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The ideal lattice of a distributive lattice with 0 is the congruence lattice of a lattice

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The congruence lattice of an arbitrary lattice is a distributive algebraic lattice, i.e. the ideal lattice of a distributive semilattice with 0. The converse of this statement is a long-standing conjecture of lattice theory. We prove the following:

Theorem. *Let L be the lattice of all ideals of a distributive lattice with* 0. *Then there exists a lattice K such that L is isomorphic to the congruence lattice of K.*

The conjecture was first established for finite distributive lattices by R. P. Dilworth. Later, it was solved for the ideal lattice of relatively pseudo-complemented join-semilattices (E. T. SCHMIDT [4], [5]).

The first section of this paper reviews the definitions and gives the outline of the proof. The basic notion is the so-called distributive homomorphism of a semilattice (see [4]). The second section proves that for every distributive lattice F with 0 there exists a generalized Boolean algebra B — considered as a semilattice — and a distributive homomorphism of B onto F. In the third section we prove the main result and in the last section we give some generalizations.

1. Preliminaries

Semilattice always means a join-semilattice in this paper. The compact elements of an algebraic lattice L form a semilattice L^c with 0, and L is isomorphic to the ideal lattice of L^c. We denote by $\mathrm{Con}(K)$ the congruence lattice of the lattice K. The compact elements of $\mathrm{Con}(K)$ are called compact congruence relations, these form the semilattice $\mathrm{Con}^c(K)$.

Let B be a sublattice of a lattice K. The connection between $\mathrm{Con}^c(B)$ and $\mathrm{Con}^c(K)$ is of course very loose. Let θ be a congruence relation of B.

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Then there exists a smallest congruence relation $\theta^0 \in \mathrm{Con}(K)$ such that $\theta^0|_B \geqq \theta$. It is easy to see that $\theta_1^0 \vee \theta_2^0 = (\theta_1 \vee \theta_2)^0$, i.e. the correspondence $\theta \to \theta^0$ is a homomorphism of $\mathrm{Con}^c(B)$ into the semilattice $\mathrm{Con}^c(K)$. If this homomorphism is onto we call K a *strong extension of B* [1]; or we say that B is a strongly large sublattice. It is an important case if $\theta^0|_B = \theta$ holds, then we write $\bar{\theta}$ instead of θ^0. $\bar{\theta}$ is called the extension of θ.

It is well known that in generalized Boolean lattices (i.e. relatively complemented distributive lattices with zero) there is a one-to-one correspondence between congruence relations and ideals and therefore if B denotes a generalized Boolean lattice then $\mathrm{Con}^c(B) \cong B$. Let F be a distributive semilattice with 0. We would like to get a lattice K such that $\mathrm{Con}^c(K) = F$ holds. Therefore we start with a generalized Boolean lattice B which has a join-homomorphism onto F and we construct a strong extension K of B such that $\theta \to \theta^0$ is the given join-homomorphism. The construction of a strong extension of this kind was developed in [4].

We will make a further assumption that B is a convex sublattice of K. In this case the homomorphism $\theta \to \theta^0$ has an additional property, formulated in the next proposition.

Proposition 1. *Let B be a convex sublattice of K and let $\theta^0 = \Phi^0 \vee \Psi^0$ where $\theta, \Phi, \Psi \in \mathrm{Con}^c(B)$. Then there exist $\Phi_1, \Psi_1 \in \mathrm{Con}^c(B)$ such that $\Phi_1 \vee \Psi_1 = \theta$ and $\Phi_1^0 \leqq \Phi^0$, $\Psi_1^0 \leqq \Psi^0$.*

Proof. θ is a compact congruence relation of B, hence $\theta = \bigvee_{i=1}^{n} \theta(a_i, b_i)$, where $a_i < b_i$, $a_i b_i \in B$. From $\theta^0 = \Phi^0 \vee \Psi^0$ we get $a_i \equiv b_i(\Phi^0 \vee \Psi^0)$, $i = 1, 2, \dots, n$. We have therefore for every i a finite chain $a_i = c_{0,i} < c_{1,i} < \dots < c_{n,i} = b_i$ such that $c_{j,i} \equiv c_{j+1,i}(\Phi^0)$ or $c_{j,i} \equiv c_{j+1,i}(\Psi^0)$. By the assumption, B is a convex sublattice, i.e $c_{j,i} \in B$. Let Φ_1 be the join of all principal congruences $\theta(c_{j,i}, c_{j+1,i}) \in \mathrm{Con}^c(B)$ with $c_{j,i} \equiv c_{j+1,i}(\Phi^0)$. In a similar way we get Ψ_1. Then $a_i \equiv b_i(\Phi_1 \vee \Psi_1)$ for every i, i.e. $\theta = \Phi_1 \vee \Psi_1$, and $\Phi_1^0 \leqq \Phi^0$, $\Psi_1^0 \leqq \Psi^0$.

This Proposition suggests the following

Definition 1. Let S, T be two distributive semilattices. A homomorphism φ of S into T is called *weak-distributive* if $\varphi(u) = \varphi(x \vee y)$ implies the existence of $x_1, y_1 \in S$ such that $x_1 \vee y_1 = u$, $\varphi(x_1) \leqq \varphi(x)$, $\varphi(y)_1 \leqq \varphi(y)$ (see Figure 1).

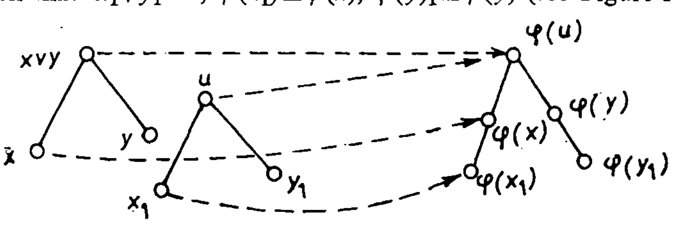

Figure 1.

The congruence relation induced by a weak-distributive homomorphism is called a *weak-distributive congruence.*

Let φ be a homomorphism of the semilattice S into the semilattice T. The congruence relation of S induced by φ is denoted by θ_φ.

Proposition 2. *Let S be a distributive semilattice. $\varphi\colon S\to T$ is a weak-distributive homomorphism if and only if $a\equiv b\vee c\ (\theta_\varphi)$, $a\geqq b\vee c$ imply the existence of elements $b_1\geqq b$, $c_1\geqq c$ such that $b\equiv b_1\ (\theta_\varphi)$, $c\equiv c_1\ (\theta_\varphi)$ and $b_1\vee c_1=a$* (Figure 2).

Proof. Let us assume that φ is a weak-distributive homomorphism and let $a\geqq b\vee c$, $\varphi(a)=\varphi(b\vee c)=\varphi(b)\vee\varphi(c)$, i.e. $a\equiv b\vee c\ (\theta_\varphi)$. φ is weak-distributive, hence we have elements $b_0, c_0\in S$ such that $b_0\vee c_0=a$, $\varphi(b_0)\leqq\varphi(b)$, $\varphi(c_0)\leqq\varphi(c)$. Let $b_1=b\vee b_0$, $c_1=c\vee c_0$ then $b_1\vee c_1=b\vee c\vee b_0\vee c_0=b\vee c\vee a=a$ and $\varphi(b_1)=\varphi(b\vee b_0)=$ $=\varphi(b)\vee\varphi(b_0)=\varphi(b)$, i.e. $b_1\equiv b\ (\theta_\varphi)$. Similarly we get $c_1\equiv c\ (\theta_\varphi)$ which proves that θ_φ satisfies the given property.

Let θ_φ be a congruence relation with the property formulated in the Proposition. Let $a[\theta_\varphi]=x[\theta_\varphi]\vee y[\theta_\varphi]$, i.e. $a\equiv x\vee y\ (\theta_\varphi)$. Then $a\vee x\vee y\equiv x\vee y\ (\theta_\varphi)$ and there exist $x_1, y_1\in S$ satisfying $x_1\vee y_1=x\vee y\vee a$, $x\equiv x_1\ (\theta_\varphi)$, $y\equiv y_1\ (\theta_\varphi)$. Therefore $x_1\vee y_1\geqq a$, hence by the distributivity of S we get elements x_2, y_2 for which $x_2\leqq x_1$, $y_2\leqq y_1$ and $x_2\vee y_2=a$. These elements satisfy $\varphi(x_2)\leqq\varphi(x_1)\leqq\varphi(x)$, i.e. φ is weak-distributive.

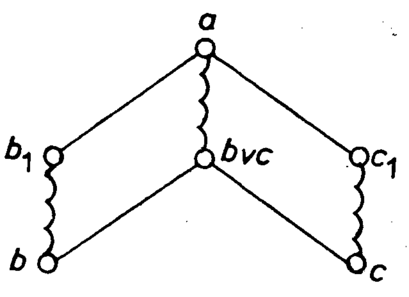

Figure 2.

It is easy to give an example for a semilattice S and $a, b\in S$ such that there is no smallest weak-distributive congruence satisfying $a\equiv b\ (\theta)$, i.e. the principal weak-distributive congruence does not exist. We follow another way to define a special weak-distributive congruence which plays the role of the principal congruence. The principal congruences of a semilattice have the property that every congruence class contains a maximal element.

Definition 2. [4] A congruence relation θ of a semilattice is called *monomial* if every θ-class has a maximal element.

The monomial congruence are special meet-representable congruences. Every congruence relation of a semilattice is the join of principal congruence relations therefore it is natural to introduce the following notion.

Definition 3. [4] A congruence relation θ of a semilattice is called *distributive* if θ is the join of weak-distributive monomial congruences. A homomorphism φ: $S \to T$ is distributive iff the congruence relation θ induced by φ is distributive.

Remark. It is easy to prove that the join of weak-distributive congruences is weak-distributive. The basic properties of distributive congruences are listed in [6].

If $(B; \vee, \wedge)$ is a generalized Boolean lattice, then the semilattice $(B; \vee)$ will be called a generalized Boolean semilattice.

For the solution of the characterization problem of congruence lattices of attices it is enough to solve the following two problems.

Problem 1. Let B be a generalized Boolean semilattice and let θ be a distributive congruence of B. Does there exist a lattice K satisfying $\mathrm{Con}^c(K) \cong B/\theta$? Does there exist a strong extension of B satisfying the same property?

This problem was solved positively in [4]. In section 3 we give the sketch of the proof.

Problem 2. Let F be a distributive semilattice with 0. Does there exist a generalized Boolean semilattice B and a distributive congruence θ of B such that F is isomorphic to B/θ?

This problem is open. We solve this problem if F is a lattice, i.e. we prove the following.

Theorem 1. *Let F be a distributive lattice with* 0. *Then there exist a generalized Boolean semilattice B and a distributive congruence θ of B such that* $F \cong B/\theta$.

The proof of this theorem will be given in the next sections. We present here the basic idea of the proof.

Let F be a semilattice, $a, b \in F$. The pseudocomplement $a*b$ of a relative to b is an element $a*b \in F$ satisfying $a \vee x \geqq b$ iff $x \leqq a*b$. If $a*b$ exists for all $a, b \in F$ then F is a relatively pseudocomplemented semilattice. (In the literature the pseudocomplement is usually defined in meet-semilattices.)

Let F be a relatively pseudocomplemented lattice (i.e. the join-semilattice $F^{\vee}$ is relatively pseudocomplemented). The proof of Theorem 1 in this case is quite easy. Let B be the Boolean lattice R-generated by F. (See [2], p. 87.) Then for every $x \in B$ there exists a smallest $\bar{x} \in F$ satisfying $x \leqq \bar{x}$. The mapping $x \to \bar{x}$ is a distributive homomorphism of B onto F. The congruence relation induced by this mapping is monomial. The converse of this statement is true: if θ is a monomial distributive congruence of B then B/θ is a relatively pseudocomplemented lattice.

If F is a relatively pseudocomplemented *semilattice* then this construction does not work. In this case we consider for every $a\in F$, $a\neq 0$ the *skeleton* of $(a]$, i.e. $S(a)=\{x*a;\ x\leqq a\}$ ([2], p. 112). $S(a)$ is a Boolean lattice. Consider the lower discrete direct product $\prod_d (S(a);\ a\in F, a\neq 0)$, i.e. the sublattice of the direct product $\prod S(a)$ of those sequences t for which $t(a)=0$ for all but finitely many $a\in F$. This is a generalized Boolean lattice B, and it is easy to show that B has a distributive congruence θ satisfying $B/\theta\cong F$ (see [4]).

To prove Theorem 1 we generalize the notion of the skeleton. Let φ be the identity $\varphi\colon S(1)\to F$. If B denotes $S(1)$ and $0, I\in B$ then this φ obviously has the following properties:

(1) φ is a $\{0, 1\}$-homomorphism of the Boolean semilattice B into the semilattice F,

(2) if $\varphi(I)=x\vee y$ in F then there exist $x_1, y_1\in B$ such that $x_1\vee y_1=I$, $\varphi(x_1)\leqq x$, $\varphi(y_1)\leqq y$.

(1) follows from the property that $S(a)$ is a subsemilattice of F, and (2) is obvious if we take $x_1=y*1$, $y_1=x_1*1$.

Definition 4. Let F be a distributive semilattice with 0, $1\in F$ and let B be a Boolean semilattice with unit element I and zero element 0. B is called a *pre-skeleton* of F if there exists a mapping φ of B into F such that conditions (1) and (2) are satisfied.

Condition (2) is related to the distributivity of φ; if (2) is satisfied for every $a\in B$ (instead of I) and φ is onto then we get that φ is distributive.

2. The pre-skeleton

To prove Theorem 1 we shall show that every bounded distributive lattice has a pre-skeleton. First we verify some simple well-known properties of free Boolean algebras. The free Boolean algebra B generated by the set G is denoted by $F(G)$. If $|G|=m$ we shall write $F(m)$ for $F(G)$. 1 denotes the unit element of $F(G)$. Let $G'=\{x'|x\in G\}$ (x' denotes the complement of x) and $G_1=G\cup G'$. For $g\in G$, g^e is either g or g'. Let k be a natural number. We consider the subset G_k of B defined by $G_0=\{1\}$ and $G_k=\{x|x\in B,\ x\neq 0,\ x=g_1^e\wedge\ldots\wedge g_k^e$, where $g_1, \ldots, g_k$ are different elements of $G\}$. From these sets G_k we get $\mathscr{H}=\bigcup_{i=0}^{\infty} G_i$. If $|G|=n$ is a natural number then G_n is the set of atoms of $F(n)$ and each $a\in F(n)$, $a\neq 0$ has a unique representation as a join of elements of G_n. If G is infinite we have no atoms, therefore we must take the whole set $\mathscr{H}$, which is of course a relative sublattice of B.

The most important properties of $\mathscr{H}$ are collected in the following definition.

Definition 5. A relative sublattice $\mathscr{H}$ of a Boolean algebra B is called a *join-base* iff the following conditions are satisfied:

(i) $0 \notin \mathscr{H}$ and $1 \in \mathscr{H}$.
(ii) Each $a \in B$, $a \neq 0$ has a representation as a join of elements of $\mathscr{H}$.
(iii) There is a dimension function δ from $\mathscr{H}$ onto an ideal of the chain of non-negative integers such that $\delta(1)=0$ and $x \prec y$ in $\mathscr{H}$ if and only if $x \leqq y$ and $\delta(x)=\delta(y)+1$. The set of all $x \in \mathscr{H}$ with $\delta(x)=i$ is denoted by $\mathscr{H}_i$.
(iv) For every finite subset $U=\{u_1, \dots, u_n\}$ of B there exists an $i \in \mathrm{N}$ such that each $\mathscr{H}_k$ $(k \geqq i)$ has a finite subset $A_k(U)$ with the property that each $u \in U$ has a unique join representation as a join of elements of $A_k(U)$.
(v) If $a \wedge b \neq 0$ in $B, a, b \in \mathscr{H}$ then $a \wedge b \in \mathscr{H}$; if $a \vee b$ exists in $\mathscr{H}$ and a, b are incomparable then $a, b \in \mathscr{H}_i$, $a \vee b \in \mathscr{H}_{i-1}$ for some $i \in \mathrm{N}$. Assume, that there exists an $a_0 \in \mathscr{H}_{i-1}$, $a_0 \neq a \vee b$, $a_0 > a$, then there is a $b_0 \in \mathscr{H}_{i-1}$ such that $a_0 \vee b_0$ exists and $a_0 \wedge (a \vee b)=a$, $b_0 \wedge (a \vee b)=b$.

Let $\mathscr{H}$ be a join-base of a Boolean semilattice B and let $f\colon \mathscr{H} \to L$ be a homomorphism into a distributive lattice (i.e. $f(a \wedge b)=f(a) \wedge f(b)$ whenever $a \wedge b$ exists, and the same for $\vee$). We want to extend f to a homomorphism $\varphi\colon B \to L$ (i.e., φ will be a join-homomorphism of the Boolean algebra B). Let $a=h_1 \vee \dots \vee h_n$ where $h_i \in \mathscr{H}$. The only way to define φ is the following: $\varphi(a)=f(h_1) \vee \dots \vee f(h_n)$. Condition (iv) yields that this definition is unique and (ii) implies that φ maps B into L.

Definition 6. The homomorphism φ of the Boolean semilattice into L is called an *L-valued homomorphism of B induced by f.*

To prove Theorem 1 we need the definition of free $\{0, 1\}$-distributive product (see G. Grätzer [2], p. 106).

Definition 7. Let D be the class of all bounded distributive lattices and let L_i, $i \in I$ be lattices in D. A lattice L in D is called a *free* $\{0, 1\}$*-distributive product* of the L_i, $i \in I$, iff every L_i has an embedding ε_i into L such that

(i) L is generated by $\bigcup(\varepsilon_i L; i \in I)$.

(ii) If K is any lattice in D and φ_i is a $\{0, 1\}$-homomorphism of L_i into K for $i \in I$, then there exists a $\{0, 1\}$-homomorphism φ of L into K satisfying $\varphi_i=\varphi\varepsilon_i$ for all i.

The free $\{0, 1\}$-distributive product is denoted by $\Pi^*(A_i; i \in I)$ or by $A * B$. The lower discrete direct product is denoted by $\Pi_d(A_i; i \in I)$ and finally if A_i are lattices with unit element then $\Pi^d(A_i; i \in I)$ is the upper discrete direct product, i.e. the sublattice of the direct product ΠA_i of those sequences t for which $t(a)=1$ for all but finitely many a.

Lemma 1. *Let L be a bounded distributive lattice and let A_i $(i\in I)$ be Boolean semilattices. If φ_i: $A_i\to L$ $(i\in I)$ are L-valued $\{0, 1\}$-homomorphisms generated by f_i: $\mathscr{H}^i\to L$ then the free $\{0, 1\}$-distributive product $\Pi^* A_i$ has a join-base $\mathscr{H}$ and a homomorphism f: $\mathscr{H}\to L$ such that $\mathscr{H}\cap A_i=\mathscr{H}^i$ for each $i\in I$. There exists an L-valued homomorphism φ of $\Pi^* A_i$ generated by f satisfying $\varphi_i=\varphi\varepsilon_i$.*

Proof. Let $\mathscr{H}$ be the set of all those elements $h\neq 0$ of $\Pi^* A_i$ which have a finite meet-representation as a meet of elements from $\bigvee\mathscr{H}^i$. (Then $\mathscr{H}$ is isomorphic to the upper direct product $\Pi^d\mathscr{H}^i$.) Obviously $\mathscr{H}^i\subseteq\mathscr{H}$, $\mathscr{H}^i=\mathscr{H}\cap A_i$. Let $u=h_1\wedge\wedge h_2\wedge\ldots\wedge h_n$ where the $h_i\in\mathscr{H}^i$ belong to different components, then this representation is unique. We have by (iii) the functions δ_i: $\mathscr{H}^i\to\mathrm{N}$. Now let δ: $\mathscr{H}\to\mathrm{N}$ be defined by $\delta(u)=\delta_1(h_1)+\ldots+\delta_n(h_n)$. It is easy to verify (iv) and (v). Assume that f_i: $\mathscr{H}^i\to L$ are homomorphisms, then we can extend them as follows: $f(u)==f_1(h_1)\wedge\ldots\wedge f_n(h_n)$. Hence $x\geqq y$ $(x, y\in\Pi^* A_i)$ implies $f(x)\geqq f(y)$. Let us assume that for incomparable $b, c\in\mathscr{H}$, $b\vee c$ exists, i.e. $b\vee c\in\mathscr{H}$. Then by (v) there exist an i and $b_0, c_0\in\mathscr{H}_i$ such that $b=b_0\wedge(b\vee c)$ and $c=c_0\wedge(b\vee c)$. Thus we get by the distributivity of L that $f(b)\vee f(c)=[f_i(b_0)\wedge f(b\vee c)]\vee[f_i(c_0)\wedge f(b\vee c)]=(f_i(b_0)\vee f_i(c_0))\wedge\wedge f(b\vee c)$. But f_i: $\mathscr{H}^i\to L$ is a homomorphism, hence $f_i(b_0\vee c_0)=f_i(b_0)\vee f_i(c_0)$. Obviously $b_0\vee c_0\geqq b\vee c$, i.e. $f(b_0\vee c_0)\geqq f(b\vee c)$. This yields $f(b)\vee f(c)=f(b\vee c)$, i.e. f is a homomorphism of $\mathscr{H}$ into L.

The free Boolean algebra on m generators is the free $\{0, 1\}$-distributive product of m copies of the free Boolean algebra on one generator, i.e. if $B_i\cong F(1)$, $i\in I$ then $F(m)\cong\Pi^* B_i$.

Corollary. *If each $B_i\cong F(1)$ has a $\{0, 1\}$-homomorphism φ_i into the distributive lattice L, then there exists an L-valued homomorphism φ of $F(m)$ into L such that $\varphi_i=\varphi\varepsilon_i$.*

Lemma 2. *Let L be a bounded distributive lattice. Then there exists a pre-skeleton B of L.*

Proof. First assume that B is a pre-skeleton and ψ: $B_1\to B$ is a lattice homomorphism of the Boolean lattice B_1 onto B. Then it is easy to see that B_1 is again a pre-skeleton and the corresponding join-homomorphism is $\varphi\psi(x)$. Therefore to prove our Lemma it is enough to take a free Boolean algebra generated by a "big" set.

We start with the set G_1 of all pairs (a, b) satisfying $a, b\in L$, $a\vee b=1$, $a, b\neq 1$. Let G be a subset of G_1 which is maximal with respect to the property: $(a, b)\in G$ iff $(b, a)\notin G$.

In the free Boolean algebra $F(G)$ we define $(a, b)'=(b, a)$, i.e. the complement of (a, b) is (b, a). The mapping $\varphi\colon F(G)\to L$ is defined as follows. For $(a, b)\in G_1$ we set $\varphi((a, b))=a$ and let $\varphi(0)=0$. Then $\varphi((a, b))\vee\varphi((b, a))=a\vee b=1$, i.e. φ is a $\{0, 1\}$-homomorphism of the semilattice $F((a, b))$ into L. Then by the Corollary to Lemma 1 there exists an extension φ of these homomorphisms. Let $x\vee y=1=\varphi(I)$, $x, y\neq 1$, where I denotes the unit element of $F(G)$. Take $x_1=(x, y)$, $y_1=(y, x)\in F(G)$. By the definition of φ we have $\varphi(x_1)=x$, $\varphi(y_1)=y$, i.e. $F(G)$ is a pre-skeleton of L.

Example 1. As an illustration consider the lattice L represented by Figure 3.

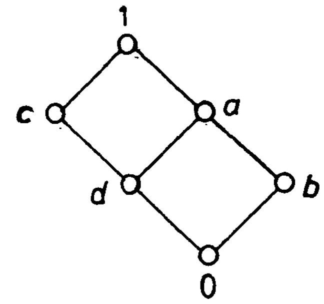

Figure 3.

The set G_1 contains the pairs (a, c), (b, c), (c, a), (c, b) and for a generating set we can choose $G=\{(a, c), (b, c)\}$; then B is the free Boolean algebra generated by two elements, i.e. $B\cong 2^4$. Figure 4 gives the join-homomorphism φ, in which the wavy line indicates congruence modulo $\theta=\operatorname{Ker}\varphi$.

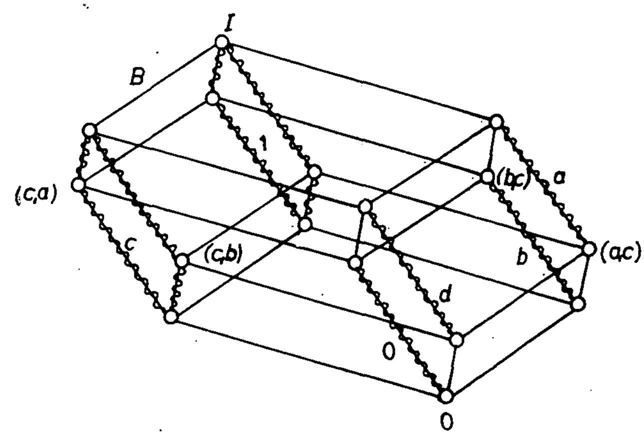

Figure 4.

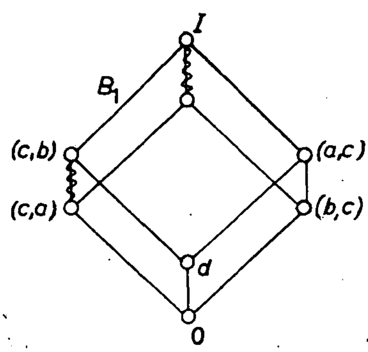

Figure 5.

Remark. The set G_1 can be made into a poset as follows: $(x, y) \leqq (u, v)$ iff $x \leqq u$ and $y \geqq v$. We adjoin 0 and I and we take the Boolean algebra B_1 freely generated by this poset. B_1 is of course the homomorphic image of B defined above. Sometimes it is easier to work with this "smaller" Boolean algebra (see Figure 5).

Example 2. Let L be the lattice shown in Figure 6.

Let $\mathbf{N} = \{0, 1, 2, \ldots\}$ be the set of all natural numbers. B is the Boolean-algebra containing all finite and cofinite subsets of $\mathbf{N}$. We define $(a_i, b) = \{x_i;\ x \geqq i\}$, $(b, a_i) = \{0, 1, \ldots, i-1\}$. Then $G = \{(a_i, b), (b, a_i);\ i = 0, 1, \ldots\}$ is a generating set. The corresponding join homomorphism is the following. Let A be a subset of $\mathbf{N}$ with the smallest element $f(A)$. If A is finite then $\varphi(A)$ is b if $f(A) = 0$ and $\varphi(A) = c_{f(A)}$ if $f(A) > 0$. For an infinite A we have $\varphi(A) = 1$ if $f(A) = 0$ and $\varphi(A) = a_{f(A)}$ if $f(A) > 0$. It is easy to see that φ is a distributive homomorphism of B onto L, which proves that $I(L) \cong L$ is the congruence lattice of a lattice. This is the simplest example to show that $\mathrm{Con}^c(K)$ need not to be relatively pseudocomplemented.

Lemma 3. *Let A_1, A_2 be Boolean semilattices and let φ_i: $A_i \to L$ be L-valued $\{0\}$-homomorphisms generated by the homomorphisms f_i: $\mathscr{H}_i \to L$ of the join-bases $\mathscr{H}_i \subseteqq A_i$ $(i = 1, 2)$. Then $\mathscr{H} = \mathscr{H}_1 \cup \mathscr{H}_2 \cup \{1\}$ is a join-base of $A_1 \times A_2$ and if φ is the homomorphism generated by f: $\mathscr{H} \to L$ then $\varphi_i = \varphi \varepsilon_i$*

Proof. The proof is obvious.

Remark. Lemma 3 is true for lower discrete direct product. In the infinite case this is a generalized Boolean algebra.

The basic idea of the proof of Theorem 1 can be illustrated by the following lattice (Figure 7).

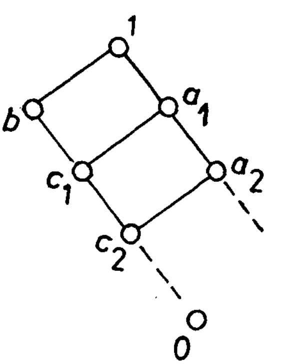

Figure 6.

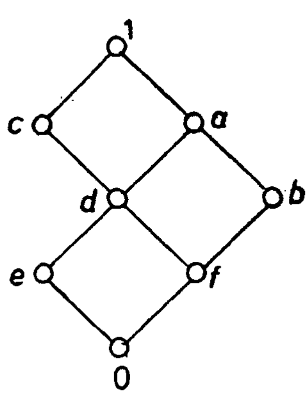

Figure 7.

Let a be an element of L. Then $(a]$ is a bounded distributive lattice. If B is a pre-skeleton of $(a]$ then we write $B=B(a)$; $B(1)$ is a pre-skeleton of L.

By Lemma 2 we have a homomorphism φ_1 of the pre-skeleton $B(1)$ onto the semilattice containing the elements $\{1, a, b, c, d, 0\}$. Applying again Lemma 2 for the principal ideal $(a]$ we get the mapping φ_a of the pre-skeleton $B(a)$ of $(a]$ onto $\{a, d, e, b, f, 0\}$. Let x be an element of $B(1)$ for which $\varphi_1(x)=a$. $B(1)$ is the direct product $(x]\times(x']$ where x' denotes the complement of x. Take the free $\{0, 1\}$-distributive product C of $(x]$ and $B(a)$. Let B be the Boolean semilattice $C\times(x']$ then by Lemmas 1 and 3 φ_1 and φ_a can be extended to a homomorphism $\varphi\colon B\to L$ which is a distributive homomorphism onto L.

We need the following

Definition 8. Let B be a Boolean semilattice and let L be a distributive lattice with 0. Let $\varphi\colon B\to L$ be a 0-preserving distributive homomorphism. (B, φ, L) is called a *saturated triple* if $\varphi(u)=x\vee y$ implies the existence of $x_1, y_1\in B$ such that $x_1\vee y_1=u$, $\varphi(x_1)\leqq x$, $\varphi(y_1)\leqq y$.

Lemma 4. *If (C,f, L), (D, g, L) are saturated triples then there exists a distributive homomorphism $h\colon C\times D\to L$ such that $h|_C=f$, $h|_D=g$ and $(C\times D, h, L)$ is saturated.*

Proof. For $(c, d)\in C\times D$ we define $h((c, d))=f(c)\vee g(d)$. Then $h((c, 0))=$ $=f(c)\vee 0=f(c)$, $h|_C=f$. Similarly $h|_D=g$. Now

$$h((a, b)\vee(c, d)) = h((a\vee c, b\vee d)) = f(a\vee c)\vee g(b\vee d) = (f(a)\vee f(c))\vee$$
$$\vee(g(b)\vee g(d)) = (f(a)\vee g(b))\vee(f(c)\vee g(d)) = h((a, b))\vee h((c, d))$$

which means that h is a homomorphism. We prove that h is distributive.

Let $h(c, d)=f(c)\vee g(d)=x\vee y$ in L. By the distributivity of L we get elements $x_1, x_2, y_1, y_2\in L$ such that $x_1\vee y_1=f(c)$, $x_2\vee y_2=g(d)$, $x_1, x_2\leqq x$, $y_1, y_2\leqq y$. Since (C, f, L) is saturated, therefore we have $c_1, c_2\in C$ such that $c_1\vee c_2=c$ and $f(c_1)\leqq x_1$, $f(c_2)\leqq y_1$. Similarly we get elements $d_1, d_2\in D$ with $d_1\vee d_2=d$, $g(d_1)\leqq x_2$, $g(d_2)\leqq y_2$. Set $\bar{x}=(c_1, d_1)$, $\bar{y}=(c_2, d_2)$. Then $\bar{x}\vee\bar{y}=(c_1\vee c_2, d_1\vee d_2)=(c, d)$, $h((c_1, d_1))=f(c_1)\vee$ $\vee g(d_1)\leqq x$, $h(c_2, d_2)\leqq y$. This proves that h is weak-distributive. Let $\theta=\operatorname{Ker} f$, $\Phi=\operatorname{Ker} g$. Then $\theta=\bigvee\theta_j$, $\Phi=\bigvee\Phi_j$; θ_j, Φ_j are monomial distributive congruences. θ_i resp. Φ_j can be extended to $C\times D$, $\bar{\theta}_i\cup\bar{\Phi}_j$ which are again monomial. It is easy to see that $\operatorname{Ker} h=\bigvee(\bar{\theta}_i\vee\bar{\Phi}_j)$.

Corollary. *Let C, D be two Boolean semilattices and f resp. g distributive homomorphisms of these Boolean semilattices into the distributive lattice L. If $f(C)$ resp. $g(D)$ are ideals of L then there exists a distributive homomorphism $h\colon C\times D\to L$ such that $h|_C=f$, $h|_D=g$.*

Remark. In Lemma 4 f and g are not necessarily L-valuations induced by some join-bases.

Let L be an arbitrary distributive lattice with 0. If $a\in L$, $a\neq 0$ the principal ideal $(a]$ is a bounded distributive lattice. Assume that for every $(a]$ we have a Boolean semilattice B_a and a distributive homomorphism φ_a of B_a onto $(a]$. Consider the lower discrete direct product $B=\Pi_d(B_a|a\in L,\ a\neq 0)$. B is a generalized Boolean semilattice. By Lemma 4 we have a distributive homomorphism $\varphi\colon B\to L$ which is onto. Consequently to prove Theorem 1 we can assume that L is a bounded distributive lattice. By Lemma 2 we have a pre-skeleton $B(1)$ with a homomorphism $\varphi_1\colon B(1)\to L$ which satisfies (2). Let u be an arbitrary non-zero element of the join-basis $H\subseteq B(1)$, $a=\varphi_1(u)$. The principal ideal $(a]$ of L is a bounded distributive lattice, therefore we can apply again Lemma 2 to get a pre-skeleton $B(a)$ and a homomorphism $\varphi_a\colon B(a)\to(a]$ into $(a]$. If u' denotes the complement of u in $B(1)$ then $B=B(1)$ is the direct product $(u']\times(u]$. Take the free $\{0,1\}$-distributive product $(u]*B(a)$ and finally the Boolean semilattice

$$B[I,u]=\big((u]*B(a)\big)\times(u'].$$

By Lemmas 1 and 3 we have a homomorphism $\varphi\colon B[I,u]\to L$, satisfying the following condition:

(*) if $r\in T=\{I,u\}$, $\varphi(r)=x\vee y$ then there exist $x_1,y_1\in B[I,u]$ with $x_1\vee y_1=r$, $\varphi(x_1)\leqq x$, $\varphi(y_1)\leqq y$.

Using the same method for an element $v\in B\subset B[I,u]$ we get from $B[I,u]$ a Boolean algebra $B[I,u,v]$ satisfying (*) for the set $T=\{I,u,v\}$.

Lemma 5. *Let* $u,v\in B$, *then* $B[I,u,v]\cong B[I,v,u]$.

Proof. If H denotes a join-base of B and $x\in H$ then we shall write $H(x)$ for $H\cap(x]$. It is easy to show that $H(x)\cup H(x')$ is again a join-base and L-valuations generated by these join-bases coincide. If $u,v\in B$ then we have therefore a join-base $H(u\wedge v)\vee H(u\wedge v')\vee H(u'\wedge v)\vee H(u'\wedge v')$. Hence we get for $B[I,u,v]$ resp. $B[I,v,u]$ the following. Let H_u resp. H_v be a join base of $B(\varphi_1(u))$ resp. $B(\varphi_1(v))$; then $(H_u^1\times H_v^1\times H^1(u\wedge v))\cup(H_u^1\times H^1(u\wedge v'))\cup(H_v^1\times H^1(u'\wedge v))\cup H^1(u'\wedge v')$ which proves the isomorphism.

Continuing this construction we get for arbitrary $u_1,u_2,\dots,u_n\in B$ a Boolean semilattice $B[I,u_1,\dots,u_n]$ and a homomorphism of this Boolean semilattice into L such that condition (*) is satisfied for $T=\{I,u_1,\dots,u_n\}$.

All these Boolean semilattices form a direct family. Let C_1 be the direct limit Then $B(1)=C_0$ is a Boolean subalgebra of C_1 and we have $\varphi\colon C_1\to L$ which satisfies (*) for all $x\in T=B(1)$. Then we start with C_1 and in the same way we get a Boolean semilattice C_2. Then C_1 is a Boolean subalgebra of C_2. Similarly, we get

C_i ($i=3, 4, \ldots$). These algebras C_i form again a direct family. Let $\bar{B}$ be the direct limit. Let $\varphi\colon \bar{B}\to L$ be the corresponding homomorphism. Then (B, φ, L) is saturated, hence φ is a weak-distributive homomorphism into L.

Lemma 6. *$\bar{B}$ has a join-base.*

Proof. This is a trivial consequence of Lemmas 1 and 3.

Lemma 7. *Let $\varphi\colon B\to L$ be a weak-distributive homomorphism of a Boolean semilattice B generated by a homomorphism $f\colon H\to L$ of a join-base H. Then φ is distributive.*

Proof. Let θ be the congruence relation induced by φ. H_k denotes the set of all $x\in H$ of dimension k. Take two elements $a, b\in B$, $a>b$ satisfying $a\equiv b$ (θ). Then a and b have join-representations as joins of elements from some H_k, say $a=h_1\vee\ldots\vee h_n\vee h_{n+1}$ and $b=h_1\vee\ldots\vee h_n$. If $c=h_1\vee\ldots\vee h_k$, $k<n$ and $d=h_i\vee\ldots\vee h_n$, $i\leqq k$ then $c\vee d=b$. By condition (iv) of Definition 5 we can assume that these representations of a, b, c, d are unique. By the weak distributivity of θ we have elements $\bar{c}\geqq c$, $\bar{d}\geqq d$ such that $\bar{c}\vee\bar{d}=a$ and $c\equiv\bar{c}\,(\theta)$, $d\equiv\bar{d}$ (θ). For $\bar{c}$, $\bar{d}$ we have the following possibilities: (i) $\bar{c}=c\vee h_{n+1}$, $\bar{d}=d$; (ii) $\bar{c}=c$, $\bar{d}=d\vee h_{n+1}$; (iii) $\bar{c}=c\vee h_{n+1}$, $\bar{d}=d\vee h_{n+1}$.

We define a binary relation θ_{ab} on B as follows: $x\equiv y$ (θ_{ab}), $x>y$ iff $x\equiv y$ (θ) and $y\leqq b$, $x\vee b=a$. Then the assumption that θ is induced by the join-base H we get that each θ_{ab}-class contains a maximal element. Let $\theta_{ab}^{\vee}$ be the smallest join congruence of B satisfying $\theta_{ab}^{\vee}\geqq\theta_{ab}$. Then $u\equiv v$ $(\theta_{ab}^{\vee})$, $u\geqq v$ iff there exist $x\geqq y$, $x\equiv y$ (θ_{ab}) such that $y\leqq v$ and $x\vee v=u$. Obviously $\theta_{ab}^{\vee}\leqq\theta$, $\vee\theta_{ab}^{\vee}=\theta$. The first part of the proof yields that $\theta_{ab}^{\vee}$ is distributive.

An element $a\in L$ is of finite order if there exists a sequence $a=x_0, x_1, x_2, \ldots, x_n$ such that $a<a\vee x_1<a\vee x_1\vee x_2<a\vee x_1\vee\ldots\vee x_{n-1}<a\vee x_1\vee\ldots\vee x_n=1$ and $a\vee x_1\vee\vee x_2\vee\ldots\vee x_{i-1}$ is incomparable with x_i ($i=1, \ldots, n$). By the construction of $\varphi\colon\bar{B}\to L$ the image of each $u\in\bar{B}$, $u\neq 0$ is the meet of elements of finite order. Now we have for every $a\in L$ a Boolean semilattice $B(a)$ and a distributive homomorphism $\varphi_a\colon B(a)\to(a]$ which maps $B(a)$ onto the set of all elements having a meet representation of elements of finite order in the lattice $(a]$. Then the triple $(B(a), \varphi_a, (a])$ is saturated. The lower discrete product of these Boolean semilattices B has by Lemma 4 a distributive homomorphism onto L which proves Theorem 1.

3. Construction of a strong extension

In this section we give the outline of the proof of the following theorem, which was proved in [4]. Combining Theorems 1 and 2 we get our main theorem.

Theorem 2. *Let θ be a distributive congruence of a generalized Boolean semilattice B. The lattice of all ideals of B/θ is the congruence lattice of a lattice.*

We denote the five element modular non-distributive lattice by M_3; M_3 with an additional atom is called M_4, etc. If α is an arbitrary cardinal number then M_α is the modular lattice of length 2 with α atoms.

Let $M=\{0<a, b, c<1\}$ be a lattice isomorphic to M_3 and let D be a bounded distributive lattice with zero element o, and unit element i. Identifying a with i and 0 with o, we get a partial lattice ${}_DM_3=D\cup M_3$ (Fig. 8), $D\cap M_3=\{0, a\}$ and D, M_3 are sublattices; $d\vee b$ resp. $d\vee c$ $(d\in D)$ is defined iff $d\in\{0, a\}$ (see Mitschke & Wille [3]). There exists a modular lattice $M_3[D]$ generated by ${}_DM_3$ such that ${}_DM_3$ is a relative sublattice of $M_3[D]$. In [3] it was proved that there exists only one modular lattice with these properties, the modular lattice $FM({}_DM_3)$ freely generated by ${}_DM_3$. This lattice was introduced in [4] and has the following description.

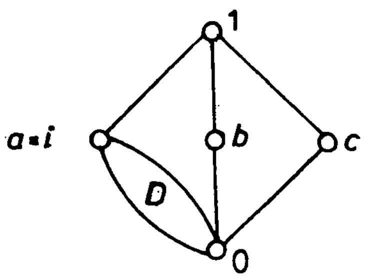

Figure 8.

An element $(x, y, z)\in D\times D\times D$ is called *normal* if $x\wedge y=x\wedge z=y\wedge z$. Let $M_3[D]$ be the poset of all normal elements, then $M_3[D]$ is a modular lattice. Let $a=(i, 0, 0)$, $b=(0, i, 0)$, $c=(0, 0, i)$, $1=(i, i, i)$, $0=(0, 0, 0)$. Then these elements form a sublattice isomorphic to M_3. The set of all elements $(x, 0, 0)$, $(x\in D)$ form a sublattice isomorphic to D. D is a strongly large sublattice of $M_3[D]$, and every congruence relation $\theta\in\mathrm{Con}(D)$ can be extended to $M_3[D]$, i.e. $\mathrm{Con}(D)\cong\mathrm{Con}(M_3[D])$. We can use the same construction for distributive lattices without unit element.

We prove Theorem 2 first for monomial congruences of Boolean semilattices i.e. for relatively pseudocomplemented lattices.

Lemma 8. *Let θ be a monomial distributive congruence of a generalized Boolean semilattice B. Then there exists a lattice N such that* $\mathrm{Con}^c(N)\cong B/\theta$.

Sketch of the proof. Consider $D=B$ and the corresponding lattice $M_3[B]$. We define a subset N of $M_3[B]$ as follows

$(**)$ $(x, y, z)\in M_3[B]$ belongs to N iff x is a maximal element of a θ-class.

Then N is a lattice and $(x, 0, 0)\in N$ iff x is a maximal element of θ-class, i.e., the ideal I generated by $(i, 0, 0)$ is isomorphic to B/θ. N is a strong extension of I, a congruence relation of I has an extension to N iff it has the form $\theta(I')$, where I' is an ideal of N. Thus $\mathrm{Con}^c(N)\cong B/\theta$, i.e. $\mathrm{Con}(N)\cong I(B/\theta)$.

The ideal J of N, generated by $(0, 0, i)$ is isomorphic to B. By the definition of I and J we have $I\cap J=0$ (Fig. 9).

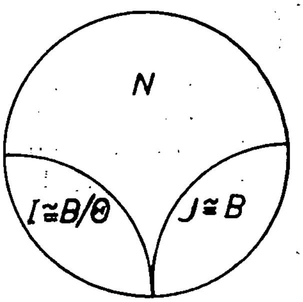

Figure 9.

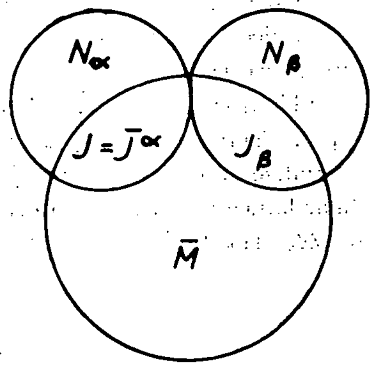

Figure 10.

Let θ be an arbitrary distributive congruence relation of the generalized Boolean semilattice B. Then θ is the join of monomial distributive congruence relations, say $\theta=\bigvee(\theta_\alpha|\alpha\in\Omega)$. We take first for every α the lattice N_α defined before. This N_α has two ideals $I_\alpha\cong B/\theta_\alpha$ and $J_\alpha\cong B$. Moreover $\mathrm{Con}^c(N_\alpha)\cong B/\theta_\alpha$.

On the other hand we consider the direct product $\Pi(B_\alpha|\alpha\in\Omega)$. M denotes the sublattice of the direct product of those normal sequences t for which $\{t(\alpha)|\alpha\in\Omega\}$ is finite, i.e. the weak direct product is normal if $\alpha, \beta, \gamma\in\Omega$, $\alpha\neq\beta$, $\alpha\neq\gamma$, $\beta\neq\gamma$ imply $t(\alpha)\wedge t(\beta)=t(\alpha)\wedge t(\gamma)=t(\beta)\wedge t(\gamma)$. Let J^α be the ideal of M consisting of all t for which $t(\beta)=0$ if $\beta\neq\alpha$. Then $J^\alpha\cong B$. M is a strong extension of J^α and $\mathrm{Con}^c(M)\cong\cong\mathrm{Con}^c(J^\alpha)\cong\mathrm{Con}^c(B)$. Let $\overline{M}$ be the dual lattice of M. Then $\bar{J}^\alpha$ is a dual of $\overline{M}$. $\bar{J}^\alpha$ is a Boolean algebra, therefore we have a natural isomorphism $\bar{J}^\alpha\cong J^\alpha$ $(x\to x')$. We use the Hall—Dilworth gluing construction for $\overline{M}$ and N_α $(\alpha\in\Omega)$, we identify for every α the dual ideal $\bar{J}^\alpha$ and the ideal J_α. In this way we get a partial lattice P (see Figure 10).

$\overline{M}$ and N_α are sublattices of P, and P is a meet-semilattice. Let $F(P)$ be the free lattice generated by P. Then $\mathrm{Con}^c(F(P))\cong B/\theta$. This proves Theorem 2.

4. Some remarks on the characterization problem

The key problem of the characterization of congruence lattices of lattices is to prove the existence of a pre-skeleton of a bounded distributive semilattice. We reformulate this problem.

Let L be a bounded distributive semilattice. Let $F(G)$ be denote the free Boolean algebra generated by the set G. If $g_i \in G$ then the elements $0, g_i, g_i', I$ form a Boolean subalgebra which is the free Boolean algebra $F(g_i)$ generated by g_i. We have remarked that $F(G)$ is the free $\{0, 1\}$-distributive product of the Boolean algebras $F(g_i)$, $g_i \in G$. Let us assume that every $F(g_i)$ has a $\{0, 1\}$-homomorphism φ_i into L. Does there exist a $\{0, 1\}$-homomorphism $\varphi\colon F(G) \to L$ such that $\varphi|_{F(g_i)} = \varphi_i$? For finite G the answer is yes, we have

Proposition 3. *Let B be a finite Boolean algebra. If $\varphi_1\colon B \to L$ and $\varphi_2\colon F(g) \to L$ are $\{0, 1\}$-homomorphisms into L then there exists a $\{0, 1\}$-homomorphism φ of the free $\{0, 1\}$-distributive product $B * F(g)$ into L such that $\varphi|B = \varphi_1$, $\varphi|_{F(g)} = \varphi_2$.*

Proof. Let $p_1, p_2, \ldots, p_n$ denote the atoms of B. The atoms of the free product are $p_1 \wedge g, \ldots, p_n \wedge g$, $p_1 \wedge g', \ldots, p_n \wedge g'$. Then $g < p_1 \vee \ldots \vee p_r = I$ yields $\varphi_2(g) < \varphi_1(p_1) \vee \ldots \vee \varphi_1(p_n) = 1 \in F$. But F is a distributive semilattice hence we have elements $a_1, a_2, \ldots, a_n \in F$ such that $\varphi_2(g) = a_1 \vee \ldots \vee a_n$, $a_i \leqq \varphi_1(p_i)$ $(i = 1, 2, \ldots, n)$. Similarly $g' < p_1 \vee \ldots \vee p_n$ therefore we have elements $b_1, \ldots, b_n \in L$ satisfying $\varphi_2(g') = b_1 \vee \ldots \vee b_n$, $b_i \leqq \varphi_1(p_i)$. On the other hand $p_i \leqq g \vee g'$ hence $\varphi_1(p_i) \leqq \varphi_2(g) \vee \varphi_2(g')$. Thus we get elements u_i, v_i such that $\varphi_1(p_i) = u_i \vee v_i$, $u_i \leqq \varphi_2(g)$, $v_i \leqq \varphi_2(g')$. Define $\varphi(p_i \wedge g) = a_i \vee u_i$, $\varphi(p_i \wedge g') = b_i \vee v_i$. Every u of $B * F(g)$ has a unique representation as a join of atoms, say $u = \vee g_i$. We define $\varphi(u) = \vee \varphi(g_i)$. This φ is obviously a homomorphism. From $p_i = (p_i \wedge g) \vee (p_i \wedge g')$ we get $\varphi(p_i) = (p_i \wedge g)$ $(p_i \wedge g') = (a_i \vee u_i) \vee (b_i \vee v_i) = a_i \vee b_i \vee \varphi_1(p_i) = \varphi_1(p_i)$. Similarly $g = \bigvee_{i=1}^{n} (p_i \wedge g) = \bigvee_i (a_i \vee u_i) = \bigvee_{i=1}^{n} a_i \vee \bigvee_{i=1}^{n} u_i = \varphi_2(g)$. (I.e. $\varphi|_B = \varphi_1$, $\varphi|_{F(g)} = \varphi_2$).

It is necessary to generalize Lemma 1 for distributive semilattice. Let B be the free Boolean algebra $F(G)$. Then the join-base is $H = \bigcup_{i=0}^{\infty} H_i \cup \{1\}$.

We have for every $g_i \in G$ a $\{0, 1\}$-homomorphism $\varphi_i\colon F(g_i) = \{0, g_i, g_i', I\} \to L$, i.e. we have a mapping $H_1 \to L$ and we want to get a $\{0, 1\}$-homomorphism $\varphi\colon B \to L$ which is a common extension of each φ_i. To define such a φ it is natural to use induction on k. If $x \in H_1$ then $x = g_i$ or $x = g_i'$ for some $g_1 \in G$ and we have $\varphi(x) = \varphi_i(x)$. Using the method of Proposition 3 it is easy to define $\varphi(x)$ for all $x \in H_2$. How can we define $\varphi(x)$ for $x \in H_3$?

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