Acta Sci. Math., 43 (1981), 41—43

On the Korovkin closure

V. KOMORNIK

Let X be a topological space and denote by $C(X)$ the vector lattice of all continuous real functions defined on X. Given a linear subspace $\mathfrak{H}\subset C(X)$, we denote by $\mathfrak{H}_0$, as usual, the set of all $\mathfrak{H}$-bounded, continuous functions:

$$\mathfrak{H}_0 \equiv \{f\in C(X):\ \exists h_1, h_2\in\mathfrak{H} \text{ with } h_1 \leqq f \leqq h_2\}.$$

We define the Korovkin closure of $\mathfrak{H}$ as the set of all functions $f\in\mathfrak{H}_0$ having the following property: "For every net $(L_i)_{i\in I}$ of positive linear maps L_i: $\mathfrak{H}_0\to\mathfrak{H}_0$ such that L_ih converges to h pointwise on X for all $h\in\mathfrak{H}$, L_if also converges to f pointwise on X."

We denote the Korovkin closure of the linear subspace $\mathfrak{H}$ by Kor ($\mathfrak{H}$). The following inclusions are obvious:

$$\mathfrak{H} \subset \operatorname{Kor}(\mathfrak{H}) \subset \mathfrak{H}_0 \subset C(X).$$

This paper is devoted to the characterization of Kor ($\mathfrak{H}$) in some general cases. We shall extend some results of H. BAUER [1] and K. DONNER [2].

To formulate our theorems, we recall the definition due to H. Bauer, of the space of $\mathfrak{H}$-affine functions. This space, denoted by $\hat{\mathfrak{H}}$, consists of all $f\in\mathfrak{H}_0$ satisfying the equality

$$\sup\{h\in\mathfrak{H}:\ h \leqq f\} = \inf\{h\in\mathfrak{H}:\ h \geqq f\}.$$

We shall prove:

Theorem 1. *If X is locally compact and Hausdorff, then for all linear subspaces $\mathfrak{H}$ of $C(X)$ the following identity holds:*

$$\operatorname{Kor}(\mathfrak{H}) = \hat{\mathfrak{H}}.$$

Remark 1. This identity was proved by H. BAUER [1], Theorem 3.3, in the special case when the linear subspace $\mathfrak{H}$ is adapted, i.e. satisfies the following three conditions:

(i) $\mathfrak{H} = \mathfrak{H}^+ - \mathfrak{H}^+$ where $\mathfrak{H}^+ \equiv \{h\in\mathfrak{H}:\ h \geqq 0\}$,
(ii) $\forall x\in X\ \exists h_x\in\mathfrak{H}:\ h_x(x) \neq 0$,
(iii) $\forall h\in\mathfrak{H}\ \exists h_1\in\mathfrak{H}\ \forall\varepsilon > 0$: the closure of $\{t\in X:\ |h(t)| > \varepsilon\cdot|h_1(t)|\}$ is compact.

Received August 30, 1980.

Remark 2. Recently K. Donner [2] proved a general theorem which can be applied to our situation when $\mathfrak{H}_0$ is a vector lattice. But $\mathfrak{H}_0$ is a vector lattice if and only if $\mathfrak{H}=\mathfrak{H}^+-\mathfrak{H}^+$. Thus Donner's result yields that special case of our theorem when the linear subspace $\mathfrak{H}$ satisfies the condition $\mathfrak{H}=\mathfrak{H}^+-\mathfrak{H}^+$.

Theorem 1 will be got as a special case of the following more general one:

Theorem 2. *If X is a topological space and $\mathfrak{H}$ is a linear subspace of $C(X)$, then each of the following five conditions implies the identity* $\operatorname{Kor}(\mathfrak{H})=\hat{\mathfrak{H}}$:

(a) *X is locally compact and totally regular,*
(b) $\mathfrak{H}=\mathfrak{H}^+-\mathfrak{H}^+$,
(c) $\dim \mathfrak{H}<\infty$,
(d) *all the functions in $\mathfrak{H}$ are bounded,*
(e) *each point of X has a neighbourhood in the weak topology, induced by $C(X)$, where all the functions from $\mathfrak{H}$ are bounded.*

In the proof we shall use the following lemma, essentially proved by H. Bauer:

Lemma. *For any $g\in\mathfrak{H}_0$, $x\in X$ and $c\in \boldsymbol{R}$ such that*

$$\sup\{h(x)\colon\ g \geqq h\in\mathfrak{H}\} \leqq c \leqq \inf\{h(x)\colon\ g \leqq h\in\mathfrak{H}\},$$

there exists a positive linear functional $\mu\colon \mathfrak{H}_0\to\boldsymbol{R}$ with

(A) $\mu(g) = c$, *and*
(B) $\mu(h) = h(x)$ *for all* $h\in\mathfrak{H}$.

Proof (compare with [1; 2.2 Lemma]). On $\mathfrak{H}_0$ the map $f\mapsto\inf\{h(x)\colon f\leqq h\in\mathfrak{H}\}$ is a sublinear functional p. This functional majorizes the linear form $\lambda\cdot g\mapsto\lambda\cdot c$ defined on the linear subspace of $\mathfrak{H}_0$ generated by g. The Hahn—Banach theorem hence implies the existence of a linear form μ on $\mathfrak{H}_0$ satisfying (A) and the relation $\mu\leqq p$. (B) and the positivity of μ follow from this latter inequality.

Proof of Theorem 2. The relation $\hat{\mathfrak{H}}\subset\operatorname{Kor}(\mathfrak{H})$ is well-known (see [1], Corollary 1.3). Conversely, we shall show that given any $g\in\mathfrak{H}_0\setminus\hat{\mathfrak{H}}$, g does not belong to $\operatorname{Kor}(\mathfrak{H})$.

As condition (e) is weaker than conditions (a), (c), (d), we treat only cases (b) and (e).

Because of $g\notin\hat{\mathfrak{H}}$ there is a point $x\in X$ and a number $c\in\boldsymbol{R}$ such that

$$\sup\{h(x)\colon\ g \geqq h\in\mathfrak{H}\} < c < \inf\{h(x)\colon\ g \leqq h\in\mathfrak{H}\},\quad c\neq g(x).$$

Let us fix by the above Lemma a positive linear functional μ satisfying (A) and (B). By the relation $g\in\mathfrak{H}_0\setminus\hat{\mathfrak{H}}$ we can choose a function h_0 with

(C) $h_0\in\mathfrak{H},\quad h_0\geqq 0,\quad h_0(x)>1.$

(Indeed, for any functions $h_1, h_2\in\mathfrak{H}$, $h_1\leqq g\leqq h_2$ we have $h_2-h_1\geqq 0$ and $h_2(x)-h_1(x)>0$.)

If condition (b) is satisfied, fix a neighbourhood base $\mathscr{B}$ of x in the weak topology induced by $C(X)$ so as to satisfy

(D) $h_0(t) > 1$ for any $t \in U \in \mathscr{B}$.

If condition (e) is satisfied, fix a neighbourhood base $\mathscr{B}$ of x in the weak topology induced by $C(X)$, satisfying over and above (D) also the following condition:

(E) Each function from $\mathfrak{H}$ is bounded on each element of $\mathscr{B}$.

Assign to every $U \in \mathscr{B}$ a function $q_U \in C(X)$ such that

(F) $0 \leqq q_U \leqq 1$, $q_U(x) = 1$, $q_U(t) = 0$ for all $t \in X \setminus U$.

(This is possible because the weak topology is totally regular.)

For $U \in \mathscr{B}$ and $f \in \mathfrak{H}_0$ we define

$$L_U f \equiv \mu(f) \cdot q_U + f - f \cdot q_U.$$

Obviously, L_U: $\mathfrak{H}_0 \to C(X)$ is a positive linear map. Moreover, L_U: $\mathfrak{H}_0 \to \mathfrak{H}_0$ is also true: being $\mathfrak{H}_0$ a linear subspace, this will follows from the two relations $q_U \in \mathfrak{H}_0$ and $f \cdot q_U \in \mathfrak{H}_0$ (for all $U \in \mathscr{B}$ and $f \in \mathfrak{H}_0$). The first relation follows from (C), (D) and (F): $0 \leqq q_U \leqq h_0$. If condition (b) is satisfied, then there is an $h \in \mathfrak{H}$ with $-h \leqq f \leqq h$ from which we get $-h \leqq f \cdot q_U \leqq h$, proving the second relation. If condition (e) is satisfied, then $f \cdot q_U$ is bounded by (E), (F) and vanishes on $X \setminus U$ then there is therefore by (C) and (D) a real number d with $-d \cdot h_0 \leqq f \cdot q_U \leqq d \cdot h_0$. Hence again $f \cdot q_U \in \mathfrak{H}_0$.

Finally, take the net $(L_U)_{U \in \mathscr{B}}$ of positive linear maps L_U: $\mathfrak{H}_0 \to \mathfrak{H}_0$. An easy computation shows that the net $(L_U h)_{U \in \mathscr{B}}$ converges to h pointwise (moreover uniformly) on X for all $h \in \mathfrak{H}$ ($\mu(h) = h(x)$ by (B)), but the net $(L_U g)_{U \in \mathscr{B}}$ does not converge to g pointwise on X because $((L_U g)(x))_{U \in \mathscr{B}}$ is a constant net with the constant $\mu(g) = c \neq g(x)$ by (A). Thus g does not belong to the Korovkin closure of $\mathfrak{H}$ and the theorem is proved.

Remark. The results of this paper (and the proofs) remain valid if we replace pointwise convergence by uniform convergence on the compact subsets of X in the definition of the Korovkin closure. The author wishes to thank Dr. Z. Sebestyén for having followed with attention these investigations.

References

[1] H. Bauer, Theorems of Korovkin type for adapted spaces, *Ann. Inst. Fourier*, **23** (1973), 245—260.

[2] K. Donner, Korovkin closures for positive linear operators, *J. Appr. Theory*, **26** (1979), 14—25.

DEPARTMENT OF MATH. ANALYSIS, II
LORÁND EÖTVÖS UNIVERSITY
H—1445 BUDAPEST 8, PF. 323