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The hyperbolic M. Riesz theorem

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1. Introduction. We shall prove the non-Euclidean hyperbolic versions of the theorems of M. RIESZ [7] and of L. FEJÉR and F. RIESZ [4] (see [8, Theorem VIII.45, p. 339 and Theorem VIII.46, p. 340], [3, p. 46]) and show a property of conformal mappings in terms of the non-Euclidean geometry in the unit disk.

Let $D=\{|z|<1\}$, $T=[0, 2\pi)$, and $K=\{e^{it}|t\in T\}$. Let

$$\sigma(z, w) = \tanh^{-1}(|z-w|/|1-\bar{z}w|)$$

be the non-Euclidean hyperbolic distance between z and w in D, and let

$$\sigma(z) \equiv \sigma(z, 0) = (1/2)\log[(1+|z|)/(1-|z|)], \quad z\in D.$$

Let B be the family of functions f, holomorphic and bounded, $|f|<1$, in D. Then $\sigma(f)$ for $f\in B$, like $|f|$, has the property that $\log\sigma(f)$ is subharmonic in D, so that $\sigma(f)^p=\exp[p\log\sigma(f)]$ is subharmonic in D for all $p>0$; see [10]. Let H^p_σ be the family of $f\in B$ such that

$$\int_T \sigma(f)^p(re^{it})\,dt$$

is bounded for $0\leqq r<1$ $(0<p<\infty)$. The class H^p_σ is the hyperbolic counterpart of the (parabolic) Hardy class H^p in D [3, p. 2], and is called the hyperbolic Hardy class. (Recently, it is observed that an "elliptic" analogue of H^p $(0<p<\infty)$, namely, a meromorphic Hardy class yields no new family [11, Theorem 1].) Each $f\in B$ has the radial limit $f^*(t)=\lim_{r\to 1-0} f(re^{it})$ at e^{it} for a.e. $t\in T$, and as will be seen, $\sigma(f^*)$ is of class $L^p(T)$ for all $f\in H^p_\sigma$ $(0<p<\infty)$. The hyperbolic M. Riesz theorem is

Theorem 1. *Let C be a rectifiable curve with the initial point a and the terminal point b (possibly, $a=b$) in the complex plane. Suppose that*

$$C\subset D\cup K \quad \text{and} \quad C\cap K\subset\{a, b\}.$$

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Set for $t \in T$, *with* $e^{it} \notin C \cap K$,

$$V(t) = \int_C |d \arg(e^{it} - w)| \quad (w \in C).$$

Then, for each $f \in H_\sigma^p$ $(0 < p < \infty)$,

$$\int_C \sigma(f)^p(z)\,|dz| \leqq \pi^{-1} \int_T \sigma(f^*)^p(t) V(t)\,dt.$$

If C is the diameter $\{xe^{is} | -1 \leqq x \leqq 1\}$ $(s \in T)$, then $V(t) \equiv \pi/2$, so that the hyperbolic Fejér and F. Riesz theorem is

Theorem 2. *For each* $f \in H_\sigma^p$ $(0 < p < \infty)$ *and each* $s \in T$,

$$\int_{-1}^{1} \sigma(f)^p(xe^{is})\,dx \leqq (1/2) \int_T \sigma(f^*)^p(t)\,dt.$$

The Fejér and F. Riesz theorem has the obvious application to conformal mappings from D onto a Jordan domain with the rectifiable boundary [4, Satz IV]; see [8, Corollary, p. 341] and [3, Corollary, p. 47]. The hyperbolic version is not so apparent as in the cited case; namely, the following theorem does not appear to be a direct consequence of Theorem 2. There is no relation between $\sigma(f)$ and $|f'|/(1-|f|^2)$ like that between $|f|$ and $|f'|$.

Theorem 3. *Let* γ *be a Jordan curve in* D *with finite non-Euclidean length* L. *Let* f *be a one-to-one conformal mapping from* D *onto the interior of* γ. *Then the non-Euclidean length of the image of each diameter by* f *is not greater than* $L/2$. *The constant* 2 *in* $L/2$ *cannot be replaced by any larger constant.*

For the proofs of Theorems 1 and 3, the principal idea is to obtain the M. Riesz theorem for subharmonic functions of class PL in the sense of E. F. Beckenbach and T. Radó [2] (see also [6, p. 9]); see Theorem 4 in Section 2.

2. Subharmonic functions of class PL. A function u defined in D is called of class PL in D if $u \geqq 0$ (possibly, $u \equiv 0$) and $\log u$ is subharmonic in D; we regard $-\infty$ as a subharmonic function. The family of all functions of class PL in D is denoted by PL again. All members of PL are subharmonic in D, and if $u \in PL$, then $u^p \in PL$ for each $p > 0$. If f is holomorphic in D, then $|f| \in PL$, and further, if $f \in B$, then $\sigma(f) \in PL$. Let PL^p be the family of all $u \in PL$ such that u^p has a harmonic majorant in D $(0 < p < \infty)$. Here, a function v subharmonic in D is said to have a harmonic majorant h in D if h is harmonic and $v \leqq h$ in D. The class H^p is the family of f holomorphic in D such that $|f| \in PL^p$, while H_σ^p is the family of $f \in B$ such that $\sigma(f) \in PL^p$ $(0 < p < \infty)$.

Theorem 4. *Let C and V be as in Theorem* 1. *Then, each* $u \in PL^p$ $(0<p<\infty)$ *has the radial limit $u^*(t)$ at e^{it} for a.e.* $t \in T$, *and*

$$\int_C u^p(z)\,|dz| \leqq \pi^{-1} \int_T (u^*)^p(t) V(t)\,dt.$$

Earlier, a special case of Theorem 4, where $p=1$, C is an arbitrary diameter, u is continuous on $D \cup K$, and $u \in PL$, was established by BECKENBACH [1, Theorem 2]. It is now an easy exercise to extend a geometric theorem of BECKENBACH [1, Theorem 3] with the aid of Theorem 4.

Theorem 1 (and consequently, Theorem 2) now follows from Theorem 4, applied to $\sigma(f) \in PL^p$; note that $\sigma(f)^* = \sigma(f^*)$. The theory of subharmonic functions of class PL thus serves for the differential geometry, as originated by Beckenbach and Radó, as well as for the hyperbolic Hardy classes.

For the proof of Theorem 4 we shall make use of

Lemma 1 [5, Theorem]. *Let* $\varphi \geqq 0$ *be a function convex and increasing on* $(-\infty, +\infty)$, *and suppose that*

$$\varphi(t)/t \to +\infty \quad as \quad t \to +\infty.$$

Set $\varphi(-\infty) = \lim_{t \to -\infty} \varphi(t)$, *and let v be a subharmonic function in D such that* $\varphi(v)$, *again subharmonic, has a harmonic majorant in D. Then the radial limit $v^*(t)$ exists at e^{it} for a.e.* $t \in T$, *and is of $L^1(T)$, such that*

$$v(z) \leqq (2\pi)^{-1} \int_T \frac{1-|z|^2}{|e^{it}-z|^2} v^*(t)\,dt.$$

Furthermore, $\varphi(v^*) \in L^1(T)$.

In effect, v admits a positive harmonic majorant in D (see [9, p. 65]), so that $v = v^\wedge - q$, where $q \geqq 0$ is a Green's potential and $v^\wedge$ is the least harmonic majorant of v in D, expressed by the Poisson integral of a signed measure

$$d\mu(t) = v^*(t)\,dt + d\mu_S(t) \quad \text{on } T,$$

where $d\mu_S$ is singular with respect to dt. Now, [5, Theorem] asserts that $d\mu_S(t) \leqq 0$ on T and $\varphi(v^*) \in L^1(T)$.

Proof of Theorem 4. Since $u^p \in PL^1$ with $(u^p)^* = (u^*)^p$, it suffices to prove the theorem in the case $p=1$. Set $\varphi(t) = e^t$ and $v = \log u$ and consider Lemma 1. Since $\varphi(v)$ has a harmonic majorant, v has the harmonic majorant

$$h(z) = (2\pi)^{-1} \int_T \frac{1-|z|^2}{|e^{it}-z|^2} v^*(t)\,dt \quad (z \in D).$$

Furthermore, $h^* = v^* = \log u^*$. Since $u^* = \varphi(v^*) \in L^1(T)$ by Lemma 1, it follows from Jensen's inequality that $e^h \leqq g$, where g is the Poisson integral of $\varphi(v^*)=u^*$. Thus,

$$f \equiv e^{h+ik} \in H^1,$$

where k is a harmonic conjugate of h in D. Therefore $|f^*|=|f|^*=e^{h^*}=u^*$ and

$$u = e^v \leqq e^h = |f| \quad \text{in } D.$$

We now apply M. Riesz's cited theorem to f of Hardy class H^1 to obtain the following chain of estimates:

$$\int_C u(z)\,|dz| \leqq \int_C |f(z)|\,|dz| \leqq \pi^{-1} \int_T |f^*(t)|\,V(t)\,dt =$$

$$= \pi^{-1} \int_T u^*(t) V(t)\,dt,$$

whence follows Theorem 4.

3. Conformal mappings. We remember that if f is holomorphic in D and if $f' \in H^1$, then f is continuous on $D \cup K$ and $f(e^{it})$ is absolutely continuous as a function of $t \in T$ with

$$\frac{d}{dt} f(e^{it}) = ie^{it}(f')^*(t) \quad \text{for a.e.} \quad t \in T, \tag{3.1}$$

where $(f')^*(t)$ is again the radial limit of f' at e^{it}; see [3, Theorem 3.11, p. 42]. For $f \in B$ we denote

$$f^\#(z) = |f'(z)|/(1-|f(z)|^2), \quad z \in D,$$

and for the proof of Theorem 3 we shall make use of

Lemma 2. *Let $f \in B$ and $f' \in H^1$, and assume that $|f(e^{it})|<1$ for all $t \in T$. Then $f^\# \in PL^1$ and*

$$(f^\#)^*(t) = \left|\frac{d}{dt} f(e^{it})\right| / (1-|f(e^{it})|^2)$$

for a.e. $t \in T$.

Proof. A calculation yields that $\Delta \log f^\# = 4(f^\#)^2 > 0$ except for the zeros of f', so that $f^\# \in PL$. Since $|f|$ is bounded by a constant $c<1$ in D, it follows from $f^\# \leqq |f'|/(1-c^2)$ in D with $f' \in H^1$ that $f^\# \in PL^1$. Since $(f^\#)^* = |(f')^*|/(1-|f|^2)$ a.e. on T, the second assertion follows from (3.1).

Proof of Theorem 3. Since γ is rectifiable (in the Euclidean sense), it follows from [3, Theorem 3.12, p. 44] that $f' \in H^1$. By Lemma 2, $f^\# \in PL^1$. Since

$$L = \int_T \frac{|df(e^{it})|}{1-|f(e^{it})|^2} = \int_T \frac{\left|\frac{d}{dt} f(e^{it})\right|}{1-|f(e^{it})|^2}\,dt,$$

it further follows from Lemma 2 that

$$L = \int_T (f^{\#})^*(t)\, dt.$$

The first assertion in Theorem 3 now follows from Theorem 4 applied to each diameter C and $f^{\#} \in PL^1$.

It remains to prove the sharpness of 2 in $L/2$. For simplicity we consider the half-plane $R=\{w|\operatorname{Re} w>0\}$ with the non-Euclidean metric $|dw|/[2 \operatorname{Re} w]$ in the differential form. The non-Euclidean length of a curve Γ in R is denoted by $\lambda(\Gamma)$. Let $\varepsilon>0$ and let $0<a<b$. Consider the Euclidean rectangle Q with the vertices $z_1=a+\varepsilon i$, $z_2=a-\varepsilon i$, $z_3=b-\varepsilon i$, and $z_4=b+\varepsilon i$. Let f_ε be a one-to-one conformal mapping from D onto Q such that f_ε maps the diameter $[-1, 1]$ onto the segment ab on the real axis. If we show that

$$\lambda(f_\varepsilon(K))/\lambda(ab) \to 2 \quad \text{as} \quad \varepsilon \to 0, \tag{3.2}$$

then the function $(f_\varepsilon-1)/(f_\varepsilon+1)$ serves as an example for the sharpness. Let z_1z_2, z_2z_3, z_3z_4, and z_4z_1 be the four sides of $f_\varepsilon(K)$. A calculation yields that

$$\lambda(z_4 z_1) = \lambda(z_2 z_3) = (1/2)\log(b/a) = \lambda(ab),$$

and as $\varepsilon \to 0$,

$$\lambda(z_1 z_2) = \varepsilon/a \to 0 \quad \text{and} \quad \lambda(z_3 z_4) = \varepsilon/b \to 0.$$

Therefore (3.2) holds.

Appendix. Tsuji's proof of M. Riesz's theorem contains an obscure point. There is a gap between (5) and (6) in [8, p. 341]; the meanings of $\partial/\partial x$ in (5) and (6) are different. Since M. Riesz did not raise his result explicitly as in [8, Theorem VIII.46, p. 340], we must avoid this difficulty. The principal point is to prove that, for f holomorphic on $D \cup K$,

$$\int_C |f(w)|\,|dw| \leqq \pi^{-1} \int_T |f(e^{it})| V(t)\, dt, \tag{A}$$

where C and V are the same as in Theorem 1. Choose points $w_0=a, w_1, \ldots, w_{n-1}, w_n=b$ on C in this order. Then

$$V(t) = \lim \sum_{k=1}^{n} |\arg(e^{it}-w_k) - \arg(e^{it}-w_{k-1})|$$

as $\max_{1\leqq k\leqq n} |w_k - w_{k-1}| \to 0$, where $\arg(e^{it}-w)$ is a fixed branch in D; $V(t)$ is Lebesgue measurable on T. Now it follows from [7, (3), p. 54] (a careful reading shows that the cited point is true even if A or B lies on K) that the following estimate of the integral on the rectilinear segment $w_{k-1}w_k$ holds:

$$\int_{w_{k-1}w_k} |f(w)|\,|dw| \leqq \pi^{-1} \int_T |f(e^{it})|\,|\arg(e^{it}-w_k)-\arg(e^{it}-w_{k-1})|\,dt,$$

$1 \leqq k \leqq n$. Summing up both sides from $k=1$ to n, and letting $\max_{1\leqq k\leqq n} |w_k - w_{k-1}| \to 0$, we obtain (A).

Remark. It might be more appropriate to call [8, Theorem VIII.46] the F. Carlson and M. Riesz theorem.

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