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On the converse of the Fuglede—Putnam theorem

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1. Let $\mathfrak{K}$ and $\mathfrak{H}$ be Hilbert spaces, and let $\mathscr{L}(\mathfrak{K}, \mathfrak{H})$ denote the space of all bounded linear operators from $\mathfrak{K}$ to $\mathfrak{H}$. (We also write $\mathscr{L}(\mathfrak{H})=\mathscr{L}(\mathfrak{H}, \mathfrak{H})$.) The well-known Fuglede—Putnam theorem [2] asserts that if $A\in\mathscr{L}(\mathfrak{H})$ and $B\in\mathscr{L}(\mathfrak{K})$ are normal, then the pair (A, B) of operators has the following property:

(FP) *If* $AX=XB$ *where* $X\in\mathscr{L}(\mathfrak{K}, \mathfrak{H})$, *then* $A^*X=XB^*$.

In this note we shall show that the normality of A and B in the above theorem is essential.

2. We say that an ordered pair (A, B) of operators $\big(A\in\mathscr{L}(\mathfrak{K})$ and $B\in\mathscr{L}(\mathfrak{H})\big)$ is *disjoint* if the only operator $X\in\mathscr{L}(\mathfrak{K}, \mathfrak{H})$ satisfying $AX=XB$ is $X=0$. Let $A=A_1\oplus A_2$ and $B=B_1\oplus B_2$. Then it is easy to see that (A, B) is disjoint if and only if (A_i, B_j) $(i, j=1, 2)$ is disjoint. Also, if (A, B) is disjoint, then it trivially satisfies the property (FP). We recall the fact that each operator A can be written uniquely $A=A_{(n)}\oplus A_{(c.n.)}$ where $A_{(n)}$ is normal and $A_{(c.n.)}$ is *completely nonnormal*, that is, no nontrivial direct summand of $A_{(c.n.)}$ is normal (see e.g. [1]).

Theorem. *Let* $A\in\mathscr{L}(\mathfrak{H})$ *and* $B\in\mathscr{L}(\mathfrak{K})$. *The following statements are equivalent:*

(i) *The pair* (A, B) *has the property* (FP).

(ii) *If* $AY=YB$ *where* $Y\in\mathscr{L}(\mathfrak{K}, \mathfrak{H})$, *then* $(\operatorname{ran} Y)^-$ *reduces* A, $(\ker Y)^\perp$ *reduces* B, *and the restrictions* $A|(\operatorname{ran} Y)^-$ *and* $B|(\ker Y)^\perp$ *are normal operators, where* ran *and* ker *denote the range and the kernel, respectively.*

(iii) *The pairs* $(A, B_{(c.n.)})$ *and* $(A_{(c.n.)}, B)$ *are disjoint.*

(iv) *A and B can be decomposed as follows:*

$A=A_1\oplus A_2$ *and* $B=B_1\oplus B_2$, *where* A_1 *and* B_1 *are normal, and the pairs* (A, B_2) *and* (A_2, B) *are disjoint.*

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Proof. (i)⇒(ii): Since $AY=YB$ and (A, B) satisfies (FP), $A^*Y=YB^*$ and so $(\mathrm{ran}\, Y)^-$ and $(\ker Y)^\perp$ are reducing subspaces for A and B, respectively. Since $A(AY)=(AY)B$, we obtain $A^*(AY)=(AY)B^*$ by (FP), and the identity $A^*Y==YB^*$ implies $A^*AY=AA^*Y$. Thus we see that $A|(\mathrm{ran}\, Y)^-$ is normal. Clearly (B^*, A^*) satisfies (FP), and $B^*Y^*=Y^*A^*$. Therefore it follows from the above argument that $B^*|(\mathrm{ran}\, Y^*)^-=(B|(\ker Y)^\perp)^*$ is normal.

(ii)⇒(iii): Let us write $A=A_{(n)}\oplus A_{(c.n.)}$ on $\mathfrak{H}=\mathfrak{H}_{(n)}\oplus\mathfrak{H}_{(c.n.)}$. Suppose that $A_{(c.n.)}X=XB$ where $X\in\mathcal{L}(\mathfrak{K}, \mathfrak{H}_{(c.n.)})$. We define $\tilde{X}\in\mathcal{L}(\mathfrak{K}, \mathfrak{H})$ by setting $\tilde{X}x=Xx$ for $x\in\mathfrak{K}$. Then $A\tilde{X}=\tilde{X}B$, and by the condition (ii) $(\mathrm{ran}\, \tilde{X})^-$ reduces A and $A|(\mathrm{ran}\, \tilde{X})^-$ is normal, that is, $(\mathrm{ran}\, X)^-$ reduces $A_{(c.n.)}$ and $A_{(c.n.)}|(\mathrm{ran}\, X)^-$ is normal. But since $A_{(c.n.)}$ has no normal direct summand, $(\mathrm{ran}\, X)^-=\{0\}$, that is, $X=0$. Thus $(A_{(c.n.)}, B)$ is disjoint. Similarly, we see that $(A, B_{(c.n.)})$ is disjoint.

(iii)⇒(iv) is trivial.

(iv)⇒(i): Let $\mathfrak{H}=\mathfrak{H}_1\oplus\mathfrak{H}_2$ and $\mathfrak{K}=\mathfrak{K}_1\oplus\mathfrak{K}_2$ be the decompositions corresponding to $A=A_1\oplus A_2$ and $B=B_1\oplus B_2$, respectively. Suppose $AX=XB$. Then by the condition (iv) X has the form $X=\begin{pmatrix} X_1 & 0 \\ 0 & 0 \end{pmatrix}$ with respect to the decompositions $\mathfrak{K}=\mathfrak{K}_1\oplus\mathfrak{K}_2$ and $\mathfrak{H}=\mathfrak{H}_1\oplus\mathfrak{H}_2$. Therefore for the proof of the equation $A^*X=XB^*$ it suffices to show $A_1^*X_1=X_1B_1^*$, but this follows from the Fuglede—Putnam theorem since A_1 and B_1 are normal.

3. The following fact is known as a corollary of the Fuglede—Putnam theorem (see [2, Theorem 1.6.4] and its proof). Let $A\in\mathcal{L}(\mathfrak{H})$ and $B\in\mathcal{L}(\mathfrak{K})$ be normal. If there exists a *quasi-affinity* $X\in\mathcal{L}(\mathfrak{K}, \mathfrak{H})$ (i.e., X is one-to-one and has dense range) such that $AX=XB$, then A and B are unitarily equivalent.

An immediate corollary of our theorem is the following.

Corollary 1. *Suppose that (A, B) has the property* (FP). *If there exists a quasi-affinity X such that $AX=XB$, then A and B are unitarily equivalent normal operators.*

An operator A is called *hyponormal* or *cohyponormal* according as $A^*A-AA^*\geqq 0$ or $\leqq 0$. Radjabalipour [3], Stampfli and Wadhwa [4] proved the following theorem (indeed, they obtained more general results there); if A is hyponormal and B is cohyponormal, and if there exists a quasi-affinity X such that $AX=XB$, then A and B are normal operators.

We can rephrase their theorem as follows;

Corollary 2. *If A is hyponormal and B is cohyponormal then the pair (A, B) has the property* (FP).

Proof. It is easy to see that every invariant subspace for a hyponormal operator T on which T is normal is reducing. From this fact and the theorem of Radjabalipour, Stampfli and Wadhwa, we see that (A, B) satisfies the condition (ii) in Theorem. Therefore (A, B) has the property (FP).

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