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Models for operators with bounded characteristic function

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1. Introduction. In the theory of B. SZ.-NAGY and C. FOIAŞ [15], the characteristic function Θ_T of a completely non-unitary contraction T is used to generate a functional model for T. In addition, if Θ is an arbitrary purely contractive analytic function, then Θ can be used to generate a contraction that has Θ as its characteristic function. The SZ.-NAGY and FOIAŞ theory provides in fact a model of the minimal unitary dilation U of the contraction: U is represented as a shift acting on a subspace of the direct sum of two vector valued L^2 spaces, and the characteristic function is identified as a projection on the dilation space. (See [15, Chapter VI].)

Now suppose that T is any bounded operator on a Hilbert space $\mathfrak{H}$. The characteristic function Θ_T of T is the operator valued analytic function

$$\Theta_T(\lambda) = [-TJ_T + \lambda J_{T^*} Q_{T^*}(I-\lambda T^*)^{-1} J_T Q_T] | \mathfrak{D}_T,$$

where $J_T = \operatorname{sgn}(I-T^*T)$, $J_{T^*} = \operatorname{sgn}(I-TT^*)$, $Q_T = |I-T^*T|^{1/2}$, $Q_{T^*} = |I-TT^*|^{1/2}$, and $\mathfrak{D}_T = J_T\mathfrak{H}$. $\Theta_T(\lambda)$ is defined for those complex numbers λ for which $I-\lambda T^*$ is boundedly invertible, and takes values which are continuous operators from $\mathfrak{D}_T$ to the space $\mathfrak{D}_{T^*} = J_{T^*}\mathfrak{H}$. (See [11]; cf. [1], [3], [4], [5], [6], [8], [10], [13], [15].)

It was shown in [11] that if $\Theta(\lambda)$ is an operator valued analytic function (defined for $\lambda \in D$, the open unit disk in the complex plane), then $\Theta(\lambda)$ coincides with the characteristic function of some operator if and only if it is purely contractive and fundamentally reducible (see Sec. 2 below). This result was obtained by using a model of BALL [1], which is much less geometric than the type constructed by SZ.-NAGY and FOIAŞ. In particular, the model in [1] does not provide the interpretation of the characteristic function as a projection.

In this paper, we restrict our attention to bounded operator valued analytic functions $\Theta(\lambda)$, i.e., for which $\sup_{\lambda\in D} \|\Theta(\lambda)\| < \infty$. We are then able to obtain a functional model of the SZ.-NAGY and FOIAŞ type, which provides the extension of

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their theory that was promised in the concluding section of [11]. In Sec. 14 we describe the relationship between this model and the model of BALL [1].

Remark. Other authors [3], [4], [5], [6] have also considered the problem of representing an arbitrary $\Theta(\lambda)$ (satisfying certain conditions) as a characteristic function, but have not used a SZ.-NAGY and FOIAŞ type model. In [9], however, a model of this type is used to represent dissipative operators, with the unit disk in the SZ.-NAGY and FOIAŞ theory replaced by the upper half plane.

2. Krein spaces. Purely contractive analytic functions. A *Krein space* is a space $\mathfrak{K}$ with an indefinite inner product $[.,.]$ (i.e., $[x, x]$ can be negative for some $x\in\mathfrak{K}$) on which is defined a *fundamental symmetry* J: $J^2=I$, $[Jx, y]=[x, Jy]$, and the J-inner product $[J.,.]$ makes $\mathfrak{K}$ a Hilbert space. The topology on $\mathfrak{K}$ is that defined by the J-norm $\|x\|_J=[Jx, x]^{1/2}$. For an operator A on $\mathfrak{K}$, we denote by A^* the adjoint of A with respect to the indefinite inner product $[.,.]$. (See [2], [11].)

If $\mathfrak{A}$ and $\mathfrak{B}$ are subsets of $\mathfrak{K}$, then we write $\mathfrak{A}\perp\mathfrak{B}$ if $[a, b]=0$ for all $a\in\mathfrak{A}$ and $b\in\mathfrak{B}$. We define $\mathfrak{A}^\perp=\{x\in\mathfrak{K}: [a, x]=0 \text{ for all } a\in\mathfrak{A}\}$ and $\mathfrak{A}\ominus\mathfrak{B}=\mathfrak{A}\cap\mathfrak{B}^\perp$. A *projection* on $\mathfrak{K}$ is a continuous operator P such that $P=P^*=P^2$. A *regular* subspace of $\mathfrak{K}$ is a subspace $\mathfrak{L}$ such that $\mathfrak{L}\oplus\mathfrak{L}^\perp=\mathfrak{K}$. The regular subspaces are precisely those that are the ranges of projections (cf. [12]).

An *operator valued analytic function* is a function Θ which is defined and analytic in D, the open unit disk in the complex plane, and which takes values that are continuous operators from a Krein space $\mathfrak{D}$ to a Krein space $\mathfrak{D}_*$. Θ is said to be *purely contractive* if, for each $\lambda\in D$,

$$[\Theta(\lambda)a, \Theta(\lambda)a] < [a, a] \quad (a\in\mathfrak{D},\ a\neq 0)$$

and

$$[\Theta(\lambda)^*a_*, \Theta(\lambda)^*a_*] < [a_*, a_*] \quad (a_*\in\mathfrak{D}_*,\ a_*\neq 0).$$

Let $\Theta_0=\Theta(0)$. We call Θ *fundamentally reducible* if there are fundamental symmetries on $\mathfrak{D}$ and $\mathfrak{D}_*$ that commute with $\Theta_0^*\Theta_0$ and $\Theta_0\Theta_0^*$, respectively [11, Sec. 3].

The spaces $\mathfrak{D}_T$ and $\mathfrak{D}_{T^*}$, defined in Sec. 1, are Krein spaces with the indefinite inner products

$$[x, y] = (J_Tx, y) \quad (x, y\in\mathfrak{D}_T) \quad \text{and} \quad [x, y] = (J_{T^*}x, y) \quad (x, y\in\mathfrak{D}_{T^*}).$$

The characteristic function Θ_T is an operator valued analytic function that is purely contractive and fundamentally reducible [11, Sec. 4].

3. Coincidence of characteristic functions. If $\mathfrak{D}$ and $\mathfrak{D}'$ are two Krein spaces, then an operator τ: $\mathfrak{D}\to\mathfrak{D}'$ is said to be *unitary* if it is continuous and invertible, and if $[\tau x, \tau x]=[x, x]$ for all $x\in\mathfrak{D}$. Two operator valued analytic functions $\Theta(\lambda)$: $\mathfrak{D}\to\mathfrak{D}_*$ and $\Theta'(\lambda)$: $\mathfrak{D}'\to\mathfrak{D}'_*$ are said to *coincide* if there are unitary operators τ: $\mathfrak{D}\to\mathfrak{D}'$ and τ_*: $\mathfrak{D}_*\to\mathfrak{D}'_*$ such that $\Theta'(\lambda)=\tau_*\Theta(\lambda)\tau^{-1}$ for all $\lambda\in D$.

As in [15, Sec. VI.1.2], we have the following result.

Proposition 3.1. *The characteristic functions of unitarily equivalent operators coincide.*

Proof. Let T_1 and T_2 be bounded operators on Hilbert spaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$, respectively, and suppose that for some unitary operator $\sigma\colon \mathfrak{H}_1\to\mathfrak{H}_2$ we have $T_2=\sigma T_1\sigma^{-1}$. Then, if we define $\tau=\sigma|\mathfrak{D}_{T_1}$ and $\tau_*=\sigma|\mathfrak{D}_{T_1^*}$, it is clear that

$$(3.1)\quad \mathfrak{D}_{T_2}=\tau\mathfrak{D}_{T_1},\quad \mathfrak{D}_{T_2^*}=\tau_*\mathfrak{D}_{T_1^*},\quad \text{and}\quad J_{T_2}=\tau J_{T_1}\tau^{-1},\quad J_{T_2^*}=\tau_* J_{T_1^*}\tau_*^{-1},$$

$$\Theta_{T_2}(\lambda)=\tau_*\Theta_{T_1}(\lambda)\tau^{-1}.$$

It follows from (3.1) that τ and τ_* are unitary operators, and thus Θ_{T_1} and Θ_{T} coincide. □

For any bounded operator T on a Hilbert space $\mathfrak{H}$ there is a unique maximal subspace $\mathfrak{H}_0$ in $\mathfrak{H}$ reducing T to a unitary operator (see, for example, [7, Sec. 4]). If $\mathfrak{H}_1=\mathfrak{H}\ominus\mathfrak{H}_0$, then $T|\mathfrak{H}_1$ is completely non-unitary, i.e. there is no non-zero subspace of $\mathfrak{H}_1$ which reduces T to a unitary operator.

Proposition 3.2. *The characteristic functions of a bounded operator and its completely non-unitary part coincide.*

Proof. Formally the same as [15, Sec. VI.1.2]. □

In Sec. 6 we will deduce (Theorem 6.1) that, for completely non-unitary operators with bounded characteristic functions, coincidence of the characteristic functions implies unitary equivalence of the operators.

4. Dilations. Fourier representations. Let T be a completely non-unitary operator on a separable Hilbert space $\mathfrak{H}$, and suppose that T has bounded characteristic function $\Theta_T(\lambda)$, i.e. $\sup_{\lambda\in D}\|\Theta_T(\lambda)\|<\infty$.

We can construct (see [7]) a Krein space $\mathfrak{K}$ containing $\mathfrak{H}$ as a subspace (with the indefinite inner product $[.,.]$ of $\mathfrak{K}$ restricting to the Hilbert space inner product $(.,.)$ on $\mathfrak{H}$) and an operator U on $\mathfrak{K}$ which is a *minimal unitary dilation* of T, i.e. U is unitary and satisfies

$$[U^n h, k]=(T^n h, k)\quad (h, k\in\mathfrak{H},\ n=1,2,\dots)\quad\text{and}\quad \bigvee_{n=-\infty}^{\infty} U^n\mathfrak{H}=\mathfrak{K}.$$

(The symbol $\vee$ denotes closed linear span.)

The following subspaces of $\mathfrak{K}$ are important in the study of the geometry of the dilation space (see [13]; cf. [15]):

$$\mathfrak{L}=\overline{(U-T)\mathfrak{H}},\quad \mathfrak{L}_*=\overline{(I-UT^*)\mathfrak{H}},\quad M(\mathfrak{L})=\bigvee_{n=-\infty}^{\infty}U^n\mathfrak{L},\quad M(\mathfrak{L}_*)=\bigvee_{n=-\infty}^{\infty}U^n\mathfrak{L}_*,$$

$$M_+(\mathfrak{L})=\bigvee_{n=0}^{\infty}U^n\mathfrak{L},\quad M_+(\mathfrak{L}_*)=\bigvee_{n=0}^{\infty}U^n\mathfrak{L}_*,\quad \mathfrak{R}=M(\mathfrak{L}_*)^{\perp},\quad \mathfrak{K}_+=\bigvee_{n=0}^{\infty}U^n\mathfrak{H}.$$

We are assuming that T is completely non-unitary and has bounded characteristic function. Therefore, by [13, Sec. 6], $M(\mathfrak{L}_*)$ is regular and, by [13, Corollary 3.2],

$$M(\mathfrak{L})\vee M(\mathfrak{L}_*) = \mathfrak{K}.$$

Hence, if P denotes the projection of $\mathfrak{K}$ onto $M(\mathfrak{L}_*)$ (i.e., the projection with range $M(\mathfrak{L}_*)$ and null space $\mathfrak{R}$), then we have

$$\overline{(I-P)M(\mathfrak{L})} = \mathfrak{R} \tag{4.1}$$

(cf. [15, Sec. VI.2.1]).

It follows from the construction of the dilation in [7] that there are unitary operators φ: $\mathfrak{L}\to\mathfrak{D}_T$ and φ_*: $\mathfrak{L}_*\to\mathfrak{D}_{T^*}$ and a fundamental symmetry J on $\mathfrak{K}$ such that

$$\varphi(U-T)h = Q_T h, \quad \varphi_*(I-UT^*)h = J_{T^*}Q_{T^*}h \quad (h\in\mathfrak{H});$$

$$\varphi J|\mathfrak{L} = J_T\varphi, \quad \varphi_* UJU^*|\mathfrak{L}_* = J_{T^*}\varphi_*;$$

$$\|\varphi l\| = \|l\|, \quad \|\varphi_* l_*\| = \|l_*\| \quad (l\in\mathfrak{L},\, l_*\in\mathfrak{L}_*).$$

(See [13, Sec. 3].)

Let $P_{\mathfrak{L}}$ denote the projection of $\mathfrak{K}$ onto $\mathfrak{L}$. If $h\in M(\mathfrak{L})$, then the *Fourier coefficients of h in $M(\mathfrak{L})$* are

$$l_n = P_{\mathfrak{L}}U^{*n}h \quad (-\infty < n < \infty).$$

When Θ_T is bounded, we have $\sum_{n=-\infty}^{\infty}\|l_n\|^2<\infty$ (see [13, Sec. 6]; cf. [8, Sec. III.1]), and thus we can define $\Phi_{\mathfrak{L}}$, the *Fourier representation of* $M(\mathfrak{L})$, by

$$(\Phi_{\mathfrak{L}}h)(t) = \sum_{n=-\infty}^{\infty} e^{int}\varphi l_n.$$

$\Phi_{\mathfrak{L}}$ is a unitary operator from $M(\mathfrak{L})$ to $L^2(\mathfrak{D}_T)$, the Krein space of square integrable $\mathfrak{D}_T$-valued functions with inner product

$$[u, v] = 1/2\pi\int_0^{2\pi}[u(t), v(t)]\,dt \quad (u, v\in L^2(\mathfrak{D}_T)).$$

Similarly, if $h\in M(\mathfrak{L}_*)$ and $l_n = P_{\mathfrak{L}_*}U^{*n}h$ are the Fourier coefficients of h in $M(\mathfrak{L}_*)$, then we define $\Phi_{\mathfrak{L}_*}$, the *Fourier representation of* $M(\mathfrak{L}_*)$, by

$$(\Phi_{\mathfrak{L}^*}h)(t) = \sum_{n=-\infty}^{\infty} e^{int}\varphi_* l_n.$$

$\Phi_{\mathfrak{L}_*}$ is a unitary operator from $M(\mathfrak{L}_*)$ to $L^2(\mathfrak{D}_{T^*})$. (See [13, Sec. 6]; cf. [15, Chapter V].)

5. Functional models for a given operator. If $\mathfrak{D}$ is a Krein space with fundamental symmetry J, then we also denote by J the fundamental symmetry induced on $L^2(\mathfrak{D})$ by $(Jv)(t)=J\cdot v(t)$. Thus we have on $L^2(\mathfrak{D}_T)$ and $L^2(\mathfrak{D}_{T*})$ the fundamental symmetries J_T and J_{T*}, respectively. As in [15, Sec. V.2] we have the operator $\Theta_T\colon L^2(\mathfrak{D}_T)\to L^2(\mathfrak{D}_{T*})$ defined by

$$(\Theta_T v)(t) = \Theta_T(e^{it})v(t) \quad \text{a.e.} \quad (v\in L^2(\mathfrak{D}_T)),$$

where $\Theta_T(e^{it})=\lim\limits_{r\to 1-}\Theta_T(re^{it})$. Since Θ_T is a purely contractive analytic function, it satisfies $[(I-\Theta_T^*\Theta_T)v, v]\geqq 0$ for all $v\in L^2(\mathfrak{D}_T)$, or in terms of the Hilbert space inner product on $L^2(\mathfrak{D}_T)$, $(J_T(I-\Theta_T^*\Theta_T)v, v)\geqq 0$. We can therefore define $\Delta_T=(J_T(I-\Theta_T^*\Theta_T))^{1/2}$, an operator on $L^2(\mathfrak{D}_T)$ that satisfies the relation $\|\Delta_T v\|^2=[(I-\Theta_T^*\Theta_T)v, v]$, for all $v\in L^2(\mathfrak{D}_T)$.

For $f\in M(\mathfrak{L})$ we have, using the fact that the Fourier representations are unitary and the relation $\Theta_T\Phi_{\mathfrak{L}}=\Phi_{\mathfrak{L}_*}P|M(\mathfrak{L})$ [13, equation (6.4)],

$$[(I-P)f,(I-P)f] = [f,f]-[Pf,Pf] = [\Phi_{\mathfrak{L}}f, \Phi_{\mathfrak{L}}f]-[\Phi_{\mathfrak{L}*}Pf, \Phi_{\mathfrak{L}*}Pf] =$$

$$= [\Phi_{\mathfrak{L}}f, \Phi_{\mathfrak{L}}f]-[\Theta_T\Phi_{\mathfrak{L}}f, \Theta_T\Phi_{\mathfrak{L}}f] = [(I-\Theta_T^*\Theta_T)\Phi_{\mathfrak{L}}f, \Phi_{\mathfrak{L}}f] = \|\Delta_T\Phi_{\mathfrak{L}}f\|^2$$

(cf. [15, Sec. VI.2.1]). Hence, by (4.1), there is a unitary operator $\Phi_{\mathfrak{R}}\colon \mathfrak{R}\to\overline{\Delta_T L^2(\mathfrak{D}_T)}$ such that

$$\Phi_{\mathfrak{R}}(I-P)f = \Delta_T\Phi_{\mathfrak{L}}f \quad (f\in M(\mathfrak{L})).$$

Here we are considering $\mathfrak{R}$ as a Hilbert space with the inner product [., .] [13, Theorem 7.1], and $\overline{\Delta_T L^2(\mathfrak{D}_T)}$ as a Hilbert space with the usual inner product on $L^2(\mathfrak{D}_T)$. *(In the sequel, $\overline{\Delta_T L^2(\mathfrak{D}_T)}$ will always be considered as a Hilbert space.)*

Since $M(\mathfrak{L}_*)$ is regular [13, Sec. 6] we can write

$$\mathfrak{K} = M(\mathfrak{L}_*)\oplus\mathfrak{R}.$$

If we make the definition

$$\mathbf{K} = L^2(\mathfrak{D}_{T*})\oplus\overline{\Delta_T L^2(\mathfrak{D}_T)},$$

then we can deduce that the operator $\Phi=\Phi_{\mathfrak{L}_*}\oplus\Phi_{\mathfrak{R}}$ is a unitary operator from $\mathfrak{K}$ to $\mathbf{K}$. Φ is known as the *Fourier representation* of $\mathfrak{K}$.

If we let e^{it} also denote multiplication by the function e^{it} then $e^{it}\Theta_T=\Theta_T e^{it}$ and $e^{it}J_T=J_Te^{it}$, and hence $e^{it}\Delta_T=\Delta_Te^{it}$. We also have $UP=PU$ and $\Phi_{\mathfrak{L}}U=e^{it}\Phi_{\mathfrak{L}}$, and so (cf. [15, Sec. VI.2.1])

$$\Phi_{\mathfrak{R}}U(I-P)f = e^{it}\Phi_{\mathfrak{R}}(I-P)f \quad (f\in M(\mathfrak{L})).$$

By continuity, it follows that $\Phi_{\mathfrak{R}}Uh=e^{it}\Phi_{\mathfrak{R}}h$ for all $h\in\mathfrak{R}$.

Let $\mathbf{U}$ denote multiplication by e^{it} on $\mathbf{K}$, i.e.

$$\mathbf{U}(u\oplus v) = e^{it}u\oplus e^{it}v \quad (u\in L^2(\mathfrak{D}_{T*}),\ v\in\overline{\Delta_T L^2(\mathfrak{D}_T)}).$$

Then, since $\Phi_{\mathfrak{R}}U=e^{it}\Phi_{\mathfrak{R}}$ and $\Phi_{\mathfrak{L}_*}U=e^{it}\Phi_{\mathfrak{L}_*}$, we have $\Phi U=\mathbf{U}\Phi$.

The subspace $M_+(\mathfrak{L}_*)$ is regular [8, Sec. III. 2], and thus, by [13, Sec. 4], we have

$$\mathfrak{K}_+ = M_+(\mathfrak{L}_*) \oplus \mathfrak{N}$$

(recall the definitions of these subspaces in Sec. 4). Consequently, Φ maps $\mathfrak{K}_+$ onto the space

$$\mathbf{K}_+ = H^2(\mathfrak{D}_{T^*}) \oplus \overline{\Delta_T L^2(\mathfrak{D}_T)},$$

where $H^2(\mathfrak{D}_{T^*})$ (the space of $\mathfrak{D}_{T^*}$-valued analytic functions with square summable Taylor coefficients) is identified with a subspace of $L^2(\mathfrak{D}_{T^*})$ in the usual manner cf. [15, Sec. V. 1.1]). Then if $U_+ = U|\mathfrak{K}_+$ and $\mathbf{U}_+ = \mathbf{U}|\mathbf{K}_+$, we have $\Phi U_+ = \mathbf{U}_+ \Phi$.

For $u \in H^2(\mathfrak{D}_{T^*})$ and $v \in \overline{\Delta_T L^2(\mathfrak{D}_T)}$ we have then

$$\mathbf{U}_+(u \oplus v) = e^{it} u \oplus e^{it} v \quad \text{and} \quad \mathbf{U}_+^*(u \oplus v) = e^{-it}(u - u_0) \oplus e^{-it} v.$$

Remark on notation. Here (and in the sequel) it is assumed that, for each n, u_n denotes the nth coefficient in the Fourier series of the function u. Thus, for $u \in H^2(\mathfrak{D}_{T^*})$, $u_0 = u(0)$. Also, we will not be distinguishing between a vector and the constant function whose range is that vector.

Let us make the definition $\mathbf{H} = \Phi \mathfrak{H}$. Since $\mathfrak{H}$ is a Hilbert space with the inner product $[.,.]$, and since Φ is unitary, $\mathbf{H}$ is also a Hilbert space. We know by [13, equation (3.3)] that $\mathfrak{K}_+ = \mathfrak{H} \oplus M_+(\mathfrak{L})$, and therefore we deduce $\mathfrak{H} = \mathfrak{K}_+ \ominus M_+(\mathfrak{L})$. Hence,

$$\mathbf{H} = \mathbf{K}_+ \ominus \Phi M_+(\mathfrak{L}).$$

We can obtain an explicit description of $\Phi M_+(\mathfrak{L})$ by making the observation that, for $g \in M_+(\mathfrak{L})$,

$$\Phi g = \Phi[Pg + (I-P)g] = \Phi_{\mathfrak{L}^*} Pg \oplus \Phi_{\mathfrak{R}}(I-P)g = \Theta_T \Phi_{\mathfrak{L}} g \oplus \Delta_T \Phi_{\mathfrak{L}} g$$

(using [13, equation (6.4)]). Hence $\Phi M_+(\mathfrak{L}) = \{\Theta_T u \oplus \Delta_T u \colon u \in H^2(\mathfrak{D}_T)\}$. Consequently we obtain

$$\mathbf{H} = \mathbf{K}_+ \ominus \{\Theta_T u \oplus \Delta_T u \colon u \in H^2(\mathfrak{D}_T)\}.$$

If we denote by $\mathbf{T}$ the operator $\Phi T \Phi^{-1}$ on $\mathbf{H}$, then we have $\mathbf{T}^* = \mathbf{U}_+^*|\mathbf{H}$, and thus we obtain the following functional model.

Theorem 5.1. (cf. [15, Theorem VI.2.3]) *Let T be a completely non-unitary operator on a separable Hilbert space $\mathfrak{H}$, with bounded characteristic function Θ_T. Then the Krein space*

$$\mathbf{H} = [H^2(\mathfrak{D}_{T^*}) \oplus \overline{\Delta_T L^2(\mathfrak{D}_T)}] \ominus \{\Theta_T u \oplus \Delta_T u \colon u \in H^2(\mathfrak{D}_T)\}$$

is a Hilbert space and T is unitarily equivalent to the operator $\mathbf{T}$ on $\mathbf{H}$ defined by

$$\mathbf{T}^*(u \oplus v) = e^{-it}(u - u_0) \oplus e^{-it} v \quad (u \oplus v \in \mathbf{H}).$$

The unitary dilation U of T constructed in [7] *is unitarily equivalent to the operator* $\mathbf{U}$ *defined on the Krein space*

$$\mathbf{K} = L^2(\mathfrak{D}_{T^*}) \oplus \overline{\Delta_T L^2(\mathfrak{D}_T)} \quad by \quad \mathbf{U}(u \oplus v) = e^{it} u \oplus e^{it} v \quad (u \oplus v \in \mathbf{K}). \quad \square$$

6. Unitary equivalence of completely non-unitary operators.

Theorem 6.1. *Let T_1 and T_2 be completely non-unitary operators with bounded characteristic functions. Then T_1 and T_2 are unitarily equivalent if and only if their characteristic functions coincide.*

Proof. By Proposition 3.1, if T_1 and T_2 are unitarily equivalent, then their characteristic functions coincide. Conversely, suppose that $\tau\colon \mathfrak{D}_{T_1} \to \mathfrak{D}_{T_2}$ and $\tau_*\colon \mathfrak{D}_{T_1^*} \to \mathfrak{D}_{T_2^*}$ are unitary operators such that $\Theta_{T_2}(\lambda) = \tau_* \Theta_{T_1}(\lambda) \tau^{-1}$ ($\lambda \in D$). Then we obtain (since $\Theta_T(0) = -TJ_T$)

$$I - T_2^* T_2 = I - \Theta_{T_2}(0)^* \Theta_{T_2}(0) = \tau(I - \Theta_{T_1}(0)^* \Theta_{T_1}(0))\tau^{-1} = \tau(I - T_1^* T_1)\tau^{-1},$$

and hence $J_{T_2} = \tau J_{T_1} \tau^{-1}$. We similarly deduce that $J_{T_2^*} = \tau_* J_{T_1^*} \tau_*^{-1}$, and thus τ and τ_* are unitary with respect to the Hilbert space inner products as well as the indefinite inner products.

We can regard τ as mapping $L^2(\mathfrak{D}_{T_1})$ to $L^2(\mathfrak{D}_{T_2})$ (and similarly for τ_*), and then we have $\Delta_{T_2} = \tau \Delta_{T_1} \tau^{-1}$.

Let $\mathbf{T}_1$ and $\mathbf{T}_2$ be the operators (on $\mathbf{H}_1$ and $\mathbf{H}_2$) defined in Theorem 5.1, unitarily equivalent to T_1 and T_2, respectively. Then, as in [15, Sec. VI. 2.3], we can deduce that the operator $\hat{\tau}$, taking $u \oplus v$ to $\tau_* u \oplus \tau v$ ($u \oplus v \in \mathbf{H}_1$), is a unitary operator from $\mathbf{H}_1$ to $\mathbf{H}_2$ such that $\mathbf{T}_2 = \hat{\tau} \mathbf{T}_1 \hat{\tau}^{-1}$. It then follows that T_1 and T_2 are unitarily equivalent. $\square$

7. Notes on functional models. When Θ_T is bounded and $\lim_{n\to\infty} T^{*^n} = 0$, then we have $\mathfrak{R} = \{0\}$ [13, Theorem 5.5] and the model of Theorem 5.1 can also be described as follows:

Let $\mathbf{K}_+$ be the space of sequences $\{h_n\}_{n \geq 0}$ with $h_n \in \mathfrak{D}_{T^*}$ ($n = 0, 1, 2, \ldots$) and $\sum_{n=0}^{\infty} \|h_n\|^2 < \infty$. The inner product on $\mathbf{K}_+$ is defined by

$$[\{h_n\}_{n \geq 0}, \{k_n\}_{n \geq 0}] = \sum_{n=0}^{\infty} [h_n, k_n] = \sum_{n=0}^{\infty} (J_{T^*} h_n, k_n).$$

Clearly, $\mathbf{K}_+$ is a Krein space, with the fundamental symmetry $J\{h_n\}_{n \geq 0} = \{J_{T^*} h_n\}_{n \geq 0}$.

We consider $\mathfrak{H}$ as a subspace of $\mathbf{K}_+$ by identifying the vector $h \in \mathfrak{H}$ and the sequence

$$\mathbf{h} = \{J_{T^*} Q_{T^*} T^{*n} h\}_{n \geq 0}.$$

By [13, Corollary 8.3], **h** is in $\mathbf{K}_+$, and we have (since $\lim_{n\to\infty} T^{*^n}=0$)

$$[\mathbf{h}, \mathbf{h}] = \sum_{n=0}^{\infty} (Q_{T^*}T^{*n}h, J_{T^*}Q_{T^*}T^{*n}h) =$$

$$= \sum_{n=0}^{\infty}(T^n(I-TT^*)T^{*n}h, h) = \big(h - \lim_{n\to\infty} T^n T^{*n}h, h\big) = \|h\|^2.$$

Thus the identification of $\mathfrak{H}$ as a subspace of $\mathbf{K}_+$ is valid.

If V is the unilateral shift on $\mathbf{K}_+$, mapping $(h_0, h_1, h_2, \ldots)$ to $(0, h_0, h_1, \ldots)$, then we have $T^* = V^*|\mathfrak{H}$. If we identify $\mathbf{K}_+$ with the space $H^2(\mathfrak{D}_{T^*})$, in the obvious manner, then V is identified with multiplication by e^{it} (thinking of H^2 as a subspace of L^2). The above model then coincides with the model of Theorem 5.1, which in the case $\lim_{n\to\infty} T^{*^n}=0$ identifies $\mathfrak{H}$ with the space

$$H^2(\mathfrak{D}_{T^*}) \ominus \Theta_T H^2(\mathfrak{D}_T)$$

(since $\mathfrak{R}=\{0\}$). (Cf. [15, p. 277].)

In [14], Rota obtains a model for operators with spectrum in the open unit disk, and this case is obviously included in the case considered above (namely, Θ_T bounded and $\lim_{n\to\infty} T^{*^n}=0$). Rota's model, however, differs somewhat from the model described above, and gives only a similarity model for T.

In the remaining sections of this paper we will be considering an arbitrary purely contractive analytic function $\Theta(\lambda)$. We will prove, by constructing a suitable functional model (based on that of Sz.-Nagy and Foiaş [15, Chapter VI]), that if Θ is bounded and fundamentally reducible then it is the characteristic function of some completely non-unitary operator (cf. [11]).

8. The functional model for a bounded purely contractive analytic function. Let $\Theta(\lambda)$: $\mathfrak{D}\to\mathfrak{D}_*$ be a bounded purely contractive analytic function. We will assume that Θ is fundamentally reducible, so that there are fundamental symmetries on $\mathfrak{D}$ and $\mathfrak{D}_*$ commuting with $\Theta_0^*\Theta_0$ and $\Theta_0\Theta_0^*$, respectively. As in [11, Sec. 5] we define the fundamental symmetries $J=\operatorname{sgn}(I-\Theta_0^*\Theta_0)$ on $\mathfrak{D}$ and $J_*=\operatorname{sgn}(I-\Theta_0\Theta_0^*)$ on $\mathfrak{D}_*$. The Hilbert space inner products and norms that we will use on $\mathfrak{D}$ and $\mathfrak{D}_*$ (and on $L^2(\mathfrak{D})$ and $L^2(\mathfrak{D}_*)$) will be the J- and J_*-inner products and norms obtained from these fundamental symmetries.

We also define the operators $Q=|I-\Theta_0^*\Theta_0|^{1/2}$ and $Q_*=|I-\Theta_0\Theta_0^*|^{1/2}$. They satisfy the relations (see [7, Sec. 2])

$$JQ^2 = I-\Theta_0{}^*\Theta_0, \quad J_*Q_*^2 = I-\Theta_0\Theta_0{}^*, \quad \Theta_0 J = J_*\Theta_0, \quad \Theta_0 Q = Q_*\Theta_0,$$
$$\Theta_0{}^*J_* = J\Theta_0{}^*, \quad \Theta_0{}^*Q_* = Q\Theta_0{}^*.$$

Since Θ is bounded and purely contractive, it can be considered as an operator from $L^2(\mathfrak{D})$ to $L^2(\mathfrak{D}_*)$, and we can define the operator $\Delta=\big(J(I-\Theta^*\Theta)\big)^{1/2}$ on

$L^2(\mathfrak{D})$. The space $\overline{\Delta L^2(\mathfrak{D})}$ will always be considered as a Hilbert space (with the J-inner product), and we have $\|\Delta v\|^2=[(I-\Theta^*\Theta)v, v]$ for $v\in L^2(\mathfrak{D})$ (cf. Sec. 5).

Consider the Krein spaces

$$\mathbf{K}=L^2(\mathfrak{D}_*)\oplus\overline{\Delta L^2(\mathfrak{D})} \quad \text{and} \quad \mathbf{K}_+=H^2(\mathfrak{D}_*)\oplus\overline{\Delta L^2(\mathfrak{D})}\subset\mathbf{K},$$

and let

$$\mathbf{G}=\{\Theta w\oplus\Delta w:\ w\in H^2(\mathfrak{D})\}\subset\mathbf{K}_+.$$

For v and w in $H^2(\mathfrak{D})$ we have

$$[\Theta v, \Theta w]+(\Delta v, \Delta w)=[\Theta^*\Theta v, w]+[(I-\Theta^*\Theta)v, w]=[v, w].$$

Hence, since Θ and Δ are continuous, the operator $\Theta\oplus\Delta$, mapping v to $\Theta v\oplus\Delta v$, is an isometry from $H^2(\mathfrak{D})$ to $\mathbf{K}$. Therefore $\mathbf{G}$, which is the range of $\Theta\oplus\Delta$, is a regular subspace, of both $\mathbf{K}$ and $\mathbf{K}_+$ [12, Theorem 5.2].

If we define $\mathbf{H}=\mathbf{K}_+\ominus\mathbf{G}$, then $\mathbf{H}$ is a regular subspace of both $\mathbf{K}$ and $\mathbf{K}_+$.

Let $\mathbf{U}$ be multiplication by e^{it} on $\mathbf{K}$. Then $\mathbf{U}$ is a unitary operator and $\mathbf{K}_+$ is invariant for $\mathbf{U}$; we define $\mathbf{U}_+=\mathbf{U}|\mathbf{K}_+$. Since $e^{it}\Theta=\Theta e^{it}$ and $e^{it}\Delta=\Delta e^{it}$, $\mathbf{G}$ is invariant for $\mathbf{U}_+$, and thus $\mathbf{H}$ is invariant for $\mathbf{U}_+^*$. We can therefore define an operator $\mathbf{T}$ on $\mathbf{H}$ by $\mathbf{T}^*=\mathbf{U}_+^*|\mathbf{H}$. If we denote by P the projection of $\mathbf{K}$ onto $\mathbf{H}$ then we have, as in [15, Sec. VI.3.1],

$$\mathbf{T}^n=P\mathbf{U}^n|\mathbf{H} \quad (n\geqq 0). \tag{8.1}$$

We also have

$$\mathbf{T}^*(u\oplus v)=e^{-it}(u-u_0)\oplus e^{-it}v \quad (u\oplus v\in\mathbf{H}). \tag{8.2}$$

It should be noted that, since the spectrum of $\mathbf{U}$ is in the unit circle, $\mathbf{T}$ has spectrum in the closed unit disk.

9. Basic properties of the model. A vector $u\oplus v$ $(u\in H^2(\mathfrak{D}_*),\ v\in\overline{\Delta L^2(\mathfrak{D})})$ is in $\mathbf{H}$ if and only if $u\oplus v\perp\Theta w\oplus\Delta w$ for all $w\in H^2(\mathfrak{D})$. Since we have the equations

$$[u\oplus v, \Theta w\oplus\Delta w]=[u, \Theta w]+(v, \Delta w)=[\Theta^*u, w]+(\Delta v, w)=[\Theta^*u+J\Delta v, w],$$

we conclude that $u\oplus v\in\mathbf{H}$ if and only if $\Theta^*u+J\Delta v\perp H^2(\mathfrak{D})$. In this case

$$\Theta^*u+J\Delta v=\sum_{n=1}^{\infty}e^{-int}f_n, \tag{9.1}$$

with f_n given by

$$f_n=1/2\pi\int_0^{2\pi}e^{int}(\Theta^*u+J\Delta v)(t)\,dt. \tag{9.2}$$

From (8.1) we deduce that, for $u\oplus v\in\mathbf{H}$,

$$\mathbf{T}(u\oplus v)=(e^{it}u-\Theta f_1)\oplus(e^{it}v-\Delta f_1) \quad \text{(cf. [15, Sec. VI. 3.5]).} \tag{9.3}$$

Proposition 9.1. *For* $u\oplus v\in\mathbf{H}$ *we have*

$$(I-\mathbf{T}^*\mathbf{T})(u\oplus v)=e^{-it}(\Theta-\Theta_0)f_1\oplus e^{-it}\Delta f_1$$

and

$$(I-\mathbf{T}\mathbf{T}^*)(u\oplus v)=(I-\Theta\Theta_0^*)u_0\oplus-\Delta\Theta_0^*u_0.$$

Proof. The first formula follows immediately from (9.3) and (8.2). For the second formula, we need to obtain the vector f_1 corresponding to $\mathbf{T}^*(u\oplus v)$, which (by (8.2)) is done by considering

$$\Theta^*[e^{-it}(u-u_0)]+J\Delta[e^{-it}v]=e^{-it}(\Theta^*u+J\Delta v)-e^{-it}\Theta^*u_0.$$

Since $\Theta^*u+J\Delta v\perp H^2(\mathfrak{D})$, we deduce that the required vector is $-\Theta_0^*u_0$ and hence, applying (9.3), we obtain

$$\mathbf{T}\mathbf{T}^*(u\oplus v)=((u-u_0)+\Theta\Theta_0^*u_0)\oplus(v+\Delta\Theta_0^*u_0)=u\oplus v-[(I-\Theta\Theta_0^*)u_0\oplus-\Delta\Theta_0^*u_0].\ \square$$

Lemma 9.2. *If* $u\oplus v$ *is given by*

$$u\oplus v=e^{-it}(\Theta-\Theta_0)f\oplus e^{-it}\Delta f,$$

where $f\in\mathfrak{D}$, *then* $u\oplus v\in\mathbf{H}$, *and the vector* f_1 *defined by* (9.2) *is* $f_1=(I-\Theta_0^*\Theta_0)f$.

Proof. (Cf. [15, Sec. VI.3.5].) Since $J\Delta^2=I-\Theta^*\Theta$, we have

$$\Theta^*u+J\Delta v=e^{-it}\Theta^*(\Theta-\Theta_0)f+e^{-it}(I-\Theta^*\Theta)f=e^{-it}(I-\Theta^*\Theta_0)f.$$

Therefore, $\Theta^*u+J\Delta v\perp H^2(\mathfrak{D})$, and so $u\oplus v\in\mathbf{H}$. We also have

$$f_1=1/2\pi\int_0^{2\pi}(I-\Theta(e^{it})^*\Theta_0)f\,dt=(I-\Theta_0^*\Theta_0)f.\qquad\square$$

Let us define the subset $\mathfrak{D}_1$ in $\mathfrak{D}$ by

$$\mathfrak{D}_1=\Big\{f_1=1/2\pi\int_0^{2\pi}e^{it}(\Theta^*u+J\Delta v)(t)\,dt:\ u\oplus v\in\mathbf{H}\Big\}.$$

Proposition 9.3. $\mathfrak{D}_1$ *is dense in* $\mathfrak{D}$.

Proof. Since Θ is purely contractive, the set $\{(I-\Theta_0^*\Theta_0)g: g\in\mathfrak{D}\}$ is dense in $\mathfrak{D}$. But Lemma 9.2 shows that $(I-\Theta_0^*\Theta_0)g$ is the vector f_1 for $u\oplus v=e^{-it}(\Theta-\Theta_0)g\oplus\oplus e^{-it}\Delta g$, and therefore $(I-\Theta_0^*\Theta_0)g\in\mathfrak{D}_1$ for all $g\in\mathfrak{D}$. $\square$

Proposition 9.4. *The set* $\{u_0: u\oplus v\in\mathbf{H}\}$ *is dense in* $\mathfrak{D}_*$.

Proof. Since Θ is purely contractive, the set $\{(I-\Theta_0\Theta_0^*)g: g\in\mathfrak{D}_*\}$ is dense in $\mathfrak{D}_*$. If $u\oplus v=(I-\Theta\Theta_0^*)g\oplus-\Delta\Theta_0^*g$, where $g\in\mathfrak{D}_*$, then we have

$$\Theta^*u+J\Delta v=\Theta^*(I-\Theta\Theta_0^*)g-(I-\Theta^*\Theta)\Theta_0^*g=(\Theta^*-\Theta_0^*)g.$$

Hence, $\Theta^* u + J\Delta v \perp H^2(\mathfrak{D})$, and so $u \oplus v \in \mathbf{H}$. The proof is completed by noting that $u_0 = (I - \Theta_0 \Theta_0^*) g$. □

10. The spaces $\mathfrak{D}_T$ and $\mathfrak{D}_{T^*}$. Since Θ is a purely contractive analytic function, the operators Q and Q_* (defined in Sec. 8) are injective. Thus, for $f \in \mathfrak{D}_1$ and $u \oplus v \in \mathbf{H}$, we can define

$$\varphi(JQf) = e^{-it}(\Theta - \Theta_0) f \oplus e^{-it} \Delta f \quad \text{and} \quad \varphi_*(Q_* u_0) = (I - \Theta \Theta_0^*) u_0 \oplus - \Delta \Theta_0^* u_0.$$

JQ and Q_* have dense range, and hence (by Propositions 9.3 and 9.4) φ and φ_* are densely defined on $\mathfrak{D}$ and $\mathfrak{D}_*$, respectively. If we define

$$\mathfrak{D}_T = \overline{(I - \mathbf{T}^* \mathbf{T})\mathbf{H}} \quad \text{and} \quad \mathfrak{D}_{T^*} = \overline{(I - \mathbf{T}\mathbf{T}^*)\mathbf{H}},$$

then Proposition 9.1 shows that the range of φ is dense in $\mathfrak{D}_T$ and the range of φ_* is dense in $\mathfrak{D}_{T^*}$.

Using the fact that $[\Theta f, \Theta_0 f] = [\Theta_0 f, \Theta_0 f]$ for $f \in \mathfrak{D}_1$, we have

$$(10.1) \qquad [\varphi JQf, \varphi JQf] = [(\Theta - \Theta_0) f, (\Theta - \Theta_0) f] + \|\Delta f\|^2 =$$
$$= [\Theta f, \Theta f] - [\Theta_0 f, \Theta_0 f] + [(I - \Theta^* \Theta) f, f] = [(I - \Theta_0^* \Theta_0) f, f] = \|JQf\|^2.$$

Also, since $[\Theta^* u_0, \Theta_0^* u_0] = [\Theta_0^* u_0, \Theta_0^* u_0]$, we have

$$(10.2) \qquad [\varphi_* Q_* u_0, \varphi_* Q_* u_0] = [(I - \Theta \Theta_0^*) u_0, (I - \Theta \Theta_0^*) u_0] + \|\Delta \Theta_0^* u_0\|^2 =$$
$$= [u_0, u_0] - 2[\Theta_0^* u_0, \Theta_0^* u_0] + [\Theta \Theta_0^* u_0, \Theta \Theta_0^* u_0] + [(I - \Theta^* \Theta) \Theta_0^* u_0, \Theta_0^* u_0] =$$
$$= [(I - \Theta_0 \Theta_0^*) u_0, u_0] = \|Q_* u_0\|^2.$$

If we put on $\mathbf{K}$ the norm obtained from the fundamental symmetry $J_* \oplus I$, then we have, using the Cauchy—Schwarz inequality [2, Lemma II.11.4],

$$\|JQf\|^2 = [\varphi JQf, \varphi JQf] \leqq \|\varphi JQf\|^2 \quad (f \in \mathfrak{D}_1)$$

and

$$\|Q_* u_0\|^2 = [\varphi_* Q_* u_0, \varphi_* Q_* u_0] \leqq \|\varphi_* Q_* u_0\|^2 \quad (u \oplus v \in \mathbf{H}).$$

Therefore φ^{-1}, defined on a dense subset of $\mathfrak{D}_T$, is continuous and has a unique continuous extension to all of $\mathfrak{D}_T$. Similarly, φ_*^{-1} has a unique continuous extension to all of $\mathfrak{D}_{T^*}$. By (10.1) and (10.2), these extensions are unitary, with $\mathfrak{D}$ and $\mathfrak{D}_*$ being considered as Hilbert spaces with the J- and J_*-inner products. The adjoints of these unitary maps are then unitary extensions of φ and φ_*, and these extensions will also be denoted by φ and φ_*.

We can now assert that

$$(10.3) \qquad \varphi(JQf) = e^{-it}(\Theta - \Theta_0) f \oplus e^{-it} \Delta f \quad \text{for all} \quad f \in \mathfrak{D},$$

and

$$(10.4) \qquad \varphi_*(Q_* g) = (I - \Theta \Theta_0^*) g \oplus - \Delta \Theta_0^* g \quad \text{for all} \quad g \in \mathfrak{D}_*.$$

Note that φ and φ_* are unitary with the inner product $[.,.]$ on $\mathfrak{D}_T$ and $\mathfrak{D}_{T*}$; and with the Hilbert space J- and J_*-inner products on $\mathfrak{D}$ and $\mathfrak{D}_*$, respectively. We conclude from this that $\mathfrak{D}_T$ and $\mathfrak{D}_{T*}$ are Hilbert spaces. Since they are the ranges of isometries, $\mathfrak{D}_T$ and $\mathfrak{D}_{T*}$ are regular subspaces of $\mathbf{K}$ [12, Theorem 5.2], and hence, by [2, Theorem V.5.2], the intrinsic topologies on $\mathfrak{D}_T$ and $\mathfrak{D}_{T*}$ (i.e., the topologies obtained from the norms $[h, h]^{1/2}$ on $\mathfrak{D}_T$ and $\mathfrak{D}_{T*}$) coincide with the strong topologies inherited from $\mathbf{K}$.

Theorem 10.1. *$(I-\mathbf{T}^*\mathbf{T})\varphi=\varphi(I-\Theta_0^*\Theta_0)$ and $(I-\mathbf{T}\mathbf{T}^*)\varphi_*=\varphi_*(I-\Theta_0\Theta_0^*)$.*

Proof. If f is in $\mathfrak{D}$, then the vector f_1 corresponding to $u\oplus v=\varphi JQf$ (given by (9.2)) is $f_1=(I-\Theta_0^*\Theta_0)f$. (This follows immediately from (10.3) and Lemma 9.2.) Hence by Proposition 9.1,

$$(I-\mathbf{T}^*\mathbf{T})\varphi JQf = e^{-it}(\Theta-\Theta_0)f_1\oplus e^{-it}\Delta f_1 = \varphi JQf_1 = \varphi JQ(I-\Theta_0^*\Theta_0)f = \\ = \varphi(I-\Theta_0^*\Theta_0)JQf.$$

The first assertion of the theorem then follows.

If g is in $\mathfrak{D}_*$, and if $u\oplus v=\varphi_*Q_*g$, then (10.4) shows that $u_0=(I-\Theta_0\Theta_0^*)g$. Hence, by Proposition 9.1,

$$(I-\mathbf{T}\mathbf{T}^*)\varphi_*Q_*g = (I-\Theta\Theta_0^*)u_0\oplus -\Delta\Theta_0^*u_0 = \varphi_*Q_*u_0 = \\ = \varphi_*Q_*(I-\Theta_0\Theta_0^*)g = \varphi_*(I-\Theta_0\Theta_0^*)Q_*g,$$

and the second assertion follows. □

Since $\mathfrak{D}_T$ and $\mathfrak{D}_{T*}$ are Hilbert spaces, we can define $J_T=\operatorname{sgn}(I-\mathbf{T}^*\mathbf{T})$ and $Q_T=|I-\mathbf{T}^*\mathbf{T}|^{1/2}$ as operators on $\mathfrak{D}_T$, and $J_{T*}=\operatorname{sgn}(I-\mathbf{T}\mathbf{T}^*)$ and $Q_{T*}=|I-\mathbf{T}\mathbf{T}^*|^{1/2}$ as operators on $\mathfrak{D}_{T*}$.

Corollary 10.2. *$J_T\varphi=\varphi J$, $Q_T\varphi=\varphi Q$, $J_{T*}\varphi_*=\varphi_*J_*$, and $Q_{T*}\varphi_*=\varphi_*Q_*$.* □

We have shown that the inner product $[.,.]$ is positive definite on $\mathfrak{D}_T$ and $\mathfrak{D}_{T*}$. With the inner products $[J_T.,.]$ and $[J_{T*}.,.]$, $\mathfrak{D}_T$ and $\mathfrak{D}_{T*}$ are Krein spaces having fundamental symmetries J_T and J_{T*}, respectively. Corollary 10.2 shows that φ and φ_* are Krein space isomorphisms intertwining the fundamental symmetries J and J_T, and J_* and J_{T*}.

11. The characteristic function. $\mathfrak{D}_T$ is regular, and so we can extend J_T and Q_T to operators on $\mathbf{H}$ by defining them to be zero on $\mathbf{H}\ominus\mathfrak{D}_T$. We similarly extend J_{T*} and Q_{T*} to operators defined on $\mathbf{H}$. It is clear that these extensions are self-adjoint, and that $J_TQ_T^2=I-\mathbf{T}^*\mathbf{T}$ and $J_{T*}Q_{T*}^2=I-\mathbf{T}\mathbf{T}^*$. We define

$$\Theta_T(\lambda) = -\mathbf{T}J_T+\lambda J_{T*}Q_{T*}(I-\lambda\mathbf{T}^*)^{-1}J_TQ_T|\mathfrak{D}_T$$

for those complex numbers λ for which $(I-\lambda\mathbf{T}^*)^{-1}$ exists. It will be shown in the next section that $\mathbf{H}$ is a Hilbert space, so that Θ_T is in fact the characteristic function of $\mathbf{T}$.

Theorem 11.1. *For* $\lambda \in D$, $\Theta_T(\lambda)\varphi = \varphi_* \Theta(\lambda)$.

Proof. It suffices to show that $-\mathbf{T}J_T\varphi = \varphi_*\Theta_0$ and, for $n=1,2,3,\dots$, $J_{T*}Q_{T*}\mathbf{T}^{*^{n-1}}J_TQ_T\varphi = \varphi_*\Theta_n$. By (10.3) and Corollary 10.2, we have for all $f \in \mathfrak{D}$,

$$-\mathbf{T}J_T\varphi(Qf) = -\mathbf{T}\varphi(JQf) = -\mathbf{T}\{e^{-it}(\Theta-\Theta_0)f \oplus e^{-it}\Delta f\}.$$

Lemma 9.2 and (9.3) then give us

$$-\mathbf{T}J_T\varphi(Qf) = -\{[(\Theta-\Theta_0)f - \Theta(I-\Theta_0^*\Theta_0)f] \oplus [\Delta f - \Delta(I-\Theta_0^*\Theta_0)f]\} =$$
$$= (I-\Theta\Theta_0^*)\Theta_0 f \oplus -\Delta\Theta_0^*\Theta_0 f = \varphi_*(Q_*\Theta_0 f) = \varphi_*\Theta_0(Qf).$$

Since vectors of the form Qf, with $f \in \mathfrak{D}$, are dense in $\mathfrak{D}$, we conclude that $-\mathbf{T}J_T\varphi = \varphi_*\Theta_0$.

Now let us assume that for all $f \in \mathfrak{D}$ and for some $n \geqq 1$ we have

$$\text{(11.1)} \qquad \mathbf{T}^{*n-1}J_TQ_T\varphi f = e^{-int}\left(\Theta - \sum_{k=0}^{n-1} e^{ikt}\Theta_k\right)f \oplus e^{-int}\Delta f.$$

By (10.3) and Corollary 10.2, (11.1) is true for $n=1$. If we let

$$u = e^{-int}(\Theta - \sum_{k=0}^{n-1} e^{ikt}\Theta_k)f,$$

then $u_0 = \Theta_n f$, and we obtain from (8.2) (assuming (11.1))

$$\mathbf{T}^{*n}J_TQ_T\varphi f = e^{-it}\left[e^{-int}\left(\Theta - \sum_{k=0}^{n-1} e^{ikt}\Theta_k\right)f - \Theta_n f\right] \oplus e^{-i(n+1)t}\Delta f =$$
$$= e^{-i(n+1)t}\left(\Theta - \sum_{k=0}^{n} e^{ikt}\Theta_k\right) f \oplus e^{-i(n+1)t}\Delta f.$$

Hence, by induction, (11.1) is true for all $n \geqq 1$.

It follows from (11.1) and Proposition 9.1 that, for $n=1,2,3,\dots$ and $f \in \mathfrak{D}$,

$$Q_{T^*}(J_{T^*}Q_{T^*}\mathbf{T}^{*n-1}J_TQ_T\varphi f) = (I-\mathbf{T}\mathbf{T}^*)\left[e^{-int}\left(\Theta - \sum_{k=0}^{n-1} e^{ikt}\Theta_k\right)f \oplus e^{-int}\Delta f\right] =$$
$$= (I-\Theta\Theta_0^*)\Theta_n f \oplus -\Delta\Theta_0^*\Theta_n f = \varphi_*(Q_*\Theta_n f) = Q_{T^*}\varphi_*\Theta_n f.$$

(The last two steps used (10.4) and Corollary 10.2.) Since Q_{T*} is injective on $\mathfrak{D}_{T*}$, we conclude that $J_{T*}Q_{T*}\mathbf{T}^{*^{n-1}}J_TQ_T\varphi = \varphi_*\Theta_n$ for $n=1,2,3,\dots$ and the theorem is proved. □

12. Positivity of H. In this section we prove that, with the inner product $[.,.]$, $\mathbf{H}$ is a Hilbert space. We will need the following results.

Lemma 12.1. (cf. [15, Sec. VI.3.2]) *Suppose that the vector* $h \in \mathbf{H}$ *satisfies*

$$(I-\mathbf{T}^*\mathbf{T})\mathbf{T}^n h = 0 = (I-\mathbf{T}\mathbf{T}^*)\mathbf{T}^{*n} h \tag{12.1}$$

for all $n=0, 1, 2, \dots$. *Then* $h=0$.

Proof. We can write h in the form $h=u\oplus v$. Take $n\geqq 0$, and assume that $u_k=0$ for all $k<n$; when $n=0$, this is assuming nothing about u. Then (8.2) shows that

$$\mathbf{T}^{*n} h = e^{-int} u \oplus e^{-int} v.$$

By (12.1) and Proposition 9.1, we deduce

$$0 = (I-\mathbf{T}\mathbf{T}^*)\mathbf{T}^{*n} h = (I-\Theta\Theta_0^*)u_n \oplus -\Delta\Theta_0^* u_n.$$

In particular, $(I-\Theta_0\Theta_0^*)u_n=0$, and since Θ is purely contractive, we have $u_n=0$. Therefore, by induction, $u=0$ and $h=0\oplus v$.

Since $h\in\mathbf{H}$, v must satisfy $J\Delta v=\sum_{k=1}^{\infty} e^{-ikt} f_k$ for some vectors $f_k\in\mathfrak{D}$ $(k=1, 2, \dots)$. Take $n\geqq 0$, and assume that $f_k=0$ for all $k\leqq n$; again, this is a null assumption when $n=0$. Then clearly we have, using (9.3), $\mathbf{T}^n h=0\oplus e^{int} v$, and also

$$J\Delta(e^{int} v) = \sum_{k=1}^{\infty} e^{-ikt} f_{n+k}.$$

By (12.1) and Proposition 9.1, we deduce

$$0 = (I-\mathbf{T}^*\mathbf{T})\mathbf{T}^n h = e^{-it}(\Theta-\Theta_0)f_{n+1} \oplus e^{-it}\Delta f_{n+1}.$$

Therefore we have $(\Theta-\Theta_0)f_{n+1}=0=\Delta f_{n+1}$ and hence

$$0 = \Theta^*(\Theta-\Theta_0)f_{n+1} + J\Delta^2 f_{n+1} = (I-\Theta^*\Theta_0)f_{n+1}.$$

In particular, $(I-\Theta_0^*\Theta_0)f_{n+1}=0$, and since Θ is purely contractive, we have $f_{n+1}=0$. We conclude (by induction) that $J\Delta v=0$, and thus $v=0$ $\left(v\in\overline{\Delta L^2(\mathfrak{D})}\right)$. Therefore $h=0$. □

Theorem 12.2. *Let* $\mathfrak{U}$ *be a neighborhood of zero contained in the unit disk* D. *Then* $\mathbf{H}$ *is the closed linear span of vectors of the form* $(I-\mu\mathbf{T}^*)^{-1}J_T Q_T \varphi f$ *and* $(I-\mu\mathbf{T})^{-1}Q_{T*}\varphi_* g$, *where* $\mu\in\mathfrak{U}$, $f\in\mathfrak{D}$, *and* $g\in\mathfrak{D}_*$.

Proof. Since $\mathbf{T}$ has spectrum in the closed unit disk (Sec. 8), both $(I-\mu\mathbf{T}^*)^{-1}$ and $(I-\mu\mathbf{T})^{-1}$ are defined for $\mu\in\mathfrak{U}$.

Suppose that the vector $h\in\mathbf{H}$ is orthogonal to $(I-\mu\mathbf{T}^*)^{-1}J_T Q_T\varphi f$ and $(I-\mu\mathbf{T})^{-1}Q_{T*}\varphi_* g$, for all $\mu\in\mathfrak{U}$, $f\in\mathfrak{D}$, and $g\in\mathfrak{D}_*$. The theorem will be proved once we show that $h=0$.

We have, for all $f\in\mathfrak{D}$ and $\mu\in\mathfrak{U}$,

$$0 = [h, (I-\mu\mathbf{T}^*)^{-1}J_T Q_T\varphi f] = [J_T Q_T(I-\bar{\mu}\mathbf{T})^{-1}h, \varphi f],$$

and thus, since $\mathfrak{D}_T$ is a Hilbert space, $J_T Q_T(I-\bar{\mu}\mathbf{T})^{-1}h=0$ for all $\mu\in\mathfrak{U}$. This is true only if $J_T Q_T \mathbf{T}^n h=0$ for $n=0, 1, 2, \ldots$, and hence

(12.2) $$(I-\mathbf{T}^*\mathbf{T})\mathbf{T}^n h = 0 \quad \text{for} \quad n = 0, 1, 2, \ldots .$$

Also, for $g\in\mathfrak{D}_*$ and $\mu\in\mathfrak{U}$, we have

$$0 = [h, (I-\mu\mathbf{T})^{-1}Q_{T^*}\varphi_* g] = [Q_{T^*}(I-\bar{\mu}\mathbf{T}^*)^{-1}h, \varphi_* g],$$

and so it follows, as above, that

(12.3) $$(I-\mathbf{T}\mathbf{T}^*)\mathbf{T}^{*n}h = 0 \quad \text{for} \quad n = 0, 1, 2, \ldots .$$

(12.2) and (12.3), together with Lemma 12.1, imply that $h=0$. □

H is known to be regular, and thus (by [2, Theorem V.3.4]) **H** is a Krein space. Therefore, in order to prove **H** is a Hilbert space it suffices to show that it is positive. Obviously we need only show that $[h, h]\geqq 0$ for a set of vectors h dense in **H**, and in particular (by Theorem 12.2) for vectors of the form

(12.4) $$h = \sum_{i=1}^{n} \{(I-\mu_i\mathbf{T}^*)^{-1}J_T Q_T\varphi f_i + (I-\mu_i\mathbf{T})^{-1}Q_{T^*}\varphi_* g_i\},$$

where $n\geqq 1$ and, for $i=1, 2, \ldots, n$, $f_i\in\mathfrak{D}$, $g_i\in\mathfrak{D}_*$, and $\mu_i\in\mathfrak{U}$, some neighborhood of zero in the unit disk.

For the vector h defined by (12.4) we have

$$\begin{aligned}
[h, h] = \sum_{i=1}^{n}\sum_{j=1}^{n} \{&[(I-\mu_i\mathbf{T}^*)^{-1}J_T Q_T\varphi f_i, (I-\mu_j\mathbf{T}^*)^{-1}J_T Q_T\varphi f_j]+\\
&+[(I-\mu_i\mathbf{T}^*)^{-1}J_T Q_T\varphi f_i, (I-\mu_j\mathbf{T})^{-1}Q_{T^*}\varphi_* g_j]+\\
&+[(I-\mu_i\mathbf{T})^{-1}Q_{T^*}\varphi_* g_i, (I-\mu_j\mathbf{T}^*)^{-1}J_T Q_T\varphi f_j]+\\
&+[(I-\mu_i\mathbf{T})^{-1}Q_{T^*}\varphi_* g_i, (I-\mu_j\mathbf{T})^{-1}Q_{T^*}\varphi_* g_j]\} =\\
= \sum_{i=1}^{n}\sum_{j=1}^{n} \{&[\varphi^{-1}Q_T(I-\bar{\mu}_j\mathbf{T})^{-1}(I-\mu_i\mathbf{T}^*)^{-1}J_T Q_T\varphi f_i, f_j]+\\
&+[\varphi_*^{-1}J_{T^*}Q_{T^*}(I-\bar{\mu}_j\mathbf{T}^*)^{-1}(I-\mu_i\mathbf{T}^*)^{-1}J_T Q_T\varphi f_i, g_j]+\\
&+[\varphi^{-1}Q_T(I-\bar{\mu}_j\mathbf{T})^{-1}(I-\mu_i\mathbf{T})^{-1}Q_{T^*}\varphi_* g_i, f_j]+\\
&+[\varphi_*^{-1}J_T Q_{T^*}(I-\bar{\mu}_j\mathbf{T}^*)^{-1}(I-\mu_i\mathbf{T})^{-1}Q_{T^*}\varphi_* g_i, g_j]\}.
\end{aligned}$$

In the above calculation it should be recalled that φ is unitary from the Krein space $\mathfrak{D}$ to the Krein space $\mathfrak{D}_T$, with the inner product $[J_T\cdot, \cdot]$. A similar observation applies to φ_*.

It can be shown ([10, Sec. IV.5]; cf. [11, Sec. 4] and [15, Sec.VI.1.1]) that, for $\lambda, \mu \in D$,

$$I-\Theta_T(\mu)^*\Theta_T(\lambda) = (1-\lambda\bar{\mu})Q_T(I-\bar{\mu}\mathbf{T})^{-1}(I-\lambda\mathbf{T}^*)^{-1}J_T Q_T,$$

$$I-\Theta_T(\bar{\mu})\Theta_T(\bar{\lambda})^* = (1-\lambda\bar{\mu})J_{T^*}Q_{T^*}(I-\bar{\mu}\mathbf{T}^*)^{-1}(I-\lambda\mathbf{T})^{-1}Q_{T^*},$$

$$\Theta_T(\lambda)-\Theta_T(\bar{\mu}) = (\lambda-\bar{\mu})J_{T^*}Q_{T^*}(I-\bar{\mu}\mathbf{T}^*)^{-1}(I-\lambda\mathbf{T}^*)^{-1}J_T Q_T,$$

and

$$\Theta_T(\bar{\lambda})^*-\Theta_T(\mu)^* = (\lambda-\bar{\mu})Q_T(I-\bar{\mu}\mathbf{T})^{-1}(I-\lambda\mathbf{T})^{-1}Q_{T^*}.$$

Hence, using Theorem 11.1, we have

$$[h, h] = \sum_{i=1}^{n}\sum_{j=1}^{n}\{[(1-\mu_i\bar{\mu}_j)^{-1}(I-\Theta(\mu_j)^*\Theta(\mu_i))f_i, f_j]+ \tag{12.5}$$
$$+[(\mu_i-\bar{\mu}_j)^{-1}(\Theta(\mu_i)-\Theta(\bar{\mu}_j))f_i, g_j]+$$
$$+[(\mu_i-\bar{\mu}_j)^{-1}(\Theta(\bar{\mu}_i)^*-\Theta(\mu_j)^*)g_i, f_j]+$$
$$+[(1-\mu_i\bar{\mu}_j)^{-1}(I-\Theta(\bar{\mu}_j)\Theta(\bar{\mu}_i)^*)g_i, g_j]\}.$$

Equation (12.5) can be rewritten in the form

$$[h, h] = \sum_{i=1}^{n}\sum_{j=1}^{n}[k(\mu_j, \mu_i)(f_i\oplus g_i), (f_j\oplus g_j)], \tag{12.6}$$

where $k(\mu, \lambda)$ is the operator matrix given by [11, Equation (6.1)]. By [11, Theorem 3], $k(\mu, \lambda)$ is positive definite when Θ is purely contractive and fundamentally reducible, and it therefore follows that $[h, h]\geqq 0$. Thus $\mathbf{H}$ is a Hilbert space.

13. The functional model for a bounded purely contractive analytic function: the main theorem.

Theorem 13.1. *Let $\Theta(\lambda)$: $\mathfrak{D}\to\mathfrak{D}_*$ be a bounded purely contractive fundamentally reducible analytic function. Then the Krein space*

$$\mathbf{H} = [H^2(\mathfrak{D}_*)\oplus\overline{\Delta L^2(\mathfrak{D})}]\ominus\{\Theta w\oplus\Delta w\colon w\in H^2(\mathfrak{D})\}$$

is a Hilbert space and the operator $\mathbf{T}$ on $\mathbf{H}$ defined by

$$\mathbf{T}^*(u\oplus v) = e^{-it}(u-u_0)\oplus e^{-it}v \quad (u\oplus v\in\mathbf{H})$$

is completely non-unitary. The function Θ coincides with the characteristic function of $\mathbf{T}$. The operator $\mathbf{U}$ on the Krein space $\mathbf{K}=L^2(\mathfrak{D}_*)\oplus\overline{\Delta L^2(\mathfrak{D})}$ defined by $\mathbf{U}(u\oplus v)= =e^{it}u\oplus e^{it}v$ $(u\oplus v\in\mathbf{K})$ is unitarily equivalent to the unitary dilation of $\mathbf{T}$ given by the construction in [7].

Proof. It was shown in Sec. 12 that $\mathbf{H}$ is a Hilbert space, and Lemma 12.1 shows that $\mathbf{T}$ is completely non-unitary. Θ coincides with Θ_T by virtue of

Theorem 11.1. Finally, Theorem 5.1 shows that $\mathbf{U}$ is unitarily equivalent to the dilation of $\mathbf{T}$ given in [7]. □

The construction of the dilation in [7] defines, in a natural way, a fundamental symmetry on the dilation space (referred to in Sec. 4 of this paper). For the space $\mathbf{K}$ above, this fundamental symmetry is not the obvious one $(J_* \oplus I)$, but the one defined as follows:

Let $\mathbf{M} = \{u \oplus 0: u \perp H^2(\mathfrak{D}_*)\}$. Then we have

$$\mathbf{K} = \mathbf{M} \oplus \mathbf{K}_+ = \mathbf{M} \oplus \mathbf{H} \oplus \mathbf{G}$$

(see Sec. 8). We can therefore define a fundamental symmetry $\mathbf{J}$ on $\mathbf{K}$ by

$$\mathbf{J}(u \oplus 0) = J_* u \oplus 0 \quad (u \oplus 0 \in \mathbf{M}), \quad \mathbf{J}(u \oplus v) = u \oplus v \quad (u \oplus v \in \mathbf{H}),$$
$$\mathbf{J}(\Theta w \oplus \Delta w) = \Theta J w \oplus \Delta J w \quad (\Theta w \oplus \Delta w \in \mathbf{G}).$$

$\mathbf{J}$ is a fundamental symmetry since $\mathbf{H}$ is a Hilbert space and, for $w \in H^2(\mathfrak{D})$, we have

$$[\Theta J w \oplus \Delta J w, \Theta w \oplus \Delta w] = [\Theta^* \Theta J w + (I - \Theta^* \Theta) J w, w] = [Jw, w] \geqq 0.$$

14. Comparison with the model of Ball. In this section we determine the relationship between the model of BALL [1] and the model described in Theorem 13.1.

Assume that Θ satisfies the conditions of Theorem 13.1 and let $k(\mu, \lambda)$ be the operator matrix given by [11, Equation (6.1)]. Then the matrix

$$k'(\mu, \lambda) = (I \oplus J_*) k(\bar{\lambda}, \bar{\mu})(J \oplus I) \tag{14.1}$$

coincides with the kernel matrix defined in [1, Theorem 2] (cf. [11, Sec. 6]). Also, as in [11], we will define $\tilde{\Theta}(\lambda) = \Theta(\bar{\lambda})^*$.

Let us now consider an element $u \oplus v$ in $\mathbf{H}$, so that $u \in H^2(\mathfrak{D}_*)$ and $v \in \overline{\Delta L^2(\mathfrak{D})}$, with $\Theta^* u + J \Delta v \perp H^2(\mathfrak{D})$. Therefore, if w is defined by

$$w(e^{it}) = e^{-it}[\Theta^* u + J \Delta v](-t),$$

then $w \in H^2(\mathfrak{D})$. Thus we can define a map Γ from $\mathbf{H}$ to $H^2(\mathfrak{D}) \oplus H^2(\mathfrak{D}_*)$ by

$$\Gamma(u \oplus v) = w \oplus J_* u.$$

We will prove that $\Gamma\mathbf{H}$, normed so that Γ is unitary, is the Hilbert space $\mathfrak{D}(B)$ considered by BALL [1, Sec. 3.1].

Let us take $f \in \mathfrak{D}$ and $\mu \in D$. We denote by f^μ and f_μ the functions $f^\mu(\lambda) = (1 - \lambda\mu)^{-1} f$ $(\lambda \in D)$ (cf. [15, Sec. V.8]) and $f_\mu(t) = (e^{it} - \mu)^{-1} f$ $(t \in [0, 2\pi])$. It is clear that $f^\mu \in H^2(\mathfrak{D})$ and $f_\mu \in L^2(\mathfrak{D})$. From the boundedness of Θ, and the fact that $(\lambda - \mu)^{-1}(\Theta(\lambda) - \Theta(\mu)) f$ is analytic for $\lambda \in D$, we conclude that the function

$$u = (\Theta - \Theta(\mu)) f_\mu \tag{14.2}$$

s in $H^2(\mathfrak{D}_*)$, and the function

$$w = \big(I - \tilde{\Theta}\Theta(\mu)\big) f^\mu \tag{14.3}$$

is in $H^2(\mathfrak{D})$. It is immediate from the definitions of k, $\tilde{\Theta}$, f^μ, and f_μ, that $w(\lambda) \oplus \oplus u(\lambda) = k(\bar{\lambda}, \mu)(f \oplus 0)$, for all $\lambda \in D$.

Let us also consider the function

$$v = \Delta f_\mu \tag{14.4}$$

in $\overline{\Delta L^2(\mathfrak{D})}$. Then we have, using (14.2) and (14.4),

$$\Theta^* u + J\Delta v = \Theta^*\big(\Theta - \Theta(\mu)\big) f_\mu + (I - \Theta^*\Theta) f_\mu = \big(I - \Theta^*\Theta(\mu)\big) f_\mu,$$

and hence

$$e^{-it}[\Theta^* u + J\Delta v](-t) = e^{-it}\big(I - \Theta(e^{-it})^*\Theta(\mu)\big)(e^{-it} - \mu)^{-1} f =$$
$$= \big(I - \tilde{\Theta}(e^{it})\Theta(\mu)\big)(1 - e^{it}\mu)^{-1} f = w(e^{it}),$$

where w is given by (14.3). Therefore $u \oplus v \in \mathbf{H}$ and $\Gamma(u \oplus v) = w \oplus J_* u$, i.e.,

$$\Gamma(u \oplus v)(\lambda) = (I \oplus J_*) k(\bar{\lambda}, \mu)(f \oplus 0).$$

By using (8.2), (14.2), and (14.4), we obtain

$$(I - \mu\mathbf{T}^*)(u \oplus v) = [u - \mu e^{-it}(u - u_0)] \oplus [v - \mu e^{-it} v] =$$
$$= e^{-it}[(e^{it} - \mu) u + \mu u_0] \oplus e^{-it}(e^{it} - \mu) v = e^{-it}\big[\big(\Theta - \Theta(\mu)\big) f - \big(\Theta_0 - \Theta(\mu)\big) f\big] \oplus e^{-it}\Delta f =$$
$$= e^{-it}(\Theta - \Theta_0) f \oplus e^{-it}\Delta f.$$

It therefore follows, using (10.3) and Corollary 10.2, that

$$(I - \mu\mathbf{T}^*)(u \oplus v) = \varphi J Q f = J_T Q_T \varphi f,$$

and thus we obtain

$$\Gamma\big((I - \mu\mathbf{T}^*)^{-1} J_T Q_T \varphi f\big)(\lambda) = (I \oplus J_*) k(\bar{\lambda}, \mu)(f \oplus 0). \tag{14.5}$$

Let us take $g \in \mathfrak{D}_*$ and $\mu \in D$, and consider the functions $u' = \big(I - \Theta\Theta(\bar{\mu})^*\big) g^\mu$, $v' = -\Delta\Theta(\bar{\mu})^* g^\mu$, and $w' = \big(\tilde{\Theta} - \tilde{\Theta}(\mu)\big) g_\mu$. Then we obtain, in a manner similar to that used in deriving (14.5), the formula

$$\Gamma\big((I - \mu\mathbf{T})^{-1} Q_{T^*} \varphi_* g\big)(\lambda) = (I \oplus J_*) k(\bar{\lambda}, \mu)(0 \oplus g). \tag{14.6}$$

By Theorem 12.2, $\mathbf{H}$ is the closed linear span of vectors of the form $(I - \mu\mathbf{T}^*)^{-1} J_T Q_T \varphi f$ and $(I - \mu\mathbf{T})^{-1} Q_{T^*} \varphi_* g$, where $f \in \mathfrak{D}$, $g \in \mathfrak{D}_*$, and $\mu \in D$. The space $\mathfrak{D}(B)$ in [1] is defined so that a dense subset is that spanned by vectors which are pairs of functions (in λ) of the form $(I \oplus J_*) k(\bar{\lambda}, \mu)(f \oplus 0)$ and $(I \oplus J_*) k(\bar{\lambda}, \mu)(0 \oplus g)$, where $f \in \mathfrak{D}$, $g \in \mathfrak{D}_*$, and $\mu \in D$. (Recall that the kernel matrix in [1] is given by (14.1).) Thus we will have $\Gamma\mathbf{H} = \mathfrak{D}(B)$, with Γ unitary, once we have checked that the norm induced by Γ, on the dense subset of $\mathfrak{D}(B)$ described above, is the same as that defined in [1].

Consider the vector $h \in \mathbf{H}$ defined by (12.4). Then, by (14.5) and (14.6),

$$(\Gamma h)(\lambda) = (I \oplus J_*) \sum_{i=1}^{n} k(\bar{\lambda}, \mu_i)(f_i \oplus g_i) = \sum_{i=1}^{n} k'(\bar{\mu}_i, \lambda)(Jf_i \oplus g_i)$$

(using (14.1)), and it follows from the definition of the inner product in [1] that

$$\|\Gamma h\|^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} \big(k'(\bar{\mu}_i, \bar{\mu}_j)(Jf_i \oplus g_i), (Jf_j \oplus g_j)\big) =$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \big((I \oplus J_*) k(\mu_j, \mu_i)(f_i \oplus g_i), (Jf_j \oplus g_j)\big) =$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \big((J \oplus J_*) k(\mu_j, \mu_i)(f_i \oplus g_i), (f_j \oplus g_j)\big).$$

But the inner product $(.,.)$ on $\mathfrak{D} \oplus \mathfrak{D}_*$ is the $J \oplus J_*$-inner product, and hence

$$\|\Gamma h\|^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} [k(\mu_j, \mu_i)(f_i \oplus g_i), (f_j \oplus g_j)].$$

Consequently, we have (by (12.6)) $\|\Gamma h\|^2 = [h, h]$, and so $\Gamma \mathbf{H} = \mathfrak{D}(B)$ with Γ unitary.

In [1] the characteristic function B of an operator T is taken to be $B = \bar{\Theta}_T$ (cf. [11, Sec. 6]), and so in comparing the two models we should take $B = \bar{\Theta}$. In Ball's model, B is shown to be the characteristic function of the operator R on $\mathfrak{D}(B)$ defined by

$$R(w \oplus u) = e^{-it}(w - w_0) \oplus (e^{it} u - \bar{B} J w_0).$$

We show now that $R\Gamma = \Gamma \mathbf{T}$.

For $u \oplus v \in \mathbf{H}$, we have defined $\Gamma(u \oplus v) = w \oplus J_* u$, where $w(e^{it}) = e^{-it}[\Theta^* u + J\Delta v](-t)$. Thus w_0 is the vector f_1 given by (9.2) and, using (9.3), we have

$$\mathbf{T}(u \oplus v) = (e^{it} u - \Theta w_0) \oplus (e^{it} v - \Delta w_0).$$

Note that

$$\Theta^*(e^{it} u - \Theta w_0) + J\Delta(e^{it} v - \Delta w_0) = e^{it}(\Theta^* u + J\Delta v) - w_0,$$

and hence

$$\Gamma \mathbf{T}(u \oplus v) = e^{-it}(w - w_0) \oplus J_*(e^{it} u - \Theta w_0).$$

Since $B = \bar{\Theta}$, we have $\bar{B} = J_* \Theta J$, and thus we conclude that

$$R\Gamma(u \oplus v) = R(w \oplus J_* u) = e^{-it}(w - w_0) \oplus (e^{it} J_* u - J_* \Theta w_0) = \Gamma \mathbf{T}(u \oplus v).$$

Theorem 14.1. *Suppose Θ satisfies the conditions of Theorem 13.1, and let $\mathbf{T}$ be the operator, defined in that theorem, having Θ as its characteristic function. Then*

T *is unitarily equivalent to the operator R defined in* [1], *with* $B=\overline{\Theta}$. *The equivalence is implemented by the unitary operator* $\Gamma\colon \mathbf{H}\to\mathfrak{D}(B)$ *given by* $\Gamma(u\oplus v)=w\oplus J_* u$ $(u\oplus v\in\mathbf{H})$, *where*

$$w(e^{it}) = e^{-it}[\Theta^* u+J\Delta v](-t). \quad \square$$

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