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# A characterization of compactness

Z. M. RAKOWSKI

The aim of this paper is to prove that if $X$ is a compact Hausdorff space and $f\colon X\to f(X)$ is a continuous function, then $f(\mathrm{Lim\,sup}\, A_n)=\mathrm{Lim\,sup}\, f(A_n)$ for each net $A_n$ of subsets of $X$ (recall that for a net $A_n$ $(n\in D)$ of subsets of a Hausdorff topological space $X$, $\mathrm{Lim\,sup}\, A_n$ is the set of all points $x\in X$, for which the set $\{n,\ A_n\cap U\neq\emptyset\}$ is cofinal in $D$ for every neighbourhood $U$ of $X$). This condition is used to obtain a characterization of compact spaces.

From now on, a *space* always means a Hausdorff topological space. The reader is referred to [2] for general results concerning nets of subsets of a space.

DUDA [1], p. 23, has proved the following fact: if $X$ is a compact metric space and a function $f\colon X\to f(X)$ is continuous, then $f(\mathrm{Lim\,sup}\, A_n)=\mathrm{Lim\,sup}\, f(A_n)$ for each sequence $A_1, A_2, \dots$ of subsets of $X$. We prove much stronger results.

Theorem 1. *A function $f\colon X\to f(X)$ is continuous if and only if $f(\mathrm{Lim\,sup}\, A_n)\subset \subset \mathrm{Lim\,sup}\, f(A_n)$ for each net $\{A_n,\ n\in D\}$ of subsets of $X$.*

Proof. Putting $A_n=A$ we obtain $f(\mathrm{cl}\, A)=f(\mathrm{Lim\,sup}\, A_n)\subset \mathrm{Lim\,sup}\, f(A_n)= \mathrm{cl}\, f(A)$, i.e., the continuity of $f$ (because $A_n=A$ implies $\mathrm{Lim\,sup}\, A_n=\mathrm{cl}\, A$). Conversely, take a net $\{A_n, n\in D\}$ and a point $y\in f(\mathrm{Lim\,sup}\, A_n)$. There is a point $x\in \mathrm{Lim\,sup}\, A_n$ such that $f(x)=y$. By [2], Proposition 2.2, p. 170, there exist a net $\{x_i,\ i\in E\}$ of points of $X$ and a function $p\colon E\to D$ such that the following conditions hold:

(*) for each element $n\in D$ there is an element $i_0\in E$ such that for each element $i\in E$ satisfying $i_0\leqq i$ we have $p(i)\geqq n$, and

$$x_i\in A_{p(i)} \quad \text{und} \quad x=\lim x_i.$$

The function $f$ is continuous, hence $y=f(x)=f(\lim x_i)=\lim f(x_i)$, and therefore $y\in \mathrm{Lim\,sup}\, f(A_n)$ ([2], ib.).

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Theorem 2. *If a space X is compact and a function* $f\colon X\to f(X)$ *is continuous, then* $f(\operatorname{Lim\,sup} A_n)=\operatorname{Lim\,sup} f(A_n)$ *for each net* $\{A_n,\ n\in D\}$ *of subsets of* $X$.

Proof. It is sufficient to prove that $\operatorname{Lim\,sup} f(A_n)\subset f(\operatorname{Lim\,sup} A_n)$. For, take a point $y\in\operatorname{Lim\,sup} f(A_n)$. By [2], ib., there exist a net $\{y_i,\ i\in E\}$ of points of $f(X)$ and a function $p\colon E\to D$ satisfying $(*)$ and the following condition:

$$y_i\in f(A_{p(i)}) \quad\text{and}\quad y=\lim y_i.$$

There are points $x_i\in A_{p(i)}$ such that $f(x_i)=y_i$. Since the space is compact, the net $\{x_i,\ i\in E\}$ has a convergent subnet, say $\{x_k,\ k\in F\}$. With $x=\lim x_k$ we obtain $x\in\operatorname{Lim\,sup} A_n$. Since the function is continuous we infer that the net $\{f(x_k),\ k\in F\}$ converges to $f(x)=y$. This implies that $y\in f(\operatorname{Lim\,sup} A_n)$.

Te following theorem gives a characterization of compact spaces.

Theorem 3. *A space X is compact if and only if there is a continuous function* $f\colon X\to Y$ *onto a compact space Y such that* $f(\operatorname{Lim\,sup} A_n)=\operatorname{Lim\,sup} f(A_n)$ *for each net* $\{A_n,\ n\in D\}$ *of subsets of* $X$.

Proof. Consider a net $\{x_n,\ n\in D\}$ of points of $X$. Since the space $Y$ is compact, the set $\operatorname{Lim\,sup} f(\{x_n\})$ is non-void. It follows that the set $\operatorname{Lim\,sup}\{x_n\}$ is non-void, and consequently, the net $\{x_n,\ n\in D\}$ has a convergent subnet [2], ib. The converse implication follows from Theorem 2.

## References

[1] R. Duda, On biconnected sets with dispersion points, *Dissert. Math.*, **37** (1964).

[2] Z. Frolik, Concerning topological convergence of sets, *Czechoslovak Math. J.*, **10** (1960), 168—180.

INSTITUTE OF MATHEMATICS, WROCŁAW UNIVERSITY
PL. GRUNWALDZKI 2/4.
50—384 WROCŁAW, POLAND