\mathfrak{H}=\bigoplus_\alpha \mathfrak{H}(m_\alpha)$. If T is reflexive we infer by Corollary 2.4 that $T|(m_1(T)\mathfrak{H})^-$ is reflexive. But $T|(m_1(T)\mathfrak{H})^-$ is quasi-similar to $S(m_0/m_1)$ and the reflexivity of $S(m_0/m_1)$ follows by Lemma 3.1 and Remark 2.8.

Conversely, let us assume that $S(m_0/m_1)$ is reflexive. Let X be any quasi-affinity such that $TX=XS$. Let us consider the spaces $\mathfrak{H}_\alpha=(X\mathfrak{H}(m_\alpha))^-$ and $\mathfrak{K}_\alpha=$ $=(X\ker m_\alpha(S|\mathfrak{H}(m_0)))^-$ for every ordinal number α. Then the restriction $T|\mathfrak{H}_0\vee\mathfrak{H}_1$ is quasi-similar to $S(m_0)\oplus S(m_1)$ and $T|\mathfrak{K}_\alpha\vee\mathfrak{H}_\alpha(\alpha\geqq 1)$ is quasi-similar to $S(m_\alpha)\oplus S(m_\alpha)$. All these restrictions are reflexive by Lemmas 2.7, 3.1 and Proposition 3.5 so that the reflexivity of T follows by Corollary 2.3 because $(\mathfrak{H}_0\vee\mathfrak{H}_1)\vee$ $\vee(\bigvee_{\alpha\geqq 1}(\mathfrak{H}_\alpha\vee\mathfrak{K}_\alpha))=\bigvee_{\alpha\geqq 0}\mathfrak{H}_\alpha=\mathfrak{H}$.

Corollary 3.6. *Let T and T' be two quasi-similar operators of class C_0. Then T is reflexive if and only if T' is reflexive.*

Proof. Two operators of class C_0 are quasi-similar if and only if they have he same Jordan model. Corollary obviously follows from Theorem A.

4. Hyper-reflexive operators

Proposition 4.1. *If the operators T and T' are quasi-similar and one of them is hyper-reflexive then so is the other.*

Proof. Let X and Y be two quasi-affinities such that $T'X=XT$ and $TY=YT'$ and let $A\in \mathrm{Alg\,Lat}\,\{T\}'$. Then $XAY\in \mathrm{Alg\,Lat}\,\{T'\}'$; indeed, for each $\mathfrak{M}\in \mathrm{Lat}\,\{T'\}'$ we have

$$\mathfrak{N} = \bigvee_{Z\in\{T\}'} ZY\mathfrak{M}\in \mathrm{Lat}\,\{T\}' \tag{4.1}$$

and $X\mathfrak{N}\subset \bigvee_{Z\in\{T\}'} XZY\mathfrak{M}\subset \bigvee_{Z'\in\{T'\}'} Z'\mathfrak{M}=\mathfrak{M}$. In particular, $XAY\mathfrak{M}\subset XA\mathfrak{N}\subset X\mathfrak{N}\subset\mathfrak{M}$ and $XAY\in \mathrm{Alg\,Lat}\,\{T'\}'$ because $\mathfrak{M}\in \mathrm{Lat}\,\{T'\}'$ is arbitrary.

If T' is hyper-reflexive it follows that $XAY\in\{T'\}'$ so that $X\cdot AT\cdot Y=XAY\cdot T'= =T'\cdot XAY=X\cdot TA\cdot Y$ and $A\in\{T\}'$ because X and Y are quasi-affinities. It follows that T is hyper-reflexive. The Proposition is proved.

Proof of Theorem B. By the preceding proposition it is enough to consider the case $T=S$. Let us assume that S is hyper-reflexive and take $A\in \mathrm{Alg\,Lat}\,S(m_0)$. Then the operator $B=\bigoplus_\alpha A_\alpha$, where $A_0=A$ and $A_\alpha=0$ for $\alpha\geqq 1$, belongs to $\mathrm{Alg\,Lat}\,\{S\}'$. Indeed, since each $\mathfrak{K}\in \mathrm{Lat}\,\{S\}'$ has the form $\bigoplus_\alpha \mathfrak{K}_\alpha$ where $\mathfrak{K}_\alpha\in \mathrm{Lat}\,S(m_\alpha)$, we have $B\mathfrak{K}\subset\mathfrak{K}$. It follows that $B\in\{S\}'$ and this implies $A\in\{S(m_0)\}'$. The reflexivity of $S(m_0)$ follows by Corollary 2.2.

Conversely, let us assume that $S(m_0)$ is reflexive. Because $S(m_\alpha)$ is unitarily equivalent to $S(m_0)|(\mathrm{ran}\, u_\alpha(S(m_0))^-$ $(u_\alpha=m_0/m_\alpha)$ it follows by Corollary 2.4 that $S(m_\alpha)$ is reflexive for every α. We consider the operators $R_{\alpha\beta}\in\{S\}'$ defined by $R_{\alpha\beta}(\bigoplus_\gamma h_\gamma)= =\bigoplus_\gamma k_\gamma$ where $k_\gamma=0$ for $\gamma\neq\alpha$ and

$$k_\alpha = V_{m_\beta,m_\alpha}h_\beta = \begin{cases} P_{\mathfrak{H}(m_\alpha)}h_\beta & \text{whenever} \quad \alpha>\beta,\\ (m_\alpha/m_\beta)h_\beta & \text{whenever} \quad \alpha\leqq\beta.\end{cases} \tag{4.2}$$

Cf. (3.1—2). Obviously, $P_\alpha=R_{\alpha\alpha}$ coincides with the orthogonal projection of $\bigoplus_\gamma\mathfrak{H}(m_\gamma)$ α-component space.

Let $A\in \mathrm{Alg\,Lat}\,\{S\}'$; we have $P_\alpha AP_\beta\in \mathrm{Alg\,Lat}\,\{S\}'$ and $A=\sum_{\alpha,\beta} P_\alpha AP_\beta$ in the strong operator topology. To conclude the proof it is enough to show that $P_\alpha AP_\beta\in\{S\}'$. Let us note that the operators $R_{\beta\alpha}P_\alpha AP_\beta$ and $P_\alpha AP_\beta R_{\beta\alpha}$ belong to $\mathrm{Alg\,Lat}\,\{S\}'$ and are of the form $\bigoplus_\gamma T_\gamma$ with $T_\gamma=0$ for $\gamma\neq\beta$ and $\gamma\neq\alpha$, respectively. Considering the spaces of the form $\ker m(S)\in \mathrm{Lat}\,\{S\}'$ for m a divisor of m_0, it is easily seen that necessarily $T_\gamma\in \mathrm{Alg\,Lat}\,S(m_\gamma)$ so that $T_\gamma\in\{S(m_\gamma)\}'$ by the reflexivity of $S(m_\gamma)$. It follows that $R_{\beta\alpha}P_\alpha AP_\beta$ and $P_\alpha AP_\beta R_{\beta\alpha}$ commute with S and therefore

$$R_{\beta\alpha}(P_\alpha AP_\beta S-SP_\alpha AP_\beta)=(P_\alpha AP_\beta S-SP_\alpha AP_\beta)R_{\beta\alpha}=0.$$

If the range of $R_{\beta\alpha}$ does not contain ran P_β it follows that $\beta<\alpha$ and therefore $R_{\beta\alpha}$ is one-to-one on ran P_α; therefore in both cases we infer $P_\alpha AP_\beta\in\{S\}'$. The Theorem is proved.

Remark 4.2. It follows from Theorems A and B that each hyper-reflexive operator of class C_0 is also reflexive. This fact can be proved directly also, by using Theorem 2.1.

References

[1] C. Apostol, R. G. Douglas, C. Foiaş, Quasi-similar models for nilpotent operators, *Trans. Amer. Math. Soc.*, **224** (1976), 407—415.

[2] H. Bercovici, Teoria operatorilor de clasa C_0, *Stud. Cerc. Mat.*, **31** (1979), 657—704.

[3] H. Bercovici, On the Jordan model of C_0 operators. II, *Acta Sci. Math.*, **42** (1980), 43—56.

[4] H. Bercovici, C. Foiaş, B. Sz.-Nagy, Compléments à l'étude des opérateurs de classe C_0. III, *Acta Sci. Math.*, **37** (1975), 313—322.

[5] J. A. Deddens, P. A. Fillmore, Reflexive linear transformations, *Linear Algebra and Appl.*, **10** (1975), 89—93.

[6] C. Foiaş, Reflexive operators (lecture), *Conference of Operator Theory* (Timişoara, June 1976).

[7] E. A. Nordgren, On the quasi-equivalence of matrices over H^∞, *Acta Sci. Math.*, **34** (1973), 301—310.

[8] D. Sarason, Generalized interpolation in H^∞, *Trans. Amer. Math. Soc.*, **127** (1967), 179—203.

[9] B. Sz.-Nagy, Diagonalization of matrices over H^∞, *Acta Sci. Math.*, **38** (1976), 223—238.

[10] B. Sz.-Nagy, C. Foiaş, *Harmonic Analysis of Operators on Hilbert Space*, North Holland—Akadémiai Kiadó (Amsterdam—Budapest, 1970).

[11] B. Sz.-Nagy, C. Foiaş, Modèle de Jordan pour une classe d'opérateurs de l'espace de Hilbert, *Acta Sci. Math.*, **31** (1970), 91—115.

[12] B. Sz.-Nagy, C. Foiaş, Compléments à l'étude des opérateurs de classe C_0. (I), *Acta Sci. Math.*, (1970), 287—296.

[13] B. Sz.-Nagy, C. Foiaş, Commutants and bicommutants of operators of class C_0, *Acta Sci. Math.*, **38** (1976), 311—315.

[14] P. Y. Wu, Commutants of $C_0(N)$ contractions, *Acta Sci. Math.*, **38** (1976), 193—202.

[15] P. Y. Wu, On the reflexivity of $C_0(N)$ contractions, *Proc. Amer. Math. Soc.*, **79** (1980), 405—409.

(H. B.)
DEPARTMENT OF MATH.
UNIVERSITY OF MICHIGAN
ANN ARBOR, MI 48109
U.S.A.

(C. F.)
DEPARTMENT OF MATH.
INDIANA UNIVERSITY
BLOOMINGTON, IN 47405
U.S.A.

(B. SZ.-N.)
BOLYAI INSTITUTE
UNIVERSITY SZEGED
6720 SZEGED
HUNGARY