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On the commutant of C_{11}-contractions

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1. We say that a Hilbert space operator T has property (P), or belongs to the operator class $\mathcal{P}$, if every injection $X\in\{T\}'$ is a quasi-affinity. B. SZ.-NAGY and C. FOIAŞ [1] proved that the operators of class C_0 and of finite multiplicity have property (P). H. BERCOVICI [2] characterized the class of all C_0-operators having property (P). Recently P. Y. WU [3] showed that every completely non-unitary (c. n. u.) C_{11}-contraction with finite defect indices belongs to the class $\mathcal{P}$. (Actually, he proved more.) The main purpose of this note is to characterize the class of all C_{11}-contractions having property (P).

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2. Only bounded linear operators on complex separable Hilbert spaces will be considered. Separability does not mean a restriction of generality, as it will turn out in section 5. We follow the notation and the terminology used in [4].

It is well-known that every contraction T of class C_{11} is quasi-similar to a unitary operator U (cf. [4], II.3.5). Moreover, since quasi-similar unitary operators are unitarily equivalent (cf. [4], II.3.4), the operator U is uniquely determined up to unitary equivalence.

If T is, moreover, a *c. n. u.* contraction of class C_{11}, then T is quasi-similar to the operator U of multiplication by e^{it} on the Hilbert space $\overline{\Delta L^2(\mathfrak{E})}$. (Cf. [4], VI.2.3.) Here Δ is the operator-valued function defined by $\Delta(e^{it})=[I-\Theta(e^{it})^*\Theta(e^{it})]^{1/2}$, where Θ denotes the characteristic function of T. This operator U has absolutely continuous spectral measure on the unit circle (i.e., is an a. c. u. operator). So U is unitarily equivalent to an operator M of the form $M=M_{E_1}\oplus M_{E_2}\oplus\dots$, where $\{E_n\}_n$ is a decreasing sequence of measurable subsets of the unit circle C of $\mathbf{C}$, and M_{E_n} denotes the operator of multiplication by e^{it} on the space $L^2(E_n)$. (We consider the normalized Lebesgue measure m on C.) For every measurable subset F of C let $F^=$ denote the closed support of the measure $m|F$, the restriction of m on the set F.

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If it is assumed that $E_n=E_n^=$ for every n, then the operator M is uniquely determined (cf. [5]). M will be called the *canonical functional model* of the a. c. u. operator U, and the *Jordan model* of the c. n. u. C_{11}-contraction T (cf. [6]).

Now we can state our main result:

Theorem 1. *Let T be a c. n. u. contraction of class C_{11} on the separable Hilbert space $\mathfrak{H}$, and let $M=M_{E_1}\oplus M_{E_2}\oplus\ldots$ be its Jordan model. Then T has property (P) if and only if $m(\bigcap_{n\geq 1} E_n)=0$.*

Sufficiency and necessity of this condition will be proved in sec. 3 and sec. 4, respectively. In sec. 5 some corollaries are treated, while in sec. 6 we consider arbitrary C_{11}-contractions.

We shall use the following notation. For an operator-valued function N let $d_N(e^{it})$ denote the rank of the operator $N(e^{it})$. If T is a c. n. u. C_{11}-contraction, then let d_T be the function defined by $d_T(e^{it})=d_\Delta(e^{it})$, where $\Delta=\Delta(e^{it})$ is the operator-valued function derived from the characteristic function $\Theta(e^{it})$ of T.

For two operators, T_1 and T_2, we denote by $\mathscr{I}(T_1, T_2)$ the set of intertwining operators $\mathscr{I}(T_1, T_2)=\{X|XT_1=T_2X\}$. Let Hyp lat (T) denote the lattice of hyperinvariant subspaces of T.

A system $\{\mathfrak{H}_n\}_{n\geq 1}$ of subspaces of $\mathfrak{H}$ will be called *basic* if, for any n, the subspaces $\mathfrak{H}_n$, $\bigvee_{k\neq n}\mathfrak{H}_k$ are complementary and $\bigcap_{n\geq 1}(\bigvee_{k\geq n}\mathfrak{H}_k)=\{0\}$ (cf. [7]).

3. We shall need some lemmas. The first one should be contrasted with [4], VI. Th.6.1.

Lemma 1. *Let $N(e^{it})$ $(0\leq t\leq 2\pi)$ be a function with values operators on a (separable) Hilbert space $\mathfrak{E}$, and measurable. Let us denote by U the restriction of the operator of multiplication by e^{it} on its reducing subspace $\mathfrak{N}=\overline{NL^2(\mathfrak{E})}$; and let $M=M_{E_1}\oplus M_{E_2}\oplus\ldots$ be its canonical functional model. Then $d_N(e^{it})=\operatorname{rank} N(e^{it})$ is a measurable function and for every $n\geq 1$ we have*

$$E_n=\{e^{it}|d_N(e^{it})\geq n\}^=.$$

Proof. Let $\{e_j\}_j$ be an orthonormal basis of $\mathfrak{E}$. We denote by f_j the bounded measurable functions $f_j(e^{it})=N(e^{it})e_j$. Obviously the set $\{f_j(e^{it})\}_j$ generates $(N(e^{it})\mathfrak{E})^-$ for every $e^{it}\in C$, and therefore by [8], Ch. II, Prop. 9 it follows that the family $\mathfrak{H}(e^{it})=(N(e^{it})\mathfrak{E})^-$, supplied with the notion of measurability induced by the constant field $\mathfrak{K}(e^{it})=\mathfrak{E}$, is a measurable field of Hilbert spaces. Now we infer by [8], Ch. II, Prop. 1 that the function d_N is measurable. Moreover, by [8], Ch. II, Prop. 7 we have $\mathfrak{N}=\int_C^{\oplus}\mathfrak{H}(e^{it})\,dm$, and so U is the diagonal operator $\int_C^{\oplus} e^{it}\,dm$. Denoting by F_m the measurable sets $F_m=\{e^{it}|d_N(e^{it})=m\}$ $(m=1, 2, \ldots; \aleph_0)$

and applying [8], Ch. II, Prop. 3, we get that

$$U \cong \int_C^{\oplus} e^{it}\,dm \cong \bigoplus_m \Big(\int_{F_m}^{\oplus} e^{it}\,dm \Big) \cong \bigoplus_m M_{F_m}^{(m)} \cong \bigoplus_n M_{E_n},$$

where $E_n = \{e^{it} | d_N(e^{it}) \geqq n\}^=$. (For an arbitrary operator S, $S^{(m)}$ denotes the direct sum of m copies of S.)

Taking into account this Lemma we get a characterization for the measurable subsets in the Jordan model of a c. n. u. C_{11}-contraction. Namely, we have

Corollary 1. *If T is a c. n. u. contraction of class C_{11} on a (separable) Hilbert space $\mathfrak{H}$ and $M = M_{E_1} \oplus M_{E_2} \oplus \dots$ is its Jordan model, then $d_T(e^{it})$ is a measurable function, and for every natural number n we have*

$$E_n = \{e^{it} \,|\, d_T(e^{it}) \geqq n\}^=.$$

We shall frequently use the following:

Lemma 2. *If $T \in \mathcal{L}(\mathfrak{H})$ and $\mathfrak{H}_n \in \operatorname{Hyp\,lat} T$ $(n=1, 2, \dots)$ are such that $\mathfrak{H} = \bigvee_{n \geqq 1} \mathfrak{H}_n$ and $T | \mathfrak{H}_n$ has property (P) for every n, then T has property (P).*

Lemma 3. *Let U be an a. c. u. operator on the separable Hilbert space $\mathfrak{H}$, let $M = M_{E_1} \oplus M_{E_2} \oplus \dots \in \mathcal{L}(\mathfrak{K})$ be its canonical functional model, and let E be the set defined by $E = \bigcap_{n \geqq 1} E_n$. Then the following conditions are equivalent:*

$$\text{(i)} \quad U \in \mathcal{P}; \qquad \text{(ii)} \quad m(E) = 0.$$

Proof.

a) Let us assume that $m(E) > 0$. Then

$$\begin{aligned} U &\cong M_{E_1} \oplus M_{E_2} \oplus \dots \cong (M_{E_1 \setminus E} \oplus M_{E \setminus E_2} \oplus \dots) \oplus (M_E \oplus M_E \oplus \dots) \cong \\ &\cong (M_{E_1 \setminus E} \oplus M_{E_2 \setminus E} \oplus \dots) \oplus (M_E \oplus M_E \oplus \dots) \oplus (M_E \oplus M_E \oplus \dots) \cong \\ &\cong (M_{E_1} \oplus M_{E_2} \oplus \dots) \oplus (M_E \oplus M_E \oplus \dots) = M \oplus M_E^{(\aleph_0)}. \end{aligned}$$

It is evident that $M_E^{(\aleph_0)} \notin \mathcal{P}$. Therefore $M \oplus M_E^{(\aleph_0)} \notin \mathcal{P}$, and so $U \notin \mathcal{P}$.

b) Let us assume that $m(E) = 0$. For every n let $\mathfrak{H}_n$ and $\mathfrak{K}_n$ be the subspaces defined by $\mathfrak{H}_n = \chi_{CE_n}(U)\mathfrak{H}$ and $\mathfrak{K}_n = \chi_{CE_n}(M)\mathfrak{K} = L^2(E_1 \setminus E_n) \oplus \dots \oplus L^2(E_{n-1} \setminus E_n)$. Since $M | \mathfrak{K}_n$ has finite multiplicity, and $U | \mathfrak{H}_n$ is unitary equivalent to $M | \mathfrak{K}_n$, we infer by [3], Lemma 2.5 that $U | \mathfrak{H}_n$ belongs to $\mathcal{P}$ for every n. On the other hand $\mathfrak{H}_n$ is a hyperinvariant subspace of U for every n, and in virtue of the assumption $\bigvee_{n \geqq 1} \mathfrak{H}_n = \mathfrak{H}$. The Proposition follows by Lemma 2.

We shall need yet the following:

Lemma 4. *If T is a c. n. u. contraction of class C_{11} on a separable Hilbert space $\mathfrak{H}$ and $\Theta_T(e^{it})^* \Theta_T(e^{it}) \geqq \delta$ holds a.e. for some constant $\delta > 0$, then T is similar to a unitary operator. (Here Θ_T denotes the characteristic function of T.)*

Proof. We infer by Propositions [4], V.7.1 and V.4.1 that Θ_T has an outer function scalar multiple u such that $|u(e^{it})|\geqq\delta^{1/2}$ a.e. Then $\|\Theta_T(\lambda)^{-1}\|$ has a bound independent of λ, and this implies by [4], Theorem IX.1.2 that T is similar to a unitary operator.

We are now able to prove the sufficiency.

Proposition 1. *Let T be a c. n. u. contraction of class C_{11} on a (separable) Hilbert space $\mathfrak{H}$, and let $M=M_{E_1}\oplus M_{E_2}\oplus\dots$ be its Jordan model. If $m(\bigcap_{n\geq 1} E_n)=0$, then $T\in\mathcal{P}$.*

Proof. Let $\Theta\in H^\infty(\mathcal{L}(\mathfrak{E}))$ coincide with the characteristic function of T. Let $N\in L^\infty(\mathcal{L}(\mathfrak{E}))$ be the function defined by $N(e^{it})=[\Theta(e^{it})^*\Theta(e^{it})]^{1/2}=[I-\Delta^2(e^{it})]^{1/2}$. In virtue of Corollary 1 we infer by the assumption that

$$d_T(e^{it}) = d_{\Delta^2}(e^{it}) < \infty \quad \text{a.e.} \tag{1}$$

On the other hand since $T\in C_{11}$, it follows that Θ is outer from both sides, therefore $N(e^{it})$ is a quasi-affinity a.e. (Cf. [4], VI.3.5 and V.2.4.) Now we infer easily from these facts that $N(e^{it})$ is invertible a.e. Therefore its lower bound function $m(e^{it})=\inf\{\langle N(e^{it})e, e\rangle | e\in\mathfrak{E}, \|e\|=1\}$ is positive

$$m(e^{it}) > 0 \quad \text{a.e.} \tag{2}$$

For every natural number n let α_n be the measurable set defined by

$$\alpha_n = \left\{e^{it} | m(e^{it}) > \frac{1}{n}\right\}. \tag{3}$$

It is evident that $\{\alpha_n\}_n$ is increasing:

$$\alpha_1 \subseteqq \alpha_2 \subseteqq \cdots. \tag{4}$$

Moreover, in virtue of (2) we have

$$m(C\setminus(\bigcup_{n\geq 1}\alpha_n)) = 0. \tag{5}$$

By the proof of Theorem VII.5.2 of [4], T has hyperinvariant subspaces $\mathfrak{H}_n$, such that

$$\mathfrak{H}_1 \subseteqq \mathfrak{H}_2 \subseteqq \cdots, \quad \bigvee_{n\geq 1} \mathfrak{H}_n = \mathfrak{H}; \tag{6}$$

$$\Theta_n(e^{it})^*\Theta_n(e^{it}) = \begin{cases}\Theta(e^{it})^*\Theta(e^{it}) & \text{a.e. on } \alpha_n\\ I & \text{a.e. on } C\alpha_n,\end{cases} \text{[1]} \tag{7}$$

[1] Here and in the sequel we also use the notation $C\alpha$ for the set $C\setminus\alpha$, where α is any subset of C.

where Θ_n denotes a contractive analytic function such that the purely contractive part of Θ_n coincides with the characteristic function of $T|\mathfrak{H}_n$;

(8) $$T_n = T|\mathfrak{H}_n \in C_{11} \quad \text{for every } n.$$

We infer by Lemma 4 that, for every n, T_n is similar to a unitary operator.

Quasi-similar unitary operators being unitarily equivalent T_n is similar to its Jordan model $M_n = M_{E_1^{(n)}} \oplus M_{E_2^{(n)}} \oplus \dots$. We infer by (7) that

(9) $$d_{T_n}(e^{it}) \leqq d_T(e^{it}) \quad \text{a.e.},$$

and it follows by (1) that

(10) $$d_{T_n}(e^{it}) < \infty \quad \text{a.e.}$$

By Corollary 1 and Lemma 3 we see that $M_n \in \mathcal{P}$. Since similarity preserves property (P), so for every n

(11) $$T_n \in \mathcal{P}.$$

Taking into account (6) and (11), we infer by Lemma 2, that $T \in \mathcal{P}$. The proof is finished.

4. Preparing for the proof of necessity we consider some Lemmas concerning a. c. u. operators.

Lemma 5. *Let U_1 and U_2 be a. c. u. operators having property (P). Then the operator $U = U_1 \oplus U_2$ has also property (P).*

Proof. Let $M_1 = M_{E_1} \oplus M_{E_2} \oplus \dots \in \mathcal{L}(\mathfrak{H}')$ and $M_2 = M_{F_1} \oplus M_{F_2} \oplus \dots \in \mathcal{L}(\mathfrak{H}'')$ be the canonical functional models of the operators U_1 and U_2 respectively. It is enough to prove that the operator $M = M_1 \oplus M_2 \in \mathcal{L}(\mathfrak{H} = \mathfrak{H}' \oplus \mathfrak{H}'')$ has the property (P).

Taking into account that the sequences $\{E_n\}_n$ and $\{F_n\}_n$ are decreasing we infer by Lemma 3 that $m(\bigcap_{n\geq 1} (E_n \cup F_n)) = 0$. Therefore the hyperinvariant subspaces $\mathfrak{H}_n$, defined by $\mathfrak{H}_n = \chi_{C(E_n \cup F_n)} \mathfrak{H}$ $(n = 1, 2, \dots)$, span the space $\mathfrak{H}$. Moreover $M|\mathfrak{H}_n$ has finite multiplicity, and so it belongs to $\mathcal{P}$ by Lemma 3. It follows that the operator M also has the property (P).

Lemma 6. *Let $U_1, U_2, \dots$ be a. c. u. operators. If $m(\bigcap_{n\geq 1} \sigma(\bigoplus_{k\geq n} U_k)) > 0$, then there exists a strictly increasing sequence $\{n_k\}_k$ of natural numbers such that $n_1 = 0$ and $m\left(\bigcap_{k\geq 1} \sigma\left(\bigoplus_{l=n_k+1}^{n_{k+1}} U_l\right)\right) > 0$.* ($\sigma(T)$ denotes the spectrum of T.)

Proof.

a) First of all we show that $\lim_{n\to\infty} m\left(\sigma\left(\bigoplus_{k=1}^{n} U_k\right)\right) = m\left(\sigma\left(\bigoplus_{k=1}^{\infty} U_k\right)\right)$. If $E_n(\cdot)$ denotes the spectral measure of U_n for every n, then $E(\cdot)=\bigoplus_{n=1}^{\infty} E_n(\cdot)$ will be the spectral measure of the a. c. u. operator $U=\bigoplus_{n=1}^{\infty} U_n$. Therefore $E\left(\sigma\left(\bigoplus_{k=1}^{\infty} U_k\right)\setminus\left(\bigcup_{n=1}^{\infty}\sigma\left(\bigoplus_{k=1}^{n} U_k\right)\right)\right)=0$, and so $m\left(\sigma\left(\bigoplus_{k=1}^{\infty} U_k\right)\setminus\left(\bigcup_{n=1}^{\infty}\sigma\left(\bigoplus_{k=1}^{n} U_k\right)\right)\right)=0$. Since $\sigma\left(\bigoplus_{k=1}^{\infty} U_k\right)=\left(\bigcup_{n=1}^{\infty}\sigma\left(\bigoplus_{k=1}^{n} U_k\right)\right)^{-}$, we have $\lim_{n\to\infty} m\left(\sigma\left(\bigoplus_{k=1}^{n} U_k\right)\right)=m\left(\sigma\left(\bigoplus_{k=1}^{\infty} U_k\right)\right)$.

b) Let σ denote the set $\sigma=\bigcap_{n\geq 1}\sigma(\bigoplus_{k\geq n} U_k)$. Let us assume that we have defined $0=n_1<n_2<\ldots<n_r$ such that for every $1\leq k\leq r-1$ we have $m\left(\sigma\setminus\sigma\left(\bigoplus_{l=n_k+1}^{n_{k+1}} U_l\right)\right)<\frac{m(\sigma)}{4^k}$. Applying the result of a) we infer that the sequence $\left\{m\left(\sigma\setminus\sigma\left(\bigoplus_{l=n_r+1}^{n} U_l\right)\right)\right\}_n$ tends to zero. Therefore there exists an index $n_{r+1}>n_r$ such that $m\left(\sigma\setminus\sigma\left(\bigoplus_{l=n_r+1}^{n_{r+1}} U_l\right)\right)<\frac{m(\sigma)}{4^r}$. The sequence defined by recursion in this way has the property that $m\left(\sigma\setminus\left(\bigcap_{k=1}^{\infty}\sigma\left(\bigoplus_{l=n_k+1}^{n_{k+1}} U_l\right)\right)\right)<\sum_{k=1}^{\infty}\frac{m(\sigma)}{4^k}<m(\sigma)$. Therefore $m\left(\bigcap_{k=1}^{\infty}\sigma\left(\bigoplus_{l=n_k+1}^{n_{k+1}} U_l\right)\right)>0$, and the proof is finished.

Lemma 7. *Let $U_1\in\mathscr{L}(\mathfrak{H}_1)$, $U_2\in\mathscr{L}(\mathfrak{H}_2)$, ... be a. c. u. operators having property (P). Then the a. c. u. operator $U=\bigoplus_{n=1}^{\infty} U_n\in\mathscr{L}(\mathfrak{H})$ has property (P) if and only if $m(\bigcap_{n\geq 1}\sigma(\bigoplus_{k\geq n} U_k))=0$.*

Proof.

a) Let us assume that $m(\bigcap_{n\geq 1}\sigma(\bigoplus_{k\geq n} U_k))>0$. In virtue of Lemma 6 there exists a sequence $\{n_k\}_k$, $(n_1=0)$, such that $m(\sigma)>0$, where $\sigma=\bigcap_{k\geq 1}\sigma(V_k)$ and $V_k=\bigoplus_{l=n_k+1}^{n_{k+1}} U_l$ for every natural number k. Then for every k we can decompose V_k into the direct sum $V_k=V_k'\oplus V_k''$ such that V_k' is unitary equivalent to M_σ. Let $X_k'\in\mathscr{I}(V_k', V_{k+1}')$ be a unitary operator, and $X_k''\in\{V_k''\}'$ be the identity operator $(k=1, 2, \ldots)$. In this way we get an injection $X\in\{U\}'$ which is not a quasi-surjection. Therefore $U\notin\mathscr{P}$.

b) Let us assume now that $m(\bigcap_{n\geq 1} F_n)=0$, where $F_n=\sigma\left(\bigoplus_{k=n}^{\infty} U_k\right)$. Then the hyperinvariant subspaces $\mathfrak{M}_n=\chi_{CF_n}(U)\mathfrak{H}$ $(n=1, 2, \ldots)$ of U span the space $\mathfrak{H}$: $\bigvee_{n\geq 1}\mathfrak{M}_n=\mathfrak{H}$. On the other hand, for every natural number k, $\chi_{CF_n}(U_k)\mathfrak{H}_k$ reduces

U_k, and so $U_k|\chi_{CF_n}(U_k)\mathfrak{H}_k \in \mathscr{P}$. Since $U|\mathfrak{M}_n \cong \bigoplus_{k=1}^{n-1} U_k|\chi_{CF_n}(U_k)\mathfrak{H}_k$, we infer by Lemma 5 that $U|\mathfrak{M}_n \in \mathscr{P}$ for every n. Therefore $U \in \mathscr{P}$, and this completes the proof.

Now we are ready to prove:

Proposition 2. *Let T be a c. n. u. contraction of class C_{11} on a (separable) Hilbert space $\mathfrak{H}$, and let $M = M_{E_1} \oplus M_{E_2} \oplus \dots$ be its Jordan model on the Hilbert space $\mathfrak{K}$. If $m(\bigcap_{n\geq 1} E_n) > 0$, then $T \notin \mathscr{P}$.*

Proof.

a) Since T is quasi-similar to the unitary operator M, we infer by [7] that there exist a basic system $\{\mathfrak{H}_n\}_n$ of invariant subspaces of T, and a reducing decomposition $\mathfrak{K} = \bigoplus_n \mathfrak{K}_n$ of $\mathfrak{K}$ such that for every n $T_n = T|\mathfrak{H}_n$ is similar to the a. c. u. operator $U_n = M|\mathfrak{K}_n$. For every n let $C_n \in \mathscr{I}(U_n, T_n)$ be an affinity, and let P_n denote the canonical projection of $\mathfrak{H}$ onto $\mathfrak{H}_n$ determined by the decomposition $\mathfrak{H} = \mathfrak{H}_n \dotplus (\bigvee_{k\neq n} \mathfrak{H}_k)$.

b) We can reduce the proof to the following two special cases:

(i) There exists an n such that $U_n \notin \mathscr{P}$.

(ii) $m(\bigcap_{n\geq 1} \sigma(U_n)) > 0$.

Indeed, assuming that $U_n \in \mathscr{P}$ for every n, and taking into account that $M = \bigoplus_{n\geq 1} U_n \notin \mathscr{P}$ (cf. Lemma 3), we infer by Lemmas 7 and 6 that there exists a sequence $\{n_k\}_k$, $(n_1 = 0)$, such that $m\left(\bigcap_{k\geq 1} \sigma\left(\bigoplus_{l=n_k+1}^{n_{k+1}} U_l\right)\right) > 0$. Replacing the basic system $\{\mathfrak{H}_n\}_n$ by $\{\mathfrak{H}'_k\}_k$, where $\mathfrak{H}'_k = \mathfrak{H}_{n_k+1} \dotplus \dots \dotplus \mathfrak{H}_{n_{k+1}}$, and the affinities C_n $(n = 1, 2, \dots)$ by $C'_k = C_{n_k+1} \oplus \dots \oplus C_{n_{k+1}}$ $(k = 1, 2, \dots)$, we gain the case (ii). (It can be easily seen that for every finite index-set N_1 the linear manifolds $\dotplus_{k\in N_1} \mathfrak{H}_k$ and $(\bigvee_{k\in N_1} \mathfrak{H}_k) \dotplus (\bigvee_{k\notin N_1} \mathfrak{H}_k)$ are closed. Therefore the operators $C'_k = \bigoplus_{l=n_k+1}^{n_{k+1}} C_l$ $(k = 1, 2, \dots)$ will be affinities, and $\{\mathfrak{H}'_k\}_k$ will be a basic system.)

c) Let us assume that there exists an n such that $U_n \notin \mathscr{P}$. It can be supposed that $n = 1$. Since similarity preserves the property (P), we infer that $T_1 \notin \mathscr{P}$. Therefore there exists an injection $X_1 \in \{T_1\}'$ which is not a quasi-surjection. Let $\{\alpha_n\}_n$ be a sequence of positive numbers such that $\sum_{n=1}^{\infty} \alpha_n \|P_n\| < \infty$, and let $X \in \{T\}'$ be the operator defined by $Xf = \alpha_1 X_1 P_1 f + \sum_{k=2}^{\infty} \alpha_k P_k f$ $(f \in \mathfrak{H})$. If $Xf = 0$ $(f \in \mathfrak{H})$, then for every n $P_n f = 0$, and we can prove by induction that $f \in (\bigvee_{k\geq n} \mathfrak{H}_k)$ for every n. Therefore $f = 0$, and so X is an injection. On the other hand, $(\operatorname{ran} X)^- =$

$=(\operatorname{ran} X_1)^- \dotplus (\bigvee_{n\geq 2} \mathfrak{H}_n) \neq \mathfrak{H}_1 \dotplus (\bigvee_{n\geq 2} \mathfrak{H}_n) = \mathfrak{H}$, that is X is not a quasi-surjection. Therefore T does not belong to the class $\mathscr{P}$.

d) Let us now suppose that $m(\sigma)>0$, where $\sigma = \bigcap_{n\geq 1} \sigma(U_n)$. Then for every n there exists a reducing decomposition $\mathfrak{K}_n = \mathfrak{K}_n' \oplus \mathfrak{K}_n''$ such that $U_n|\mathfrak{K}_n'$ is unitary equivalent to the operator M_σ. Let $\mathfrak{H}_n'$ and $\mathfrak{H}_n''$ denote the subspaces defined by $\mathfrak{H}_n' = C_n \mathfrak{K}_n'$, $\mathfrak{H}_n'' = C_n \mathfrak{K}_n''$. Then $\mathfrak{H}_n' \dotplus \mathfrak{H}_n'' = \mathfrak{H}_n$ and $T_n' = T_n|\mathfrak{H}_n'$ is similar to $T_{n+1}' = T_{n+1}|\mathfrak{H}_{n+1}'$ for every n.

Let $X_n \in \mathscr{I}(T_n', T_{n+1}')$ be an affinity, and let P_n' denote the canonical projection of $\mathfrak{H}_n$ onto $\mathfrak{H}_n'$ determined by the decomposition $\mathfrak{H}_n = \mathfrak{H}_n' \dotplus \mathfrak{H}_n''$, moreover let P_n'' be the projection: $P_n'' = I_{\mathfrak{H}_n} - P_n'$. Let $\{\alpha_n\}_n$ be a sequence of positive numbers such that $\sum_{n=1}^{\infty} \alpha_n (\|X_n\| \|P_n'\| + \|P_n''\|) \|P_n\| < \infty$, and let $X \in \{T\}'$ denote the operator defined by $Xf = \sum_{n=1}^{\infty} \alpha_n (X_n P_n' + P_n'') P_n f$ $(f \in \mathfrak{H})$. As in the preceding point, it can be easily seen that X is an injection. On the other hand $(\operatorname{ran} X)^- = \mathfrak{H}_1'' \dotplus (\bigvee_{n\geq 2} \mathfrak{H}_n) \neq \mathfrak{H}_1 \dotplus (\bigvee_{n\geq 2} \mathfrak{H}_n) = \mathfrak{H}$, that is X is not a quasi-surjection. Therefore T does not have property (P), and the proof is completed.

5. In this section we consider some corollaries of Theorem 1.

Corollary 2. *Let T be a c. n. u. contraction of class C_{11}. Then T belongs to $\mathscr{P}$ if and only if its Jordan model $M = M_{E_1} \oplus M_{E_2} \oplus \dots$ does.*

Proof. Cf. Theorem 1 and Lemma 3.

Corollary 3. *Property (P) is a quasi-similarity invariant for c. n. u. C_{11}-contractions.*

Corollary 4. *If T is a c. n. u. C_{11}-contraction having property (P), then its adjoint T^* also has property (P).*

Proof. We have only to note that the adjoint of an operator of the form M_E is unitary equivalent to the operator $M_{E^\sim}$, where $E^\sim = \{e^{it} | e^{-it} \in E\}$.

Corollary 5. *Let T be a c. n. u. contraction of class C_{11} on the non-necessarily separable Hilbert space $\mathfrak{H}$. If T has property (P), then the space $\mathfrak{H}$ is separable.*

Proof. Let us assume that T has property (P) and the space $\mathfrak{H}$ is non-separable. Then there exists a decomposition $\mathfrak{H} = \bigoplus_{\alpha<\beta} \mathfrak{H}_\alpha$ reducing for T, such that for every ordinal α less than the ordinal β the space $\mathfrak{H}_\alpha$ is separable. Let $M_\alpha = \bigoplus_{n\geq 1} M_{E_{\alpha,n}}$ be the Jordan model of the operator $T_\alpha = T|\mathfrak{H}_\alpha$. Since $m(E_{\alpha,1})>0$ for every $\alpha>\beta$, and β is non-denumerable, there exist a positive number $\varepsilon>0$ and a sequence $\{\alpha_n\}_{n=1}^{\infty}$ of ordinals less than β, such that for every n we have $m(E_{\alpha_n,1})>\varepsilon$.

Let T' be the operator defined by $T'=\bigoplus_{n=1}^{\infty} T_{\alpha_n}$ on the separable Hilbert space $\mathfrak{H}'=\bigoplus_{n=1}^{\infty}\mathfrak{H}_{\alpha_n}$. Taking into account that $T\in\mathcal{P}$, we infer that $T'\in\mathcal{P}$, and $T_\alpha\in\mathcal{P}$ for every $\alpha<\beta$. T' being quasi-similar to the unitary operator $\bigoplus_{n=1}^{\infty} M_{\alpha_n}$, it follows that $\bigoplus_{n=1}^{\infty} M_{\alpha_n}$ is unitary equivalent to the Jordan model of T'. By Corollary 2 we infer that $\bigoplus_{n=1}^{\infty} M_{\alpha_n}\in\mathcal{P}$, and $M_{\alpha_n}\in\mathcal{P}$ for every n. Now it follows by Lemma 7 that $\lim_{n\to\infty} m(\sigma(\bigoplus_{k\geqq n} M_{\alpha_k}))=0$.

On the other hand for every n we have $m(\sigma(\bigoplus_{k\geqq n} M_{\alpha_k}))\geqq m(\sigma(M_{\alpha_n}))=m(E_{\alpha_n,1})>\varepsilon$, what is a contradiction. Therefore the space $\mathfrak{H}$ can't be separable, and the proof is completed.

Corollary 6. *Let T be a c. n. u. contraction of class C_{11}. If T has property (P) and $\mathfrak{L}$ is an invariant subspace of T such that $T|\mathfrak{L}\in C_{11}$, then $T|\mathfrak{L}$ has property (P) also.*

Proof. We infer by [4], VI.2.3, VII.1.1, VII.2.1 and VII.3.3 that $d_{T|\mathfrak{L}}(e^{it})\leqq d_T(e^{it})$ a.e. Now it follows by Corollary 1 and Theorem 1 that $T|\mathfrak{L}$ has property (P).

Corollary 7. *Let T_1 and T_2 be c. n. u. contractions of class C_{11}. If T_1 and T_2 belong to the class $\mathcal{P}$, then the direct sum $T_1\oplus T_2$ has property (P) also.*

Proof. We have only to refer to Corollary 2 and Lemma 5.

Corollary 8. *Let $T_1, T_2, \ldots$ be c. n. u. contractions of class C_{11} having property (P). Then the contraction $T=\bigoplus_{n=1}^{\infty} T_n$ belongs to the class $\mathcal{P}$ if and only if the series $\sum_{n=1}^{\infty} d_{T_n}(e^{it})$ converges a.e.*

Proof. Since $T_n\in\mathcal{P}$, it follows that $d_{T_n}(e^{it})<\infty$ a.e., and the Jordan model M_n of T_n has property (P). (Cf. Theorem 1, Corollary 1 and Lemma 3.) On the other hand we infer by Corollary 2 that the condition $T\in\mathcal{P}$ is equivalent to the condition $\bigoplus_{n=1}^{\infty} M_n\in\mathcal{P}$. But this latter is equivalent to $m(\bigcap_{n\geqq 1}\sigma(\bigoplus_{k\geqq n} M_k))=0$ by Lemma 7. On account of Corollary 1 and the proof of Lemma 6 we see that $m(\bigcap_{n\geqq 1}\sigma(\bigoplus_{k\geqq n} M_k))=0$ holds if and only if $\sum_{n=1}^{\infty} d_{T_n}(e^{it})<\infty$ a.e., and this completes the proof.

6. Finally we intend to characterize the non-necessarily c.n.u. contractions of class C_{11} having property (P). First of all we prove the following:

Lemma 8. *Let $T\in\mathcal{L}(\mathfrak{H})$ be a c. n. u. contraction of class C_{11}, and let $U\in\mathcal{L}(\mathfrak{N})$ be an a. c. u. operator. If both T and U have property (P), then their direct sum $S=T\oplus U\in\mathcal{L}(\mathfrak{E})$ belongs to $\mathcal{P}$ also.*

Proof. Let $M\in\mathcal{L}(\mathfrak{K})$ denote the Jordan model of T. By [7] there exist a basic system $\{\mathfrak{L}_k\}_k$ of invariant subspaces of T, and a decomposition $\mathfrak{K}=\bigoplus_{k\geq 1}\mathfrak{B}_k$ of $\mathfrak{K}$ reducing for M, such that for every k $T|\mathfrak{L}_k$ is similar to $M_k'=M|\mathfrak{B}_k$. Let $C_k\in\mathcal{I}(M_k', T|\mathfrak{L}_k)$ be an affinity $(k=1, 2, \ldots)$.

Since $T\in\mathcal{P}$, we infer by Corollary 2 that $M\in\mathcal{P}$ also. Now by Lemma 7 it follows that $m(\bigcap_{n\geq 1} C\sigma_n)=0$, where $\sigma_n=C\sigma(\bigoplus_{k>n} M_k')$ $(n=1, 2, \ldots)$. For every n let $\mathfrak{K}_n$, $\mathfrak{K}_n'$, $\mathfrak{H}_n$, $\mathfrak{H}_n'$ be defined by $\mathfrak{K}_n=\chi_{\sigma_n}(M)\mathfrak{K}=\bigoplus_{k\geq 1}\mathfrak{K}_{n,k}$, $\mathfrak{K}_n'=\chi_{C\sigma_n}(M)\mathfrak{K}=\bigoplus_{k\geq 1}\mathfrak{K}_{n,k}'$, where $\mathfrak{K}_{n,k}=\chi_{\sigma_n}(M_k')\mathfrak{B}_k$, $\mathfrak{K}_{n,k}'=\chi_{C\sigma_n}(M_k')\mathfrak{B}_k$ $(k=1, 2, \ldots)$, and $\mathfrak{H}_n=\bigvee_{k\geq 1}\mathfrak{H}_{n,k}$, $\mathfrak{H}_n'=\bigvee_{k\geq 1}\mathfrak{H}_{n,k}'$, where $\mathfrak{H}_{n,k}=C_k\mathfrak{K}_{n,k}$, $\mathfrak{H}_{n,k}'=C_k\mathfrak{K}_{n,k}'$ $(k=1, 2, \ldots)$. It is clear that for every n $\mathfrak{K}_{n,k}=\{0\}$ if $k>n$, and so $\mathfrak{K}_n=\bigoplus_{k=1}^{n}\mathfrak{K}_{n,k}$. It follows that $\mathfrak{H}_{n,k}=\{0\}$ if $k>n$, that is $\mathfrak{H}_n=\mathfrak{H}_{n,1}\dotplus\ldots\dotplus\mathfrak{H}_{n,n}$. Therefore the subspaces $\mathfrak{H}_n$ and $\mathfrak{H}_n'$ are complementary: $\mathfrak{H}_n\dotplus\mathfrak{H}_n'=\mathfrak{H}$, and $T_n=T|\mathfrak{H}_n$ is similar to $M_n=M|\mathfrak{K}_n$. Moreover $T_n'=T|\mathfrak{H}_n'$ is quasi-similar to $M_n'=M|\mathfrak{K}_n'$, and $m(\sigma(M_n)\Delta\sigma_n)=m(\sigma(M_n')\Delta C\sigma_n)=0$ for every n.

For every n let the subspaces $\mathfrak{N}_n$, $\mathfrak{N}_n'$, $\mathfrak{E}_n$, $\mathfrak{E}_n'$ be defined by $\mathfrak{N}_n=\chi_{\sigma_n}(U)\mathfrak{N}$, $\mathfrak{N}_n'=\chi_{C\sigma_n}(U)\mathfrak{N}$, $\mathfrak{E}_n=\mathfrak{H}_n\oplus\mathfrak{N}_n$ and $\mathfrak{E}_n'=\mathfrak{H}_n'\oplus\mathfrak{N}_n'$. Then the decomposition $\mathfrak{E}=\mathfrak{E}_n\dotplus\mathfrak{E}_n'$ reduces S, moreover the restriction $S_n=S|\mathfrak{E}_n=T_n\oplus U_n$ $(U_n=U|\mathfrak{N}_n)$ of S onto $\mathfrak{E}_n$ is similar to $M_n\oplus U_n$, and the restriction $S_n'=S|\mathfrak{E}_n'=T_n'\oplus U_n'$ $(U_n'=U|\mathfrak{N}_n')$ of S onto $\mathfrak{E}_n'$ is quasi-similar to $M_n'\oplus U_n'$.

Let $X\in\{S\}'$ be an arbitrary operator, and let n be a natural number. Let

$$\begin{bmatrix} X_{11}^{(n)} & X_{12}^{(n)} \\ X_{21}^{(n)} & X_{22}^{(n)} \end{bmatrix}$$

be the matrix of X in the decomposition $\mathfrak{E}=\mathfrak{E}_n\dotplus\mathfrak{E}_n'$. On account of $X\in\{S\}'$ we infer that $X_{21}^{(n)}\in\mathcal{I}(S_n, S_n')$. Let $Y_n\in\mathcal{I}(M_n\oplus U_n, S_n)$ and $Z_n\in\mathcal{I}(S_n', M_n'\oplus U_n')$ be quasi-affinities. Then the operator $X_n'=Z_nX_{21}^{(n)}Y_n$ belongs to $\mathcal{I}(M_n\oplus U_n, M_n'\oplus U_n')$ and we infer by [9], Lemma 4.1 that $(\ker X_n')^\perp$ and $(\operatorname{ran} X_n')^-$ are reducing subspaces of $M_n\oplus U_n$ and $M_n'\oplus U_n'$ respectively, and $(M_n\oplus U_n)|(\ker X_n')^\perp$ is unitary equivalent to $(M_n'\oplus U_n')|(\operatorname{ran} X_n')^-$. Since $m(\sigma(M_n\oplus U_n)\Delta\sigma_n)=m(\sigma(M_n'\oplus U_n')\Delta C\sigma_n)=0$, and $M_n\oplus U_n$, $M_n'\oplus U_n'$ are a. c. u. operators, it follows that $X_n'=0$, and so $X_{21}^{(n)}=0$. Therefore $\mathfrak{E}_n\in\operatorname{Hyp\,lat}(S)$.

On the other hand we infer by the equation $m(\bigcap_{n\geq 1} C\sigma_n)=0$ that $\bigvee_{n\geq 1} \mathfrak{E}_n=\mathfrak{E}$. Since we have that $S_n\in\mathcal{P}$ for every n (cf. Lemma 5), an argument similar to the end of the proof of Proposition 1 completes the proof.

It is well-known that for every contraction T of class C_{11} on the Hilbert space $\mathfrak{H}$ there exists a (unique) "canonical" decomposition $\mathfrak{H}=\mathfrak{H}_1\oplus\mathfrak{H}_2\oplus\mathfrak{H}_3$ of $\mathfrak{H}$ reducing for T, such that $T_1=T|\mathfrak{H}_1$ is a c. n. u. contraction of class C_{11}, $T_2=T|\mathfrak{H}_2$ is an a. c. u. operator and $T_3=T|\mathfrak{H}_3$ is a singular unitary operator. (Cf. [4], I.3.2.)

Theorem 2. *Let T be a contraction of class C_{11}, and let $T=T_1\oplus T_2\oplus T_3$ be its "canonical" decomposition. Then T has property (P) if and only if T_i belongs to $\mathcal{P}$ for $i=1, 2, 3$.*

Proof. Let us assume that $T_i\in\mathcal{P}$ for $i=1, 2, 3$. (The other part of the proof is trivial.)

Let $X\in\{T\}'$ be an arbitrary operator, and let $\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22}\end{bmatrix}$ be its matrix in the decomposition $\mathfrak{H}=(\mathfrak{H}_1\oplus\mathfrak{H}_2)\oplus\mathfrak{H}_3$. Then $X_{21}Z\in\mathcal{I}(M_1\oplus T_2, T_3)$, where M_1 is the Jordan model of T_1, and $Z\in\mathcal{I}(M_1\oplus T_2, T_1\oplus T_2)$ is a quasi-affinity. Since $M_1\oplus T_2$ is an a. c. u. operator and T_3 is a singular unitary operator, we infer by [9], Lemma 4.1 that $X_{21}Z=0$, and so $X_{21}=0$. A similar argument shows that $X_{12}=0$ also holds, therefore $\mathfrak{H}_1\oplus\mathfrak{H}_2$ and $\mathfrak{H}_3$ belong to Hyp lat (T). Applying Lemma 8 the Theorem follows.

It can be given a "canonical" functional model for an arbitrary singular unitary operator also. Now a singular measure μ plays the role of the Lebesgue measure, and the form of the space of the functional model is $L^2_\mu(E_1)\oplus L^2_\mu(E_2)\oplus\ldots$, $(E_1\supseteq E_2\supseteq\ldots)$. Lemma 3 also holds its validity if condition (ii) is replaced by $\mu(E)=0$. Taking into account the previous theorems, it can be easily seen that Corollaries 2—8 hold for arbitrary contractions of class C_{11} also.

*

In a subsequent paper shall continue the study of the class $\mathcal{P}\cap C_{11}$. Among others we shall show that, for quasi-similar $\mathcal{P}\cap C_{11}$-contractions, the lattices of C_{11}-invariant subspaces are isomorphic. (An invariant subspace $\mathfrak{L}$ for T is called C_{11}-invariant if $T|\mathfrak{L}\in C_{11}$.)

References

1] B. Sz.-Nagy and C. Foiaş, On injections, intertwining contractions of class C_0, *Acta Sci. Math.*, **40** (1978), 163—167.

[2] H. Bercovici, C_0-Fredholm operators. II, *Acta Sci. Math.*, **42** (1980), 3—42.

[3] P. Y. Wu, On a conjecture of Sz.-Nagy and Foiaş , *Acta Sci. Math.*, **42** (1980), 331—338.

[4] B. Sz.-Nagy, C. Foiaş, *Harmonic Analysis of Operators on Hilbert Space*, North Holland — Akadémiai Kiadó (Amsterdam—Budapest, 1970).

[5] N. Dunford and J. T. Schwartz, *Linear operators*. II, Interscience (New York, London, 1963).

[6] P. Y. Wu, Jordan model for weak contractions, *Acta Sci. Math.*, **40** (1978), 189—196.

[7] C. Apostol, Operators quasi-similar to a normal operator, *Proc. Amer. Math. Soc.*, **53** (1975), 104—106.

[8] J. Dixmier, *Les algèbres d'opérateurs dans l'espace Hilbertien*, Gauthier-Villars (Paris, 1969).

[9] R. G. Douglas, On the operator equation $S^*XT=X$ and related topics, *Acta Sci. Math.*, **30** (1960), 19—32.

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