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On the radical classes determined by regularities

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1. Introduction

All rings considered in this paper are associative. For a given class $\mathscr{C}$ of rings each ring $A\in\mathscr{C}$ is called a *$\mathscr{C}$-ring*, and an ideal B of a ring A is called *ideal* if B (as ring) is a $\mathscr{C}$-ring.

It is well known that a non-empty subclass $\mathscr{C}$ of rings is a *radical class* or briefly a *radical* (relative to the class of all associative rings) in the sense of KUROŠ [13] and AMITSUR [1] if it satisfies the following conditions:

(i) $\mathscr{C}$ is homomorphically closed, that is, every homomorphic image of a $\mathscr{C}$-ring is a $\mathscr{C}$-ring.

(ii) The sum of all $\mathscr{C}$-ideas of a ring A is a $\mathscr{C}$-ideal.

(iii) $\mathscr{C}$ is closed under extensions, that is, if both B and A/B are $\mathscr{C}$-rings, then A is also a $\mathscr{C}$-ring.

In ring theory many so-called *regularities* determine radical classes, for instance the von Neumann regularity [17], quasi-regularity [18], G-regularity [6], strong regularity [3], and so on.

The aim of this paper is to give the definition of regularity of associative rings in the common terminology of polynomials and formal power serieses, and to show the radical characteristic of regularities in this sense. At the same time we shall get a diagram to define radicals by regularities. In view of our results it become clear that well known regularities and ring properties considered in [14], [17], [22] and [25], are radicals.

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2. Regularities determined by polynomials

$Z[x_0, x_1, ..., x_n]$ denotes the set of polynomials in non-commutative indeterminates $x_0, x_1, x_2, ..., x_n$ with integer coefficients.

Definition 1. Suppose $f(x_0, x_1, ..., x_n)$ is in $Z[x_0, x_1, ..., x_n]$. An element a_0 of a ring A is said to be *f regular* if there exist elements $a_1, a_2, ..., a_n$ in A such that the equality

$$f(a_0, a_1, ..., a_n) = 0$$

is valid in A.

A ring A is said to be f-regular if every element of A is f-regular. An ideal B of a ring A is an f-regular ideal if B is an f-regular ring.

The following theorem characterizes the radical property for f-regularities.

Theorem 1. *Suppose $f(x_0, x_1, ..., x_n) \in Z[x_0, x_1, ..., x_n]$, then the class of all f-regular rings is a radical class if and only if the following conditions are satisfied:*

1) *$f(x_0, x_1, ..., x_n)$ has no constant term.*

2) *If B is an f-regular ideal of a ring A, and for every $a_0 \in A$ there exist elements $a_1, a_2, ..., a_n$ in A such that $f(a_0, a_1, ..., a_n) \in B$, then A is an f-regular ring.*

Proof. Assume that the class $\mathscr{C}$ of all f-regular rings is a radical class. Since the zero ideal is a $\mathscr{C}$-ideal in every ring, the first condition is always satisfied.

Now suppose that B is an f-regular ideal of a ring and for every $a_0 \in A$ there exist elements $a_1, a_2, ..., a_n$ such that $f(a_0, a_1, ..., a_n) \in B$. We have to show that the ring A is f-regular. Let us consider the factor ring A/B. Take any element $\bar{a} \in A/B$. Let an element a_0 be in the coset $\bar{a}$. By hypothesis there exist elements $a_1, a_2, ..., a_n$ such that $f(a_0, a_1, ..., a_n) \in B$. So in the factor ring A/B the equality

$$f(\bar{a}, \bar{a}_1, ..., \bar{a}_n) = 0$$

holds. Hence the element $\bar{a}$ is f-regular. This implies the f-regularity of A/B. Since radicals are closed under extensions, A is f-regular. Thus the second condition is valid.

Conversely, assume that $f(x_0, x_1, ..., x_n)$ satisfies the conditions of the theorem. Clearly, $\mathscr{C}$ is homomorphically closed. Now, suppose that for an ideal J of a ring A, both J and A/J are $\mathscr{C}$-rings. Since A/J is f-regular, therefore for every element $a_0 \in A$ there exist elements $a_1, a_2, ..., a_n$ in A such that the cosets $\bar{a}_0, \bar{a}_1, ..., \bar{a}_n$ satisfy the equality

$$f(\bar{a}_0, \bar{a}_1, \bar{a}_2, ..., \bar{a}_n) = 0$$

in the factor ring A/J. This implies $f(a_0, a_1, ..., a_n) \in J$. By the second condition of the theorem, the ring A is f-regular. Hence the class $\mathscr{C}$ is closed under extensions.

Suppose both B_1 and B_2 be $\mathscr{C}$-ideals of a ring A. By the second isomorphism theorem we have

$$\frac{B_1+B_2}{B_2} \cong \frac{B_1}{B_1\cap B_2}.$$

Since the class $\mathscr{C}$ is homomorphically closed and closed under extensions the above isomorphism implies that B_1+B_2 is a $\mathscr{C}$-ring. By a simple induction we can prove that the sum of any finite number of $\mathscr{C}$-ideals of a ring A is again a $\mathscr{C}$-ideal.

Finally, it is easy to see that the sum $\mathscr{C}(A)$ of all $\mathscr{C}$-ideals of a ring A is a $\mathscr{C}$-ideal. This completes the proof of the theorem.

As a radical criterion of f-regularities we have the following

Corollary 1. *For a polynomial $f(x_0, x_1, \dots, x_n)$ in $Z[x_0, x_1, \dots, x_n]$ without constant term, the class of all f-regular rings is a radical class if one of the following two conditions is satisfied.*

(A) *For arbitrary elements $a_0, a_1, \dots, a_n$ in a ring A, if the element $f(a_0, a_1, \dots, a_n)$ is f-regular then the element a_0 is also f-regular.*

(B) *Let B be an f-regular ideal of a ring A; if the coset $\bar{a}_0$ containing $a_0 \in A$ is f-regular in the factor ring A/B, then the element a_0 is f-regular in the ring A.*

Proof. The assertion is an immediate consequence of Theorem 1. It is easy to check that the conditions of Theorem 1 are satisfied.

Remark. By Corollary 1, the conditions (A) and (B) are sufficient for an f-regularity to be a radical. It is not known whether the converse is true.

3. Regularities determined by formal power serieses

We shall use the following notations: $X_i = (x_{i1}, x_{i2}, \dots, x_{ik}, \dots)$ for $i = 1, 2, \dots, n$; $Z\{X_1, X_2, \dots, X_n\}$ denotes the set of all formal power series in infinite number of non-commutative indeterminates $x_{i1}, x_{i2}, \dots;\ i = 1, 2, \dots, n$, and with integer coefficients; that is, every $f[X_1, X_2, \dots, X_n] \in \tilde{Z}\{X_1, X_2, \dots, X_n\}$ may be written in the form

$$f[X_1, X_2, \dots, X_n] = \sum_{\alpha} m_\alpha \prod_{k=1}^{n_\alpha} x_{i_k l_k}^{\alpha_{i_k l_k}}$$

where $m_\alpha \in Z$, and $x_{ik}x_{jl} \neq x_{jl}x_{ik}$ if $(i, k) \neq (j, l)$.

For arbitrary natural numbers α_i, $i = 1, 2, \dots, n$, $f|_{\langle \alpha_1, \alpha_2, \dots, \alpha_n \rangle}$ denotes the expression which is obtained from $f[X_1, X_2, \dots, X_n]$ by putting $x_{ik} = 0$ for $k > \alpha_i$; $i = 1, 2, \dots, n$.

Definition 2. The formal power series $f[X_1, X_2, \ldots, X_n] \in \tilde{Z}\{X_1, X_2, \ldots, X_n\}$ is said to be *admissible* if for arbitrary natural numbers α_i, $i=1, 2, \ldots, n$, we always have

$$f|_{\langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle} \in Z[x_{11}, \ldots, x_{1\alpha_1}, \ldots, x_{n_1}, \ldots, x_{n\alpha_n}].$$

Examples. Let us consider the simple case $n=2$. Let

$$X_1 = X = (x_0, x_1, x_2, \ldots), \quad X_2 = Y = (y_0, y_1, y_2, \ldots).$$

a) Consider

$$f_1(X, Y) = \sum_{i=0}^{\infty} (a_i x_i y_i + b_i y_i x_i).$$

For arbitrary natural numbers α_i, $i=1, 2$, we have

$$f_1|_{\langle \alpha_1, \alpha_2 \rangle} = \sum_{i=0}^{\alpha} a_i x_i y_i + b_i y_i x_i$$

where $\alpha = \min\{\alpha_1, \alpha_2\}$. Therefore, $f_1|_{\langle \alpha_1, \alpha_2 \rangle}$ is in $Z[x_0, \ldots, x_{\alpha_1}, y_0, \ldots, y_{\alpha_2}]$ and $f_1[X, Y]$ is admissible.

b) Let

$$f_2[X, Y] = \sum_{k=0}^{\infty} x_0^k + y_0^k + x_k y_k.$$

For any natural numbers α_i, $i=1, 2$, we have

$$f_2|_{\langle \alpha_1, \alpha_2 \rangle} = \sum_{k=0}^{\infty} x_0^k + y_0^k + \sum_{i=0}^{\alpha} x_i y_i$$

where $\alpha = \min\{\alpha_1, \alpha_2\}$. Clearly, $f_2|_{\langle \alpha_1, \alpha_2 \rangle}$ is not in $Z[x_0, \ldots, x_{\alpha_1}, y_0, \ldots, y_{\alpha_2}]$. Therefore, $f_2[X, Y]$ is not admissible.

Definition 3. Suppose that the formal power series

$$f[X_1, X_2, \ldots, X_n] \in \tilde{Z}\{X_1, X_2, \ldots, X_n]$$

is admissible. An element a_0 of a ring A is called *f-regular* in A if there exist natural numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that the polynomial $f|_{\langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle}$ has at least one solution in A with $x_{i1} = a_0$.

A ring A is said to be f-regular if every element of A is f-regular. An ideal B of a ring A is f-regular if B is an f-regular ring.

The following assertions are analogous to the corresponding assertions in section 2. The proof of the following theorem is a minor modification of the above proof of Theorem 1 therefore, we omit it.

Theorem 1′. *Suppose that the formal power series*

$$f[X_1, X_2, \ldots, X_n] \in \tilde{Z}\{X_1, X_2, \ldots, X_n\}$$

is admissible. Then the class of all f-regular rings is a radical class if and only if the following conditions are satisfied.

1)′ $f[X_1, X_2, \ldots, X_n]$ *has no constant term.*

2)′ *If B is an f-regular ideal of A, and for every element* $a_{11} \in A$ *there exist natural numbers* $\alpha_1, \alpha_2, \ldots, \alpha_n$ *and elements* $a_{i1}, \ldots, a_{i\alpha_i}$, $i=1, 2, \ldots, n$, *in A such that* $f|_{\langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle}(a_{11}, \ldots, a_{n\alpha_n}) \in B$, *then the ring A is f-regular.*

Corollary 1′. *Suppose that the formal power series*

$$f[X_1, X_2, \ldots, X_n] \in \tilde{Z}\{X_1, X_2, \ldots, X_n\}$$

without constant terms is admissible. Then the class of all f-regular rings is a radical class if one of the following two conditions is satisfied.

(A)′ *For arbitrary elements* $a_{i1}, a_{i2}, \ldots, a_{i\alpha_i}$, $i=1, 2, \ldots, n$, *in a ring A, if the element* $f|_{\langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle}(a_{11}, \ldots, a_{n\alpha_n})$ *is f-regular, then the element* a_{11} *is also f-regular.*

(B)′ *Let J be an f-regular ideal of a ring A; if the coset* $\bar{a}_0$ *containing* $a_0 \in A$ *is f-regular in the factor ring A/J, then the element* a_0 *is f-regular in A.*

4. Applications

For the sake of brevity we shall call a polynomial or an admissible formal power series f a *radical expression* if the class of all f-regular rings is a radical class. Next we shall give some radical expressions.

Proposition 2. *The following formal power series are radical expressions.*

a) $$G(m_1, m_2, m_3, m_4) = x_0 + m_1 x_1 + m_2 x_0 x_1 + \sum_{i=1}^{\infty} m_3 y_i x_0 Z_i + m_4 y_i Z_i$$

where m_i, $i=1, \ldots, 4$, *are integers satisfying the condition* $m_1 m_3 = m_2 m_4$.

b) $$F(m_1, m_2, m_3) = x_0 + m_1 x_1 x_0 + m_2 x_0 x_2 + \sum_{i=1}^{\infty} m_s y_i x_0 Z_i$$

where m_i, $i=1, 2, 3$, *are integers satisfying the condition* $m_1 m_2 = 0$ *or* $m_1 m_2 = m_3$.

c) $$H(n, k) = x_0 + \sum_{i=0}^{\infty} k y_{1i} x_0 y_{2i} \ldots x_0 y_{ni}$$

d) $$P_n[p_1(x_0), p_2(x_0)] = x_0 + \sum_{i=1}^{\infty} p_1(x_0) y_{1i} p_1(x_0) \ldots y_{n_i} p_2(x_0)$$

where $p_i(x) \in Z[x]$, $i=1, 2$.

Proof. In order to prove that $G(m_1, m_2, m_3, m_4)$,

$$F(m_1, m_2, m_3), \quad P_n[p_1(x_0), p_2(x_0)] \quad \text{and} \quad H(n, k)$$

are radical expressions, we shall show that each of them satisfies one of the conditions of Corollary 1'.

First we prove that $G(m_1, m_2, m_3, m_4)$ satisfies condition (A)' of Corollary 1'. Suppose $a_0, \ldots, a_{\alpha_1}, b_0, \ldots, b_{\alpha_2}, c_0, \ldots, c_{\alpha_3}$ are elements of a ring A such that the element

$$G(m_1, m_2, m_3, m_4)|_{\langle\alpha_1, \alpha_2, \alpha_3\rangle}(a_0, \ldots, c_{\alpha_3}) = a_0 + m_1 a_1 + m_2 a_0 a_1 + \sum_{i=1}^{\alpha} m_3 b_i a_0 c_i + m_4 b_i c_i,$$

where $\alpha = \min\{\alpha_1, \alpha_2\}$, is $G(m_1, m_2, m_3, m_4)$-regular. By Definition 3, there exist elements $a_1', b_0', b_1', \ldots, b_{\alpha_2}', c_0', c_1', \ldots, c_{\alpha_3}'$ in A such that the following equality is satisfied:

$$\left[a_0 + m_1 a_1 + m_2 a_0 a_1 + \sum_{i=1}^{\alpha} m_3 b_i a_0 c_i + m_4 b_i c_i\right] + m_1 a_1' +$$
$$+ m_2 \left[a_0 + m_1 a_1 + m_2 a_0 a_1 + \sum_{i=1}^{\alpha} m_3 b_i a_0 c_i + m_4 b_i c_i\right] a_1' +$$
$$+ \sum_{j=1}^{\alpha'} \left[m_3 b_j' \left(a_0 + m_1 a_1 + m_2 a_0 a_1 + \sum_{i=1}^{\alpha} m_3 b_i a_0 c_i + m_4 b_i c_i\right) c_j' + m_4 b_j' c_j'\right] = 0$$

where $\alpha' = \min\{\alpha_2', \alpha_3'\}$.

A straightforward calculation shows that

$$a_0 + m_1 a_1'' + m_2 a_0 a_1'' + \sum_{k=1}^{\alpha''} m_3 b_k'' a_0 c_k'' + m_4 b_k'' c_k'' = 0$$

where

$$\alpha'' = 2(\alpha + \alpha') + \alpha\alpha',$$
$$a_1'' = a_1 + a_1' + m_2 a_1 a_1',$$

$$b_k'' = \begin{cases} b_k & \text{if } 0 < k \leqq \alpha, \\ b_i & \text{if } \alpha < k = \alpha + i \leqq 2\alpha, \\ b_i' & \text{if } 2\alpha < k = 2\alpha + i \leqq 2\alpha + \alpha', \\ m_2 b_i' & \text{if } 2\alpha + \alpha' < k = 2\alpha + \alpha' + i \leqq 2(\alpha + \alpha'), \\ m_3 b_j' b_i & \text{if } 2(\alpha+\alpha') + (j-1)\alpha < k = 2(\alpha+\alpha') + (j-1)\alpha + i \leqq \\ & \qquad \leqq 2(\alpha+\alpha') + j\alpha \quad \text{for } j = 1, 2, \ldots, \alpha', \end{cases}$$

$$c_k'' = \begin{cases} c_k & \text{if } 0 < k \leqq \alpha, \\ c_i a_1' & \text{if } \alpha < k = \alpha + i \leqq 2\alpha, \\ c_i' & \text{if } 2\alpha < k = 2\alpha + i \leqq 2\alpha + \alpha', \\ a_1 c_i' & \text{if } 2\alpha + \alpha' < k = 2\alpha + \alpha' + i \leqq 2(\alpha + \alpha'), \\ c_i c_j' & \text{if } 2(\alpha+\alpha') + (j-1)\alpha < k = 2(\alpha+\alpha') + (j-1)\alpha + i \leqq 2(\alpha+\alpha') + j\alpha \\ & \quad \text{for } j = 1, 2, \ldots, \alpha'. \end{cases}$$

Hence the element a_0 is $G(m_1, m_2, m_3, m_4)$-regular, and condition (A)′ is satisfied. Thus $G(m_1, m_2, m_3, m_4)$ is a radical expression. The remaining assertions are proved similarly.

Now let us survey some well-known regularities and ring properties which have already been shown to be radicals.

1) An element a_0 of a ring A is said to be regular in the sense of VON NEUMANN [17], if $a_0 \in a_0 A a_0$. If in a) in Proposition 2 we take $n=1$, $p_i(x_0)=x_0$, $i=1, 2$, then, clearly, $P_1(x_0, x_0)$-regularity coincides with the regularity in the sense of von Neumann. Therefore, the class of all von Neumann regular rings is a radical class.

2) An element a_0 of a ring A is said to be right quasi-regular, as defined by PERLIS [18] and later studied by BAER [4] and JACOBSON [11], if $a_0+a_1+a_0a_1=0$, for some element a_1 of A. By a) and d) in Proposition 2 we have

$$G(1, 1, 0, 0) = x_0+x_1+x_0x_1.$$

Hence right quasi-regularity is nothing else than $G(1, 1, 0, 0)$-regularity. Thus, right quasi-regularity is a radical property, namely, the Jacobson radical.

3) BROWN and MCCOY [6] have introduced the notion of G-regularity. An element a_0 of a ring A is said to be G-regular if the element a_0 is in $G(a_0)$, where

$$G(a_0) = A(1+a_0)+A(1+a_0)A.$$

By a) in Proposition 2 it is clear that $G(1, 1, 1, 1)$-regularity coincides with G-regularity. Thus the Brown—McCoy radical may be determined by the radical expression $G(1, 1, 1, 1)$.

4) The notion of strongly regular rings had been introduced by ARENS and KAPLANSKY [3] and was later studied by KANDÔ [12], LAJOS and SZÁSZ [14] and others. A ring A is strongly regular if $a \in a^2 A$ for every $a \in A$. If in d) in Proposition 2 we take $n=1$, $p_1(x_0)=x_0^2$, $p_2(x_0)=1$, then it is clear that $P_1(x_0^2, 1)$-regularity is the same as strong regularity. Thus, strong regularity is a radical property.

5) DE LA ROSE [19] has introduced the notion of λ-regularity. An element a_0 of a ring A is λ-regular if $a_0 \in A a_0 A$. By a) and d) in Proposition 2 we have

$$G(0, 0, 1, 0) = F(0, 0, 1) = x_0 + \sum_{i=1}^{\infty} y_i x_i Z_i.$$

Clearly, λ-regularity can be defined by the radical expression $G(0, 0, 1, 0)$. Thus the class of λ-regular rings is a radical class.

6) DIVINSKY [8] has introduced left pseudo-regularity. An element a_0 of a ring A is left pseudo-regular if $a_0+a_1a_0+a_1a_0^2=0$ for some element $a_1 \in A$. If in d) in

Proposition 2 we take $n=1$, $p_1(x_0)=1$, $p_2(x_0)=x_0+x_0^2$, then it is easy to see that $P_1(1, x_0+x_0^2)$-regularity coincides with left pseudo-regularity. Therefore left pseudo-regularity is a radical property.

7) Following Szász [22] a ring A is called an E_5-ring if every homomorphic image of A has no non-zero left annihilators. As is proved in [22], a ring A is an E_5-ring if and only if $a \in Aa+AaA$ holds for every $a \in A$. By b) in Proposition 2, the class of E_5-rings is the class of all $F(1, 0, 1)$-regular rings, so it is a radical class.

8) Following Szász [25] a ring A is called an E_6-ring if every homomorphic image of A has no non-zero two-sided annihilators. A ring A is an E_6-ring if and only if $a \in aA+Aa+AaA$ holds for every $a \in A$. By b) in Proposition 2 the class of E_6-rings coincides with the class of $F(1, 1, 1)$-regular rings. Thus, it is a radical class.

9) Blair [5] introduced the notion of f-regularity, which was later studied by Andrunakievič [2]. An element a of a ring A is said to be f-regular (in the sense of Blair) if $a \in (a)^2$, where (a) denotes the principal ideal of A generated by a. Blair has shown that an element a in a ring A is f-regular if and only if there exist elements u_i, v_i and w_i in A such that $a=\sum_{i=1}^{n} u_i a_0 v_i a w_i$. Hence, by c) in Proposition 2, f-regularity in the sense of Blair is the same as $H(3, 1)$-regularity. Thus it is a radical property.

10) By b) in Proposition 2 we have

$$F(1,0,0) = x_0+x_1x_0.$$

Therefore $F(1, 0, 0)$-regularity is D-regularity of Divinsky [9].

11) If in d) in Proposition 2 we take $n=1$, $p_1(x_0)=q(x_0)$, $p_2(x_0)=q(x_0)$, then we see that $P_1(p(x_0), q(x_0))$-regularity is nothing else than (p, q)-regularity, introduced by McKnight [15] and also studied by others [10], [16].

Remark. By means of Proposition 2, we can get a great variety of radical classes. In order to show that consider for instance the radical expressions $G(p, -p, k, -k)$, where p is a prime number. Denote by $S_{(p,k)}$ the class of $G(p, -p, k, -k)$-regular rings. Take a fixed set of symbols $M=\{\alpha, \beta, \ldots\}$. Let A_p be the (associative and noncommutative) ring on the set M over the field Z_p of integers modulo p with the relation: $\alpha\beta=\alpha$, $\alpha, \beta \in M$.

One can prove easily that A_p is not in $S_{(p,k)}$ but it belongs to $S_{(p',k)}$ for every prime $p'>p$. Hence $S_{(p,k)} \neq S_{(p',k)}$ if $p \neq p'$.

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