Acta Sci. Math., 43 (1981), 239—242

On approximation by unitary operators

JOSÉ BARRÍA and RAMÓN BRUZUAL

Introduction. Let $\mathfrak{H}$ be a separable infinite dimensional complex Hilbert space. Let $\mathscr{L}(\mathfrak{H})$ be the algebra of all bounded linear operators on $\mathfrak{H}$. In [3] D. D. ROGERS has determined the distance from an arbitrary operator in $\mathscr{L}(\mathfrak{H})$ to the set of unitary operators in $\mathscr{L}(\mathfrak{H})$. He also has given sufficient conditions for the existence of unitary approximants.

In this paper we prove the density of the operators with unitary approximants; the existence of unitary approximants for n-normal operators is established. The proofs of these results rely on the paper [3]. For easy reference we state below the results that are needed. Let $T\in\mathscr{L}(\mathfrak{H})$. Write $|T|=(T^*T)^{1/2}$, and let $\sigma_e(T)$ be the essential spectrum of T. Let $m_e(T)$ denote the infimum of the set $\sigma_e(|T|)$. The index of T is defined by $\operatorname{ind} T=\dim\ker T-\dim\ker T^*$ with the convention that $\operatorname{ind} T=0$ if $\ker T$ and $\ker T^*$ have infinite dimension. A unitary approximant of T is a unitary operator U_0 such that $\|T-U_0\|=\inf\{\|T-U\|:\ U \text{ unitary in } \mathscr{L}(\mathfrak{H})\}$.

Theorem A [3]. *Let $T\in\mathscr{L}(\mathfrak{H})$.*

(i) *If $T=U|T|$ with U unitary, then U is a unitary approximant of T.*

(ii) *If* $\operatorname{ind} T<0$ *and $m_e(T)$ is an eigenvalue of $|T|$ of infinite multiplicity, then T has a unitary approximant.*

1. Density. In [3] it is proved that an operator $T\in\mathscr{L}(\mathfrak{H})$ has no unitary approximant if T has negative index and its distance to the set of unitary operators is equal to one. However, we have the following theorem.

Theorem 1. *The set of operators in $\mathscr{L}(\mathfrak{H})$ with unitary approximants is dense in $\mathscr{L}(\mathfrak{H})$.*

Proof. Let $\mathscr{A}$ be the set of operators with unitary approximants. Let $\bar{\mathscr{A}}$ be the closure of $\mathscr{A}$ in the norm topology. Let $T\in\mathscr{L}(\mathfrak{H})$. Clearly $T\in\bar{\mathscr{A}}$ if and only if $T^*\in\bar{\mathscr{A}}$. If $\operatorname{ind} T=0$, then Theorem A(i) implies that $T\in\mathscr{A}$. Therefore,

Received July 20, 1980. The second author was partially supported by F. G. M. de Ayacucho.

it is enough to show that $T\in\mathscr{A}$ whenever T has negative index. If $\operatorname{ind} T<0$, then $T=S|T|$ where S is an isometry. Let $\varepsilon>0$. Let $\lambda=m_e(|T|)$. Since $\lambda\in\sigma_e(|T|)$, from [1, Theorem 2.18] there exists a closed infinite dimensional subspace $\mathfrak{H}_0$ such that $|T|$ has the representation:

$$\begin{bmatrix} \lambda+K_1 & X \\ & K_2 & Y \end{bmatrix}$$

with respect to the decomposition $\mathfrak{H}=\mathfrak{H}_0\oplus\mathfrak{H}_0^{\perp}$, where K_i is a compact operator and $\|K_i\|\leqq\varepsilon$, for $i=1, 2$. Since $|T|$ is a positive operator we have $X=K_2^*$ and $Y\geqq 0$. Hence $|T|=P+K$ where $P=\lambda\oplus Y\geqq 0$, K is a compact operator and $\|K\|\leqq 2\varepsilon$.

Let $T_0=SP$. We now show that T_0 has a unitary approximant. Since $S(\ker T_0)\subseteq\ker T_0^*$, then $\operatorname{ind} T_0\leqq 0$. If $\operatorname{ind} T_0=0$, then $T_0\in\mathscr{A}$ from Theorem A(i). Now we assume that $\operatorname{ind} T_0<0$. Since $T-T_0=SK$ is compact, it follows that $m_e(T)=m_e(T_0)=\lambda$. Furthermore, $|T_0|=P$ implies that $m_e(T_0)$ is an eigenvalue of $|T_0|$ of infinite multiplicity. Then from Theorem A(ii) we have $T_0\in\mathscr{A}$. Since $\|T-T_0\|=\|K\|\leqq 2\varepsilon$, the proof that $T\in\mathscr{A}$ is complete. □

2. *n*-normal operators. An $n\times n$ operator matrix $T=(T_{ij})_{i,j=1}^n$ acting on the direct sum of n copies of $\mathfrak{H}$ is *n-normal* if the set $\{T_{ij}\}_{i,j=1}^n$ consists of mutually commuting normal operators on $\mathfrak{H}$. It is well known (see [2]) that an n-normal operator matrix is unitarily equivalent to an n-normal operator matrix in upper triangular form.

Lemma 1. *Let $T=(T_{ij})_{i,j=1}^n$ be an n-normal operator matrix in upper triangular form* (i.e. $T_{ij}=0$ *for* $i>j$). *If* $\ker T_{11}=\{0\}$, *then*

$$\dim\ker T = \dim\ker T_0, \quad \textit{and} \quad \dim\ker T^* = \dim\ker T^*$$

where T_0 is the $(n-1)$-normal operator matrix obtained from T by deleting the first row and the first column.

Proof. Let $S=(-T_{11})\oplus I\oplus\ldots\oplus I$ acting on the direct sum of n copies of $\mathfrak{H}$. Let $\mathfrak{M}=\{\langle T_{12}x_2+\ldots+T_{1n}x_n, x_2, \ldots, x_n\rangle : \langle x_2, \ldots, x_n\rangle\in\ker T_0\}$. Clearly $\mathfrak{M}$ is a closed subspace and $\dim\mathfrak{M}=\dim\ker T_0$. Now we show that $S(\ker T)^-=\mathfrak{M}$. From this equality and $\ker S=\{0\}$ the first assertion of the lemma follows.

Let $\langle x_1, \ldots, x_n\rangle\in\ker T$. Then $T_{12}x_2+\ldots+T_{1n}x_n=-T_{11}x_1$ and $\langle x_2, \ldots, x_n\rangle\in\ker T_0$. Therefore, $S\langle x_1, \ldots, x_n\rangle=\langle -T_{11}x_1, x_2, \ldots, x_n\rangle$ is an element of $\mathfrak{M}$, and $S(\ker T)\subseteq\mathfrak{M}$.

Let E be the spectral measure of T_{11}. For $k\geqq 1$ we define $P_k=E\left(\left\{z\in C\colon |z|\geqq\frac{1}{k}\right\}\right)$.

For x in $\mathfrak{H}$ we have $\|x-P_kx\|^2=\left\|E\left(\left\{z\in C\colon |z|<\frac{1}{k}\right\}\right)x\right\|^2\to\|E(\{0\})x\|^2$ $(k\to\infty)$. Since $E(\{0\})=\ker T_{11}=\{0\}$, then $\|x-P_kx\|\to 0$ $(k\to\infty)$.

Now we are ready to prove that $\mathfrak{M}\subseteq S(\ker T)^-$. Let $\langle y, x_2, \ldots, x_n\rangle\in\mathfrak{M}$. Then $y=T_{12}x_2+\ldots+T_{1n}x_n$ and $\langle x_2, \ldots, x_n\rangle\in\ker T_0$. Let $y^{(k)}=P_k y$ and $x_i^{(k)}=P_k x_i$ $(i=2, \ldots, n)$. Since P_k and T_{ij} commute, then $y^{(k)}=T_{12}x_2^{(k)}+\ldots+T_{1n}x_n^{(k)}$ and $\langle x_2^{(k)}, \ldots, x_n^{(k)}\rangle\in\ker T_0$. Therefore $\langle y^{(k)}, x_2^{(k)}, \ldots, x_n^{(k)}\rangle\in\mathfrak{M}$ and $\langle y^{(k)}, x_2^{(k)}, \ldots, x_n^{(k)}\rangle\to\langle y, x_2, \ldots, x_n\rangle$ $(k\to\infty)$. If $\mathfrak{H}_k=P_k\mathfrak{H}$, then $\mathfrak{H}_k$ reduces T_{11} and T_{11} is bounded below on $\mathfrak{H}_k$ $\left(\|T_{11}x\|\geqq\frac{1}{k}\|x\| \text{ for } x \text{ in } \mathfrak{H}_k\right)$; therefore the restriction of T_{11} to $\mathfrak{H}_k$ is an invertible operator. Since $y^{(k)}\in\mathfrak{H}_k$, there exists $x_1^{(k)}\in\mathfrak{H}_k$, such that $T_{11}x_1^{(k)}=-y^{(k)}$. Therefore $T_{11}x_1^{(k)}+\ldots+T_{1n}x_n^{(k)}=0$ and $\langle x_1^{(k)}, \ldots, x_n^{(k)}\rangle\in\ker T$. Furthermore, $S\langle x_1^{(k)}, \ldots, x_n^{(k)}\rangle=\langle y^{(k)}, x_2^{(k)}, \ldots, x_n^{(k)}\rangle\to\langle y, x_2, \ldots, x_n\rangle$ $(k\to\infty)$. Hence $\langle y, x_2, \ldots, x_n\rangle\in S(\ker T)^-$ and $\mathfrak{M}\subseteq S(\ker T)^-$. This completes the proof that $\mathfrak{M}=S(\ker T)^-$.

The second assertion of the lemma follows from the fact that $\operatorname{Ker} T^*=\{\langle 0, x_2, x_n\rangle:\langle x_2, \ldots, x_n\rangle\in\ker T_0^*\}$. □

Theorem 2. *Let* $T=(T_{ij})_{i,j=1}^n$ *be an n-normal operator matrix. Then*

$$\dim\ker T=\dim\ker T^*.$$

Proof. We need to consider only the case when T is in upper triangular form. Then we assume that $T_{ij}=0$ for $i>j$. Now the proof will proceed by induction on n. If $n=1$, then T is a normal operator and $\ker T=\ker T^*$. Next we assume that the theorem holds for all $(n-1)$-normal upper triangular operator matrices. Let $\mathfrak{H}_1=\ker T_{11}\cap\ker T_{nn}$, $\mathfrak{H}_2=(\ker T_{nn}\ominus\mathfrak{H}_1)\oplus(\mathfrak{H}\ominus\ker T_{11}T_{nn})$ and $\mathfrak{H}_3=\ker T_{11}T_{nn}\ominus\ker T_{nn}$. Then the subspaces $\mathfrak{H}_1$, $\mathfrak{H}_2$, $\mathfrak{H}_3$ are pairwise orthogonal and $\mathfrak{H}=\mathfrak{H}_1\oplus\mathfrak{H}_2\oplus\mathfrak{H}_3$. Since $\ker T_{11}$, $\ker T_{nn}$ and $\ker T_{11}T_{nn}$ reduce T_{ij}, then $\mathfrak{H}_k$ $(k=1, 2, 3)$ reduces T_{ij} for all i and j. Furthermore, if $T_{ij}^{(k)}=T_{ij}|\mathfrak{H}_k$ and $T_k=(T_{ij}^{(k)})_{i,j=1}^n$ $(k=1, 2, 3)$, then T_k is an n-normal operator matrix in upper triangular form and $T=T_1\oplus T_2\oplus T_3$ with respect to the decomposition $\bigoplus_{k=1}^{3}(\mathfrak{H}_k\oplus\ldots\oplus\mathfrak{H}_k)$ (n copies) of $\mathfrak{H}\oplus\ldots\oplus\mathfrak{H}$ (n copies). The next and last step in the proof is to show that $\dim\ker T_k=\dim\ker T_k^*$ for $k=1, 2, 3$.

We consider first the operator T_1. If the dimension of $\mathfrak{H}_1$ is finite, then T_1 is acting in a finite dimensional space and the assertion is true. Assume that $\mathfrak{H}_1$ has infinite dimension. From the definition of $\mathfrak{H}_1$ we have $T_{11}^{(1)}=T_{nn}^{(1)}=0$. Therefore, $\mathfrak{H}_1\oplus\{0\}\oplus\ldots\oplus\{0\}\subseteq\ker T_1$ and $\{0\}\oplus\ldots\oplus\{0\}\oplus\mathfrak{H}_1\subseteq\ker T_1^*$. Therefore $\dim\ker T_1=\dim\ker T_1^*=\dim\mathfrak{H}_1$.

Now we consider the operator T_2. From the definition of $\mathfrak{H}_2$ it is clear that $\ker T_{11}^{(2)}=\{0\}$. From Lemma 2.1 we have $\dim\ker T_2=\dim\ker T_{2,0}$ and $\dim\ker T_2^*=\dim\ker T_{2,0}^*$, where $T_{2,0}$ is the $(n-1)$-normal operator matrix obtained from T_2 by deleting the first row and the first column. Now from induction $\dim\ker T_{2,0}=\dim\ker T_{2,0}^*$. Therefore, $\dim\ker T_2=\dim\ker T_2^*$.

Finally, we consider the operator T_3. From the definition of $\mathfrak{H}_3$ it is clear that $\ker T_{nn}^{(3)}=\{0\}$. It is easy to see that T_3^* is unitarily equivalent to the n-normal, upper triangular operator matrix $(T'_{ij})_{i,j=0}^{n-1}$ with entries $T'_{ij}=T_{n-j,n-i}^{(3)*}$ (for $0\leqq i\leqq j\leqq n-1$) and $=0$ for $i<j$. Since the $(0, 0)$-entry of this operator is one-to-one, the argument given for the operator T_2 applies in this case. Therefore, $\dim\ker T_2^*=\dim\ker T_2$. □

Corollary 1. *Let* $T=(T_{ij})_{i,j=1}^n$ *be an n-normal operator matrix. Then* T *has a unitary approximant.*

Proof. The result is immediate from Theorem 2 and Theorem A(i). □

References

[1] C. M. Pearcy, *Some recent developments in Operator Theory*, NSF—CBMS Lecture Notes, № 36, Amer. Math. Soc. (Providence, R. I., 1978).

[2] H. Radjavi, P. Rosenthal, *Invariant Subspaces*, Springer-Verlag (New York, 1973).

[3] D. D. Rogers, Approximation by unitary and essentially unitary operators, *Acta Sci. Math.*, 39 (1977), 141—151.

INSTITUTO VENEZOLANO DE INVESTIGACIONES CIENTIFICAS
DEPARTMENTO DE MATEMÁTICAS
APARTADO 1827
CARACAS 1010-A, VENEZUELA