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A maximum principle for interpolation in H^∞

N. J. YOUNG

There are several contexts in which it is desirable to calculate norms in quotient algebras of H^∞, the algebra of all bounded analytic functions in the open unit disc U, with supremum norm. For example, one version of the Nevanlinna—Pick problem (see [3] or [7]) is to find a function $f\in H^\infty$ which takes prescribed values at given points of U and minimises $\|f\|_{H^\infty}$: the minimum value of $\|f\|_{H^\infty}$ can be expressed as the quotient norm $\|g+\varphi H^\infty\|_{H^\infty/\varphi H^\infty}$ where g is some function and φ is a Blaschke product, and, once this quotient norm is known, an algorithm due to Schur and Nevanlinna enables the construction of the desired minimising function f [7]. It is less immediate that quotient norms of H^∞ are also important in an extremal problem for matrices raised by V. PTÁK [4]: to find the maximum value of $\|A^m\|$ over all contractions A on n-dimensional Hilbert space, $n\leqq m$, subject to $|A|_\sigma\leqq r<1$ ($|A|_\sigma$ is the spectral radius of A). An account of this problem is given in [5].

The second example motivates the study of the quantity $\|\psi+\varphi H^\infty\|_{H^\infty/\varphi H^\infty}$ as a function of the zeros of the Blaschke product φ, for fixed ψ. It is a plausible guess that some sort of maximum principle should hold for this quantity: the closer the zeros of φ are allowed to approach the unit circle, the larger should be $\|\psi+\varphi H^\infty\|$. This is in fact true in the case $\psi(z)=z^n$, where n is the degree of φ, as was established by PTÁK [4] in 1968. The purpose of this paper is to prove a generalization of Pták's result: a maximum principle is true whenever ψ is a Blaschke product of the same degree as φ. The method is quite different from Pták's, and offers some hope of further generalization.

We shall say that *the maximum principle holds for* f: $\Omega\subseteqq \mathbf{C}\to\mathbf{R}$ if, for any compact set $K\subseteqq\Omega$, the supremum of f on K is attained at some point of the boundary of K relative to Ω. And, if Ω is a subset of a complex vector space V, we say that the maximum principle holds for f: $\Omega\to\mathbf{R}$ if, for any plane π in V, the maximum principle holds for the restriction of f to $\pi\cap\Omega$; or, to put it more precisely, if, for all $a\in\Omega$ and $b\in V$, then maximum principle holds for the function

$$g: \{\lambda\in\mathbf{C}: a+\lambda b\in\Omega\}\to\mathbf{R} \quad \text{defined by} \quad g(\lambda)=f(a+\lambda b).$$

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Theorem: *Let n be a natural number and let*

$$F(\alpha, \beta) = \|\psi + \varphi H^\infty\|_{H^\infty/\varphi H^\infty}$$

where $\alpha = (\alpha_1, \ldots, \alpha_n) \in U^n$, $\beta = (\beta_1, \ldots, \beta_n) \in U_n$,

$$\varphi(z) = \prod_{i=1}^{n} \frac{z-\alpha_i}{1-\bar{\alpha}_i z} \quad \text{and} \quad \psi(z) = \prod_{i=1}^{n} \frac{z-\beta_i}{1-\bar{\beta}_i z}.$$

Then $F(\alpha, \beta) = F(\beta, \alpha)$ *and, for any* $\alpha, \beta \in U^n$, *the maximum principle holds for the functions* $F(\alpha, \cdot)$, $F(\cdot, \beta)$ *on* U^n.

We shall show that F is the composition of a strictly increasing and a separately plurisubharmonic function. This will follow from an interesting formula which expresses F in terms of the norm of an analytic operator-valued function. This formula also makes it easy to see the surprising symmetry property of F, which was earlier established by a different method in [5, Sec. 7].

The whole is based the well known theorem of SARASON [6] which interprets $\|\psi + \varphi H^\infty\|$ operator-theoretically. Let T denote the unilateral backward shift on l^2: that is,

$$T(x_0, x_1, x_2, \ldots) = (x_1, x_2, x_3, \ldots).$$

Sarason's theorem asserts (among other things) that, for any $\psi \in H^\infty$ and inner function φ,

(1) $$\|\psi + \varphi H^\infty\|_{H^\infty/\varphi H^\infty} = \|\psi(T)|\mathrm{Ker}\,\varphi(T)\|,$$

the symbol $\|\cdot\|$ denoting the operator norm on l^2 and the vertical bar denoting restriction.

Let P^*: $\mathrm{Ker}\,\varphi(T) \to l^2$, Q^*: $\mathrm{Ker}\,\psi(T) \to l^2$ be the natural injection mappings. Note that PP^*, QQ^* are the identity operators on $\mathrm{Ker}\,\varphi(T)$, $\mathrm{Ker}\,\psi(T)$ respectively, while P^*P, Q^*Q are the Hermitian projections with ranges $\mathrm{Ker}\,\varphi(T)$, $\mathrm{Ker}\,\psi(T)$.

Lemma 1. *QP^*: $\mathrm{Ker}\,\varphi(T) \to \mathrm{Ker}\,\psi(T)$ is invertible.*

Proof. Since both kernels have dimension n it suffices to show that QP^* is injective, or equivalently, that $\mathrm{Ker}\,\varphi(T) \cap \mathrm{Ker}\,\psi(T)^\perp = \{0\}$. For this purpose it is convenient to identify l^2 with the Hardy space H^2 of analytic functions in U in the usual way [2]. T acts on H^2 by

$$Th(z) = \frac{1}{z}\{h(z) - h(0)\}$$

while T^* acts as multiplication by z. Thus

$$\mathrm{Ker}\,\psi(T)^\perp = \mathrm{Range}\,\psi(T)^* = \psi^* H^2,$$

where $\psi^*(z) = \psi(\bar{z})^-$. Thus every element of $\mathrm{Ker}\,\psi(T)^\perp$ has at least n zeros in U. It is not hard to show that $\mathrm{Ker}\,\varphi(T)$ consists of all functions of the form g/k where

g is a polynomial of degree less than n and $k(z)=(1-a_1z)\dots(1-a_nz)$ (this can be done by induction on n). It follows that any element of $\operatorname{Ker}\varphi(T)$ which does not vanish identically has at most $n-1$ zeros in U, and therefore does not belong to $\operatorname{Ker}\psi(T)^\perp$.

Lemma 2. *Let*

$$p(z)=(z-\alpha_1)\dots(z-\alpha_n),\quad p_0(z)=(1-\bar\alpha_1 z)\dots(1-\bar\alpha_n z),$$
$$q(z)=(z-\beta_1)\dots(z-\beta_n),\quad q_0(z)=(1-\bar\beta_1 z)\dots(1-\bar\beta_n z).$$

Then

$$F(\alpha,\beta)^2=1-\|I-q_0(T)^{*-1}p(T)^*q(T)p_0(T)^{-1}\|^{-2}.$$

Proof. $\psi(T)^*$ is the operation of multiplication by the Blaschke product ψ^* and is therefore an isometry, so that $\psi(T)\psi(T)^*=I$, Hence $I-\psi(T)^*\psi(T)$ is a Hermitian projection, and its range is easily seen to be $\operatorname{Ker}\psi(T)$, Thus

$$I-\psi(T)^*\psi(T)=Q^*Q.$$

Equation (1) now gives

$$F(\alpha,\beta)^2=\|\psi(T)|\operatorname{Ker}\varphi(T)\|^2=\|\psi(T)P^*\|^2=\|P\psi(T)^*\psi(T)P^*\|=$$
$$=\|P(I-Q^*Q)P^*\|=|I_{\operatorname{Ker}\varphi(T)}-PQ^*QP^*|_\sigma.$$

Since $0\leqq PQ^*QP^*\leqq I$, this implies

$$F(\alpha,\beta)^2=1-\inf\sigma(PQ^*QP^*),$$

and since PQ^*, QP^* are invertible (by Lemma 1) this can be written

$$F(\alpha,\beta)^2=1-|(QP^*)^{-1}(PQ^*)^{-1}|_\sigma^{-1}. \tag{2}$$

We can make further progress through the use of non-orthogonal projections. Recall that if l^2 is the direct sum of subspaces E and F then the *projection of l^2 on E along F* is defined to be the operator $R\colon l^2\to l^2$ given by $R(x+y)=x$ $(x\in E, y\in F)$. If is easy to see that R is characterized by the three properties (a) $R^2=R$ (b) Range $R\subseteqq E$ (c) Range $R^*\subseteqq F^\perp$. Thus, in particular, the projection on $\operatorname{Ker}\psi(T)$ along $\operatorname{Ker}\varphi(T)^\perp$ is characterized by the three properties

$$\text{(a)}\quad R^2=R,\qquad \text{(b)}\quad \psi(T)R=0,\qquad \text{(c)}\quad \varphi(T)R^*=0. \tag{3}$$

Lemma 3. *The following are equivalent for an operator R on l^2:*

(i) *R is the projection on* $\operatorname{Ker}\psi(T)$ *along* $\operatorname{Ker}\varphi(T)^\perp$;

(ii) $R=Q^*(PQ^*)^{-1}P$;

(iii) $R=I-q_0(T)^{*-1}p(T)^*q(T)p_0(T)^{-1}$.

Proof. It is clear that $R=Q^*(PQ^*)^{-1}P$ satisfies conditions (a)—(c) of (3). These conditions may also be verified for R in (iii) with the aid of the identities

$$q(T)p(T)^*=p_0(T)q_0(T)^*,\quad p(T)q(T)^*=q_0(T)p_0(T)^*. \tag{4}$$

To prove the former, note that, since $TT^*=I$,

$$(T-\beta_j I)(T-\alpha_j I)^* = (I-\bar{\alpha}_j T)(I-\bar{\beta}_j T)^*.$$

Induction on n now establishes (4).

We return to the proof of Lemma 2. Let R be the projection on $\operatorname{Ker}\psi(T)$ along $\operatorname{Ker}\varphi(T)^\perp$. By Lemma 3,

$$(PQ^*)^{-1} = QQ^*(PQ^*)^{-1}PP^* = QRP^*$$

and hence (2) becomes

$$F(\alpha,\beta)^2 = 1-|PR^*Q^*QRP^*|_\sigma^{-1}. \tag{5}$$

From the definition of R and the fact that Q^*Q, P^*P are the orthogonal projections onto $\operatorname{Ker}\psi(T)$, $\operatorname{Ker}\varphi(T)$ respectively, it follows that $Q^*QR=R=RP^*P$. In view of the fact that $|AB|_\sigma=|BA|_\sigma$ whenever AB and BA are both defined, (5) yields

$$F(\alpha,\beta)^2 = 1-|PR^*RP^*|_\sigma^{-1} = 1-|RP^*PR^*|_\sigma^{-1} = 1-|RR^*|_\sigma^{-1} =$$
$$= 1-\|R\|^{-2} = 1-\|I-q_0(T)^{*-1}p(T)^*q(T)p_0(T)^{-1}\|^{-2}.$$

Lemma 2 is proved.

Lemma 4. *Let*

$$G(\alpha,\beta) = \|I-q_0(T)^{*-1}p(T)^*q(T)p_0(T)^{-1}\|.$$

For any $\alpha,\beta\in U^n$ the functions $G(\alpha,\cdot)$, $G(\cdot,\beta)$ are plurisubharmonic on U^n, and $G(\alpha,\beta)=G(\beta,\alpha)$.

Proof. If R has the same meaning as above then $G(\alpha,\beta)=\|R\|$ and

$$G(\beta,\alpha) = \|I-p_0(T)^{*-1}q(T)^*p(T)q_0(T)^{-1}\| = \|R^*\| = \|R\| = G(\alpha,\beta).$$

A function on U^n is plurisubharmonic provided it is upper semi-continuous and its restriction to the intersection with U^n of any plane is subharmonic. Now

$$q(T) = T^n-(\beta_1+\ldots+\beta_n)T^{n-1}+\ldots+(-1)^n\beta_1\ldots\beta_n I,$$
$$q_0(T)^* = I-(\beta_1+\ldots+\beta_n)T^*+\ldots+(-1)^n\beta_1\ldots\beta_n T^{*n}.$$

Hence $q(T)$ and $q_0(T)^*$ are analytic operator-valued functions of β in U^n, and the same is therefore true of R (for fixed α). Certainly $G(\alpha,\cdot)$ is continuous on U^n. Furthermore, for any analytic function f defined on an open set $\Omega\subseteqq\mathbf{C}$ and taking values in any Banach space, the real-valued function $z\to\|f(z)\|$ is subharmonic on Ω [1, Thm. 3.12.1]. It follows that, for a fixed α, $G(\alpha,\cdot)$ is plurisubharmonic on U^n, and so, by symmetry, is $G(\cdot,\beta)$ for fixed β.

We can now conclude the proof of the theorem. Since the maximum principle holds for any subharmonic function defined in an open subset of the plane [1], the maximum principle also holds for any plurisubharmonic function on U^n, hence for $G(\alpha, \cdot)$. By Lemma 2 $F(\alpha, \cdot)=h\circ G(\alpha, \cdot)$ where $h(t)=(1-t^{-2})^{1/2}$. Since h is a strictly increasing function on $[1, \infty)$, the maximum principle holds for $F(\alpha, \cdot)$. And since G is symmetric in α and β, so is F.

Let us consider the implications of the theorem for Pták's problem. Suppose we wish to find an operator A on n-dimensional Hilbert space which maximises $\|\psi(A)\|$ subject to the constraints $\|A\|\leqq 1$, $|A|_\sigma\leqq r$. The search can be split into two steps:

I. For each polynomial p having all its zeros in the disc $\{z\colon |z|\leqq r\}$ find an $n\times n$ matrix which maximises $\|\psi(A)\|$ subject to $\|A\|\leqq 1$, $p(A)=0$.

II. Among all such polynomials p find the one for which the corresponding maximum of $\|\psi(A)\|$ is the largest.

Problem I has been completely solved — see [5, Sec. 3]. The solution is made simpler than might be expected by the fact that an extremal matrix can be given which is independent of ψ.

Problem II, called in [5, Sec. 2] the "problem of the worst polynomial", is difficult. It has been solved only in the case that $\psi(z)=z^n$, when a worst polynomial is $p(z)=(z-\varepsilon r)^n$, for any ε, $|\varepsilon|=1$. One can hardly doubt that the same polynomial will be extremal for higher powers of z also, and it is even conceivable that $(z-\varepsilon r)^n$ is the worst polynomial for *all* $\psi\in H^\infty$. However, twelve years have elapsed since Pták solved the case $\psi(z)=z^n$ by a very special method, and attempts by several mathematicians to extend Pták's conclusion to other functions have had no success. It is not even known whether the worst polynomial has its zeros on the circle $|z|=r$ in general. The present paper shows that, if ψ is a Blaschke product of degree n, then there is a worst polynomial with all its zeros on $|z|=r$. The question remains as to whether these zeros can all be taken to coincide, as in the known case. It would also be desirable to extend the theorem to higher powers of z or, more generally, to Blaschke products ψ of arbitrary degree.

I will indicate the difficulties involved in extending the above method to the case that ψ is a Blaschke product of degree m, $m>n$. Exactly as before we have

$$\|\psi+\varphi H^\infty\|^2 = 1-\inf\sigma(PQ^*QP^*),$$

but it is no longer true that QP^* is invertible (it acts between spaces of different dimension). To get round this we can introduce the space $E=\text{Range } Q^*QP^*$, the orthogonal projection of Ker $\varphi(T)$ on Ker $\psi(T)$, and let Q_1^* be the natural injection of E into l^2. Then Q_1P^* is invertible and

$$\|\psi+\varphi H^\infty\|^2 = 1-\inf\sigma(PQ_1^*Q_1P^*),$$

and if we denote by R_1 the projection on E along Ker $\varphi(T)^\perp$ then we have as before

$$\|\psi+\varphi H^\infty\|^2 = 1-\|R_1\|^{-2}.$$

It is not hard to show that

$$E = \operatorname{Ker} \psi(T) \cap (\operatorname{Ker} \varphi(T) + \psi^* H^2)$$

and that R_1 can be expressed by the formula

$$R_1^* = I - X[\varphi(T)X]^{-1}\varphi(T), \quad \text{where} \quad X = pq_0^{-1}(T)^* F + q(T)^* T^{m-n},$$

$\psi = q/q_0$ and F is the Hermitian projection with range Ker T^{m-n}. The problem is that R_1 is no longer an analytic function of α. It is of course still possible that $\|R_1\|$ is plurisubharmonic, but I have not been able to prove it.

Note that the formula in Lemma 2 can be used to give an upper bound for $\|\psi + \varphi H^\infty\|$ which is strictly less than one.

I conclude with an observation about the intriguing fact that $\|\psi + \varphi H^\infty\| = \|\varphi + \psi H^\infty\|$ when φ, ψ are Blaschke products of the same degree. In fact a stronger statement is known to folklore: if u is a continuous unimodular function on the circle having winding number zero then u and $1/u$ are equally distant from H^∞. Paul Koosis has provided a neat and simple proof of this fact in a personal communication. His proof is based on the observation that, if $|1-w| \leqq d \leqq 1$, $w \in \mathbf{C}$, then $\left|1 - \frac{1-d^2}{w}\right| \leqq d$. If $g \in H^\infty$ and $\|u-g\| \leqq d < 1$ then an application of Rouché's theorem establishes that $1/g \in H^\infty$, and

$$\left\| \frac{1}{u} - \frac{1-d^2}{g} \right\| \leqq d.$$

The symmetry of $F(\alpha, \beta)$ is deduced by putting $u = \psi/\varphi$.

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MATHEMATICS DEPARTMENT
UNIVERSITY GARDENS
GLASGOW G12 8QW