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Functional models and extended spectral dominance

H. BERCOVICI, C. FOIAŞ, C. M. PEARCY, and B. SZ.-NAGY

In the paper [4], SCOTT BROWN showed that every subnormal operator on Hilbert space has nontrivial invariant subspaces, and thereby originated techniques which could be applied to broader classes of operators also; from the rapidly growing number of pertinent papers let us only mention a few: say [1], [2], [5], [7]. Two further papers, [9] and [10], took the first steps to exploit similar techniques in the setting of the functional model of contractions. The present paper is a partly expository synthesis and a continuation of these two papers, with some applications to invariant subspace problems. We have chosen to reproduce here, with some rearrangement and simplifications, the pertinent parts of [9] and [10] because of some shortcomings in their redaction (in particular the definition of the functional η in [9]), which unnecessarily restricted the applicability of the results.

1. Function spaces. Dominance of sets. Convex hulls

In this paper we shall have to do with Lebesgue and Hardy spaces L^p, H^p $(1 \leqq p \leqq \infty)$ relative to the unit-circle $C = \{e^{it}\colon 0 \leqq t < 2\pi\}$ and the normalized Lebesgue measure $dm = dt/(2\pi)$ on C; the general reference may be, e.g., [6]. For any measurable subset s of C, L^p_s will denote the subspace of L^p consisting of functions vanishing outside of s. Every function $f \in L^p$ admits a harmonic "extension" $\tilde{f}$ to the unit disc $D = \{\lambda\colon |\lambda| < 1\}$, defined by

$$\tilde{f}(\mu) = \int f P_\mu \, dm, \tag{1.1}$$

where P_μ is the Poisson kernel function on C corresponding to the point $\mu \in D$, i.e.,

$$P_\mu(e^{it}) = (1-|\mu|^2)\,|1-\bar{\mu}e^{it}|^{-2}; \quad \|P_\mu\|_{L^1} = 1. \tag{1.2}$$

The function f can be recovered from $\tilde{f}$ almost everywhere on C, as a "non-tangential

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limit": $f(e^{it})=\lim \tilde{f}(\mu)$ as $\mu\to e^{it}$, non-tangentially to C. If $f\in H^p$, $\tilde{f}$ is analytic in D, and in this case it is customary not to distinguish between f and $\tilde{f}$. We denote by H_0^p the subspace of H^p, consisting of the functions vanishing at the point 0.

Recall that H^∞ is the Banach dual of the space L^1/H_0^1 through the bilinear form $\langle f^\cdot, u\rangle=\int fu\,dm$ ($f\in L^1$, $u\in H^\infty$), $f\to f^\cdot$ denoting the natural map of L^1 onto L^1/H_0^1. For the sake of simplicity, we shall also write, for any $f\in L^1$, $\|f\|_{L^1/H_0^1}$ instead of $\|f^\cdot\|_{L^1/H_0^1}$, and $\|f\|_{L^1(s)}$ instead of $\|f|s\|_{L^1(s)}$. By the definition of the norm in a Banach quotient space, we have

$$\|f\|_{L^1/H_0^1}=\inf_{g\in H_0^1}\|f+g\|_{L^1},\quad\text{and hence,}\quad \|f\|_{L^1/H_0^1}\leqq\|f\|_{L^1}. \tag{1.3}$$

A subset S of the unit disc D is called *dominating for a* (measurable) *subset* s of the circle C if almost every point of s is the non-tangential limit of a sequence of points of S. (It is easily seen that for any set $S\subset D$ the set of *all* non-tangential limits of S on C is measurable, indeed an $F_{\sigma\delta\sigma}$.) A set S dominating for the whole circle C will be also called simply *dominant.* Such a set enjoys the property

$$\sup_{\lambda\in S}|u(\lambda)|=\|u\|_\infty\quad\text{for all}\quad u\in H^\infty;\ \text{cf. [3].} \tag{1.4}$$

Consider now an arbitrary complex Banach space X, its (closed) unit ball X_1, and an arbitrary subset E of X. The *absolutely convex hull* of E (aco E) is defined by

$$\operatorname{aco}E=\Big\{\sum_i c_ix_i\ \text{(finite sums)}:\ x_i\in E,\ c_i\in\mathbf{C},\ \sum_i|c_i|\leqq 1\Big\};$$

its closure will be denoted by $\overline{\operatorname{aco}}\,E$.

We shall need the following standard consequence of the Hahn—Banach theorem (cf., e.g., [5], Prop. 2.8):

Lemma 1.1. *Let the subset E of the unit ball X_1 of the complex Banach space X satisfy*

$$\sup_{x\in E}|\varphi(x)|=\|\varphi\|\quad\text{for all } \varphi \text{ in the dual space } X^*. \tag{1.5}$$

Then $\overline{\operatorname{aco}}\,E=X_1$.

We consider two special cases:

a) $X=L^1(s)$ and $E=\{P_\mu|s\colon \mu\in S\}$, where S is a subset of the unit disc D, dominating for the measurable set s on C.

b) $X=L^1/H_0^1$ and $E=\{P_\mu^\cdot\colon \mu\in S\}$, where S is a dominant subset of D.

In case a) we have $X^*=L^\infty(s)$ and we infer for any $\xi\in L^\infty(s)$, using Fatou's theorem,

$$\sup_{\mu\in S}\left|\int\xi P_\mu\,dm\right|=\sup_{\mu\in S}|\tilde{\xi}(\mu)|=\operatorname*{ess\,sup}_s|\xi|=\|\xi\|_{L^\infty(s)}.$$

In case b) we have $X^* = H^\infty$ and we deduce for any $\xi \in H^\infty$, again by using Fatou's theorem,

$$\sup_{\mu \in S} |\langle P_\mu^\cdot, \xi \rangle| = \sup_{\mu \in S} \left| \int P_\mu \xi \, dm \right| = \sup_{\mu \in S} |\tilde{\xi}(\mu)| = \operatorname*{ess\,sup}_{C} |\xi| = \|\xi\|_{H^\infty}.$$

Thus condition (1.6) holds in both cases, and we deduce from Lemma 1.1:

Lemma 1.2.
a) *If the set* $S \subset D$ *is dominating for the measurable set* $s \subset C$, *then*

$$\overline{\mathrm{aco}}\,\{P_\mu | s \colon \mu \in S\} = (L^1(s))_1.$$

b) *If the set* $S \subset D$ *is dominating for* C *then*

$$\overline{\mathrm{aco}}\,\{P_\mu^\cdot \colon \mu \in S\} = (L^1/H_0^1)_1.$$

2. Functional model and representation theorem for L^1 and L^1/H_0^1

Preliminaries. Denote by (CNU) the class of *completely nonunitary* contraction operators T on a separable complex Hilbert space $\mathfrak{H}$. The (unitarily equivalent) "functional model" of an operator $T \in$ (CNU) is the operator $S(\Theta)$ on the Hilbert space $\mathfrak{H}(\Theta)$ associated with a purely contractive analytic function $\{\mathfrak{E}, \mathfrak{E}_*, \Theta(\lambda)\}$, on the unit disc D ($\mathfrak{E}$ and $\mathfrak{E}_*$ being separable Hilbert spaces) in the following way. $\Theta(e^{it})$ being defined as the a.e. existent radial limit of $\Theta(\lambda)$ on C, and setting $\Delta(e^{it}) = [I - \Theta(e^{it})^* \Theta(e^{it})]^{1/2}$, consider the Hilbert function spaces

(2.1) $\mathfrak{K}_+(\Theta) = H^2(\mathfrak{E}_*) \oplus \overline{\Delta L^2(\mathfrak{E})}$ and $\mathfrak{H}(\Theta) = \mathfrak{K}_+(\Theta) \ominus \{\Theta w \oplus \Delta w \colon w \in H^2(\mathfrak{E})\}$

and the orthogonal projection operator $P_{\mathfrak{H}(\Theta)} \colon \mathfrak{K}_+(\Theta) \to \mathfrak{H}(\Theta)$. Then the operator $S(\Theta)$ defined on $\mathfrak{H}(\Theta)$ by

(2.2) $$S(\Theta)(u \oplus v) = P_{\mathfrak{H}(\Theta)}(e^{it} u \oplus e^{it} v) \quad (u \oplus v \in \mathfrak{H}(\Theta))$$

is in (CNU). It is unitarily equivalent to a given operator $T \in$ (CNU) on $\mathfrak{H}$ if Θ coincides with the characteristic function Θ_T of T, i.e., with the function $\{\mathfrak{D}_T, \mathfrak{D}_{T^*}, \Theta_T(\lambda)\}$ defined by

(2.3) $$\Theta_T(\lambda) = [-T + \lambda D_{T^*}(I - \lambda T^*)^{-1} D_T] | \mathfrak{D}_T,$$

where

(2.4) $$D_T = (I - T^* T)^{1/2}, \; D_{T^*} = (I - TT^*)^{1/2}, \; \mathfrak{D}_T = \overline{D_T \mathfrak{H}}, \; \mathfrak{D}_{T^*} = \overline{D_{T^*} \mathfrak{H}}.$$

Note that $\Theta_T(0) = -T | \mathfrak{D}_T$, and hence $\Theta_T(0)^* = -T^* | \mathfrak{D}_{T^*}$ so that $\Theta_T(0)^* \Theta_T(0) = T^* T | \mathfrak{D}_T$ and $\Theta_T(0) \Theta_T(0)^* = TT^* | \mathfrak{D}_{T^*}$. Further, note that $T^* T | \mathfrak{H} \ominus \mathfrak{D}_T = I_{\mathfrak{H} \ominus \mathfrak{D}_T}$ and $TT^* | \mathfrak{H} \ominus \mathfrak{D}_{T^*} = I_{\mathfrak{H} \ominus \mathfrak{D}_{T^*}}$, whence we infer that $\Theta_T(0)^* \Theta_T(0)$ and

T^*T, and hence their positive square-roots also, have on $[0, 1)$ the same spectra σ and the same essential spectra σ_e. The same holds for the other two products with the factors in the reverse order. It is also known (we refer for all these facts to Chapter VI of [8]) that for any $\mu \in D$ the characteristic function of the Möbius transform

(2.5) $$T_\mu = (T-\mu I)(I-\bar{\mu}T)^{-1}$$

coincides with $\left\{\mathfrak{D}_T, \mathfrak{D}_{T*}, \Theta_T\left(\frac{\lambda+\mu}{1+\bar{\mu}\lambda}\right)\right\}$; whence it follows as above that the parts on $[0, 1)$ of the spectra and of the essential spectra of $(\Theta(\mu)^*\Theta(\mu))^{1/2}$ and $(T_\mu^* T_\mu)^{1/2}$ are equal, and the same holds for the factors in the reverse order. We shall only need that, in particular,

(2.6) $$\inf \sigma_e(\Theta_T(\mu)\Theta_T(\mu)^*)^{1/2} = \inf \sigma_e(T_\mu T_\mu^*)^{1/2},$$

where, in case $\dim \mathfrak{D}_{T*} < \infty$, the left hand side is taken to be 1.

Let us add the remark that, for any selfadjoint operator R on an infinite dimensional Hilbert space $\mathfrak{R}$, we have

$$\inf \sigma_e(R) = \sup_{\mathfrak{A}\in\Phi} \inf_{\substack{a\in\mathfrak{A}\\ \|a\|=1}} \|Ra\|,$$

where Φ denotes the family of finite codimensional subspaces of $\mathfrak{R}$. As a consequence,

(2.7) $$\inf \sigma_e(SS^*)^{1/2} = \sup_{\mathfrak{A}\in\Phi} \inf_{\substack{a\in\mathfrak{A}\\ \|a\|=1}} \|S^*a\|$$

for any operator $S\colon \mathfrak{H}\to\mathfrak{H}'$ where $\mathfrak{H}'$ may be another Hilbert space. Thus (2.6) may be written in the form

(2.6)′ $$\sup_{\mathfrak{A}\in\Phi} \inf_{\substack{a\in\mathfrak{A}\\ \|a\|=1}} \|\Theta_T(\mu)^*a\| = \sup_{\mathfrak{A}'\in\Phi'} \inf_{\substack{a'\in\mathfrak{A}'\\ \|a'\|=1}} \|T_\mu^* a'\|,$$

where Φ and Φ' denote the families of finite codimensional subspaces of $\mathfrak{D}_{T*}$ and $\mathfrak{H}$, respectively.

The product $h\cdot h'^*$ and some of its properties. Starting from a purely contractive analytic function $\{\mathfrak{E}, \mathfrak{E}_*, \Theta(\lambda)\}$ we define, for $h=u\oplus v$, $h'=u'\oplus v'\in\mathfrak{H}(\Theta)$, the "product" $h\cdot h'^*$ by

(2.8) $$(h\cdot h'^*)(e^{it}) = (h(e^{it}), h'(e^{it}))_{\mathfrak{E}_*\oplus\mathfrak{E}} = (u(e^{it}), u'(e^{it}))_{\mathfrak{E}_*} + (v(e^{it}), v'(e^{it}))_{\mathfrak{E}};$$

it is clear that

(2.9) $$h\cdot h'^* = \overline{h'\cdot h^*} \in L^1.$$

We are looking for conditions under which every function f in L^1 can be represented in the form $f=h\cdot h'^*$ on C or on a given subset s of C, or on C in the form $f\equiv h\cdot h'^*$ modulo H_0^1. In order to do so we use elements of $\mathfrak{H}(\Theta)$ associated with points $\mu\in D$ and vectors $a\in\mathfrak{E}_*$ in the following way

$$\mu\circ a=P_{\mathfrak{H}(\Theta)}(p_\mu a\oplus 0), \tag{2.10}$$

where

$$p_\mu(\lambda)=(1-|\mu|^2)^{1/2}(1-\bar\mu\lambda)^{-1}\in H^\infty. \tag{2.11}$$

A straightforward calculation yields that

$$\mu\circ a=(p_\mu a-\Theta w)\oplus(-\Delta w),\quad \text{where}\quad w\in H^\infty(\mathfrak{E})\quad \text{is given by} \tag{2.12}$$

$$w(e^{it})=[p_\mu(e^{it})\,\Theta(e^{it})^*a]_+=p_\mu(e^{it})\,\Theta(\mu)^*a; \tag{2.13}$$

$[\,]_+$ and $[\,]_-$ denote the natural orthogonal projections of any (scalar or vector valued) function space L^2 onto its subspaces H^2 and $L^2\ominus H^2$, respectively.

For any $h=u\oplus v\in\mathfrak{H}(\Theta)$ we have then, using the second representation of w in (2.13),

$$(\mu\circ a)\cdot h^*=(p_\mu a-\Theta w,u)_{\mathfrak{E}_*}-(\Delta w,v)_{\mathfrak{E}}=(p_\mu a,x)_{\mathfrak{E}_*}=(a,\overline{p_\mu}x)_{\mathfrak{E}_*}, \tag{2.14}$$

where $x=u-\Theta(\mu)(\Theta^*u+\Delta v)$.

If $\{a_n\}_1^\infty$ is any orthonormal sequence in $\mathfrak{E}_*$ then (2.14) implies that $(\mu\circ a_n)\cdot h^*\to 0$ pointwise on C, as $n\to\infty$. Moreover, we have

$$|(\mu\circ a_n)\cdot h^*|\leqq|p_\mu|\,\|x\|_{\mathfrak{E}_*}\in L^1,\quad\text{because}\quad p_\mu\in L^2,\quad x\in L^2(\mathfrak{E}_*);$$

by virtue of the Lebesgue dominated convergence theorem we infer $\|(\mu\circ a_n)\cdot h^*\|_{L^1}\to 0$ as $n\to\infty$. Recalling (2.9) also, we have proved:

Lemma 2.1. *If* $\dim\mathfrak{E}_*=\infty$ *and* $\{a_n\}_1^\infty$ *is an orthonormal sequence in* $\mathfrak{E}_*$ *then, for any* $\mu\in D$ *and* $h\in\mathfrak{H}(\Theta)$,

$$(\mu\circ a_n)\cdot h^*\to 0\quad\textit{and}\quad h\cdot(\mu\circ a_n)^*\to 0$$

in L^1, *and* a fortiori *in every* $L^1(s)$ *and in* L^1/H_0^1. (Cf. (1.3).)

Next we derive from (2.12) and (2.13) the following relations for $\mu\in D$, $a\in\mathfrak{E}_*$, $\|a\|_{\mathfrak{E}_*}=1$,

$$\begin{aligned}(\mu\circ a)\cdot(\mu\circ a)^*&=\|p_\mu a-\Theta w\|_{\mathfrak{E}_*}^2+\|\Delta w\|_{\mathfrak{E}}^2=\\&=|p_\mu|^2-(p_\mu a,\Theta w)_{\mathfrak{E}_*}-(\Theta w,p_\mu a)_{\mathfrak{E}_*}+\|w\|_{\mathfrak{E}}^2=|p_\mu|^2-(\Theta^*p_\mu a,w)_{\mathfrak{E}}-y,\end{aligned} \tag{2.15}$$

where, using the first one of the representations in (2.13) for w, we have

$$y=(w,\Theta^*p_\mu a)_{\mathfrak{E}}-(w,[\Theta^*p_\mu a]_+)_{\mathfrak{E}}=(w,[\Theta^*p_\mu a]_-)_{\mathfrak{E}}.$$

As $w\in H^\infty$ we infer that $y\in H_0^1$, while

$$\begin{aligned}\|y\|_{L^1}&\leqq\int\|w\|_{\mathfrak{E}}\|[\Theta^*p_\mu a]_-\|_{\mathfrak{E}}\,dm\leqq\|w\|_{L^2(\mathfrak{E})}\|[\Theta^*p_\mu a]_-\|_{L^2(\mathfrak{E})}\leqq\\&\leqq\|w\|_{L^2(\mathfrak{E})}\|\Theta^*p_\mu a\|_{L^2(\mathfrak{E})}\leqq\|w\|_{L^2(\mathfrak{E})}\|p_\mu a\|_{L^2(\mathfrak{E}_*)}=\|w\|_{L^2(\mathfrak{E})}.\end{aligned}$$

For the middle term in the last member of (2.15) we have the same evaluation. Observe that $|p_\mu|^2$ equals the Poisson kernel function P_μ (see (1.2) and that, by (2.13),

$$\|w\|_{L^2(\mathfrak{E})} = \|p_\mu \Theta(\mu)^* a\|_{L^2(\mathfrak{E})} \leqq \|p_\mu\|_{L^2} \|\Theta(\mu)^* a\|_{\mathfrak{E}} = \|\Theta(\mu)^* a\|_{\mathfrak{E}};$$

we conclude:

Lemma 2.2. *For any $a \in \mathfrak{E}_*$ of norm 1, and any $\mu \in D$ we have*

$$\|(\mu \circ a) \cdot (\mu \circ a)^* - P_\mu\|_{L^1} \leqq 2\|\Theta(\mu)^* a\|_{\mathfrak{E}}, \tag{2.16}$$

$$\|(\mu \circ a) \cdot (\mu \circ a)^* - P_\mu\|_{L^1/H_0^1} \leqq \|\Theta(\mu)^* a\|_{\mathfrak{E}}. \tag{2.16$^\cdot$}$$

The representation theorems. From now on we shall always assume that $\dim \mathfrak{E}_* = \infty$ (this was tacitly assumed in Lemma 2.1), and consider the quantity, already appearing in (2.6) and (2.6)$'$:

$$\eta_\Theta(\mu) = \sup_{\mathfrak{A} \in \Phi} \inf_{\substack{a \in \mathfrak{A} \\ \|a\|=1}} \|\Theta(\mu)^* a\|_{\mathfrak{E}} \left(= \inf \sigma_e[(\Theta(\mu)\Theta(\mu)^*)^{1/2}]\right); \quad \mu \in D, \tag{2.17}$$

Φ denoting the family of finite codimensional subspaces of $\mathfrak{E}_*$.

Lemma 2.3. *For any given $\mu \in D$, $\mathfrak{A}_0 \in \Phi$, and $\varepsilon > 0$, there exists an orthonormal sequence $\{a_n\}_1^\infty$ in $\mathfrak{A}_0$ such that*

$$\|\Theta(\mu)^* a_n\|_{\mathfrak{E}} \leqq \eta_\Theta(\mu) + \varepsilon. \tag{2.18}$$

Proof. By induction: Suppose that for some $m \geqq 1$ the vectors $a \in \mathfrak{A}_0$ with $n < m$ have been already chosen so that they form an orthonormal system and satisfy (2.18) (these conditions are void if $m = 1$). The subspace $\mathfrak{A}_m = \mathfrak{A}_0 \ominus (\bigvee_{n<m} a_n)$ belongs to Φ, so by (2.17) we have $\inf_{\substack{a \in \mathfrak{A}_m \\ \|a\|=1}} \|\Theta(\mu)^* a\| \leqq \eta_0(\mu)$; and hence there exists a unit vector $a_m \in \mathfrak{A}_m$ satisfying (2.18) for $n = m$. Clearly $\{a_n\}_1^m$ is orthonormal. The proof is done.

In the sequel we shall be concerned, for any $\vartheta \in [0, 1)$, about the set

$$S_\vartheta = \{\mu \in D\colon\ \eta_\Theta(\mu) \leqq \vartheta\}. \tag{2.19}$$

Lemma 2.4. *Suppose that, for some $\vartheta \in [0, \frac{1}{2})$, the set S_ϑ is dominating for some measurable set $s \subset C$, and take a $\vartheta' \in (2\vartheta, 1)$. Suppose we have*

$$\|f - h \cdot k^*\|_{L^1(s)} \leqq \omega \quad \textit{for some } f \textit{ in } L^1, \textit{ and } h, k \textit{ in } \mathfrak{H}(\Theta).$$

Then there exist h', k' in $\mathfrak{H}(\Theta)$ such that

$$\|f - h' \cdot k'^*\|_{L^1(s)} \leqq \vartheta' \omega, \tag{2.20}$$

$$\|h - h'\| \leqq \omega^{1/2}, \quad \|k - k'\| \leqq \omega^{1/2}. \tag{2.21}$$

Proof. Fix an $\varepsilon>0$, to be specified later. By Lemma 1.2 there exists a finite sum $\sum_1^r c_m P_{\mu_m}$ with $\mu_m \in S$ such that

$$\left\|f-h\cdot k^*-\sum_1^r c_m P_{\mu_m}\right\|_{L^1(s)} \leqq \varepsilon \quad \text{and} \quad \sum_1^r |c_m| \leqq \omega. \tag{2.22}$$

According to Lemma 2.3 we can choose an orthonormal sequence $\{a_n\}_1^\infty$ in $\mathfrak{E}_*$, satisfying (2.18) for $\mu=\mu_1$. By Lemma 2.1 we know that $l\cdot(\mu_1\circ a_n)^*\to 0$ in L^1 as $n\to\infty$, for any fixed $l\in\mathfrak{H}(\Theta)$. Therefore we can find b_1, equal to some a_n, such that

$$\|h\cdot(\mu_1\circ b_1)^*\|_{L^1} \leqq \varepsilon, \quad \|k\cdot(\mu_1\circ b_1)^*\|_{L^1} \leqq \varepsilon$$

(indeed every a_n with n large enough does it). Next, again by Lemmas 2.3 and 2.1, we can choose a unit vector b_2 in $\mathfrak{E}_*\ominus(\bigvee b_1)$ such that

$$\|\Theta(\mu_2)^* b_2\|_{\mathfrak{E}} \leqq \eta_\Theta(\mu_2)+\varepsilon \quad \text{and} \quad \|l\cdot(\mu_2\circ b_2)^*\|_{L^1} \leqq \varepsilon \quad \text{for} \quad l=h, k, \mu_1\circ b_1.$$

Continuing, we find step by step an orthonormal sequence $\{b_n\}_1^r$ in $\mathfrak{E}_*$ such that

$$\|\Theta(\mu_m)^* b_m\|_{\mathfrak{E}} \leqq \eta_\Theta(\mu_m)+\varepsilon \quad \text{and} \quad \|l\cdot(\mu_m\circ b_m)^*\|_{L^1} \leqq \varepsilon \tag{2.23}$$

for

$$l=h, k, \mu_n\circ b_n \quad (n<m).$$

Now choose complex numbers d_m, e_m such that $c_m=d_m\bar{e}_m$, $|d_m|=|e_m|=|c_m|^{1/2}$, and set

$$h'=h+\sum_1^r d_m\cdot(\mu_m\circ b_m), \quad k'=k+\sum_1^r e_m\cdot(\mu_m\circ b_m).$$

Inequalities (2.21) are easily verified; it suffices to look at the first one. Indeed, using (2.10) we have

$$\begin{aligned}\|h-h'\| &= \left\|\sum d_m(\mu_m\circ b_m)\right\| = \left\|P_{\mathfrak{H}(\Theta)}\sum d_m(p_{\mu_m}b_m\oplus 0)\right\| \leqq \\ &\leqq \left\|\sum d_m p_{\mu_m} b_m\right\|_{H^2(\mathfrak{E}_*)} = \left(\sum |d_m|^2\right)^{1/2} = \left(\sum |c_m|\right)^{1/2} \leqq \omega^{1/2}.\end{aligned}$$

For the difference $\Omega=f-h'\cdot k'^*$ we begin with the following rearrangement:

$$\begin{aligned}\Omega &= \left(f-h\cdot k^*-\sum c_m P_{\mu_m}\right)-\sum c_m[(\mu_m\circ b_m)\cdot(\mu_m\circ b_m)^*-P_{\mu_m}]- \\ &-\sum d_m(\mu_m\circ b_m)\cdot k^*-\sum \overline{e_m}h\cdot(\mu_m\circ b_m)^*-\sum_{n\neq m}\sum d_m\overline{e_n}(\mu_n\circ b_n)\cdot(\mu_m\circ b_m)^*.\end{aligned}$$

From inequalities (2.22), (2.16), and (2.23) we deduce:

$$\begin{aligned}\|\Omega\|_{L^1(s)} &\leqq \varepsilon+\sum |c_m| 2\|\Theta(\mu_m)^* b_m\|+\sum |d_m|\varepsilon+\sum |e_m|\varepsilon+\sum_{n\neq m}\sum |d_m||e_n|\varepsilon \leqq \\ &\leqq \varepsilon+\omega\cdot 2\vartheta+(\omega^{1/2}r^{1/2}+\omega^{1/2}r^{1/2}+\omega r)\varepsilon,\end{aligned}$$

and this is obviously $\leqq\vartheta'\omega$ if ε was chosen appropriately small; thus (2.19) holds. The proof is done.

In the case $s=C$ a similar result can be obtained even under a milder condition, namely that S be dominant for some $\vartheta\in[0, 1)$ (instead of $\vartheta\in[0, \frac{1}{2})$). However, we get then, for any $\vartheta'\in(\vartheta, 1)$, evaluations in the quotient space L^1/H_0^1 (instead of the space $L^1(C)=L^1$). The method of proof is the same except that we can now refer to part b) of Lemma 1.1 (instead of part a)), and in particular, to the estimate (2.16)$^\bullet$ in Lemma 2.2 (instead of the estimate (2.16)).

Let us formulate the result so obtained, without repeating the details of the proof:

Lemma 2.4$^\bullet$. *Suppose that, for some* $\vartheta\in[0, 1)$, *the set* S_ϑ *is dominant and take a* $\vartheta'\in(\vartheta, 1)$. *Suppose we have*

$$\|f-h\cdot k^*\|_{L^1/H_0^1}\leqq\omega \text{ for some } f \text{ in } L^1, \text{ and } h,\ k \text{ in } \mathfrak{H}(\Theta).$$

Then there exist h' *and* k' *in* $\mathfrak{H}(\Theta)$ *such that*

$$\|f-h'\cdot k'^*\|_{L^1/H_0^1}\leqq\vartheta'\omega,$$

$$\|h-h'\|\leqq\omega^{1/2},\quad \|k-k'\|\leqq\omega^{1/2}.$$

Now we can turn to our main "representation theorems".

Theorem A. *Suppose that, for some* $\vartheta\in[0, \frac{1}{2})$, *the set* S_ϑ *is dominating for some measurable subset s of C, and take* $\vartheta'\in(2\vartheta, 1)$. *For every* $f\in L^1$ *and* $h, k\in\mathfrak{H}(\Theta)$ *there exist* $h', k'\in\mathfrak{H}(\Theta)$ *such that*

$$f=h'\cdot k'^* \quad \text{a.e. on } s, \text{ and}$$

$$\|h-h'\|,\quad \|k-k'\|\leqq(1-\vartheta'^{1/2})^{-1}\|f-h\cdot k^*\|_{L^1(s)}^{1/2}$$

Proof. Repeated application of Lemma 2.4, with $\omega=\|f-h\cdot k^*\|_{L^1(s)}$, shows the existence of sequences $\{h_n\}_0^\infty$, $\{k_n\}_0^\infty$ in $\mathfrak{H}(\Theta)$ such that $h_0=h$, $k_0=k$ and

$$\|f-h_n\cdot k_n^*\|_{L^1(s)}\leqq\vartheta'^n\omega,\quad\text{and}\quad \|h_n-h_{n+1}\|,\ \|k_n-k_{n+1}\|\leqq(\vartheta'^n\omega)^{1/2}\quad(n=0, 1, \ldots).$$

This obviously implies that the limits $h'=\lim_n h_n$, $k'=\lim_n k_n$ exist, satisfy $\|f-h'\cdot k'^*\|_{L^1(s)}=\lim_n\|f-h_n\cdot k_n^*\|_{L^1(s)}=0$, and

$$\|h-h'\|=\left\|\sum_0^\infty(h_n-h_{n+1})\right\|\leqq\sum_0^\infty(\vartheta'^n\omega)^{1/2}=(1-\vartheta'^{1/2})^{-1}\omega^{1/2};$$

similarly for $\|k-k'\|$. The proof is complete.

An almost identical proof, based on Lemma 2.4$^\bullet$, yields:

Theorem A$^\bullet$. *Suppose that, for some* $\vartheta\in[0, 1)$ *the set* S_ϑ *is dominant and take* $\vartheta'\in(\vartheta, 1)$. *Then, for every* $f\in L^1$ *and* $h, k\in\mathfrak{H}(\Theta)$ *there exist* $h', k'\in\mathfrak{H}(\Theta)$ *such that*

$$f\equiv h'\cdot k'^* \mod H_0^1 \quad \text{on } C, \text{ and}$$

$$\|h-h'\|,\quad \|k-k'\|\leqq(1-\vartheta'^{1/2})^{-1}\|f-h\cdot k^*\|_{L^1/H_0^1}^{1/2}.$$

Corollary A. *Under the hypotheses of Theorem* A *the set*

$$Z = \{h \in \mathfrak{H}(\Theta):\ h \cdot k^* = 0 \text{ a.e. on } s \text{ for some nonzero } k \in \mathfrak{H}(\Theta)\}$$

is dense in $\mathfrak{H}(\Theta)$.

Corollary A$^\bullet$. *Under the hypotheses of Theorem* A$^\bullet$ *the set*

$$Z^\bullet = \{h \in \mathfrak{H}(\Theta):\ h \cdot k^* \equiv 0 \bmod H_0^1 \text{ for some nonzero } k \in \mathfrak{H}(\Theta)\}$$

is dense in $\mathfrak{H}(\Theta)$.

Proof. Choose ϑ and ϑ' as required in the respective Theorem. For a fixed $\mu \in S_\vartheta$ choose, as in Lemma 2.3, an orthonormal sequence $\{a_n\}_1^\infty$ such that $\|\Theta(\mu)^* a_n\|_{\mathfrak{E}} \leqq \vartheta'$. Using also (2.12) and (2.13) we have

$$\|\mu \circ a_n\|^2 = \|p_\mu a_n\|^2_{H^2(\mathfrak{E}_*)} - \|p_\mu \Theta(\mu)^* a_n\|^2_{H^2(\mathfrak{E})} = 1 - \|\Theta(\mu)^* a_n\|^2_{\mathfrak{E}} \geqq 1 - \vartheta',$$

and hence, $\mu \circ a_n \neq 0$. Now apply Theorem A or A$^\bullet$, respectively, with $f=0$, $k=\mu\circ a_n$, and an arbitrarily chosen $h \in \mathfrak{H}(\Theta)$. We infer the existence of sequences $\{h_n'\}$, $\{k_n'\}$ in $\mathfrak{H}(\Theta)$ such that

$$h_n' \cdot k_n'^* = 0 \quad \text{a.e. on } s, \text{ or } \quad h_n' \cdot k_n'^* \equiv 0 \bmod H_0^1 \text{ on } C,$$

respectively, and moreover,

$$\|h - h_n'\|, \quad \|k - k_n'\| \leqq (1 - \vartheta'^{1/2})^{-1} \|h \cdot (\mu \circ a_n)^*\|^{1/2}_{L^1(s) \text{ or } L^1/H_0^1}.$$

By Lemma 2.1, $\|h \cdot (\mu \circ a_n)^*\|_{L^1} \to 0$ as $n \to \infty$, which implies the same in the metrics of $L^1(s)$ and L^1/H_0^1 as well. This concludes the proof of both corollaries.

*

The first interest of these corollaries lies in their implication to the existence of non-cyclic vectors for the "model" operator $S(\Theta)$ defined on $\mathfrak{H}(\Theta)$ by $S(\Theta)h = P_{\mathfrak{H}(\Theta)}(e^{it}h)$, cf. (2.2).

Indeed, if the set S_ϑ is dominant for some $\vartheta \in [0, 1)$, then no vector $h \in Z^\bullet$ is cyclic for $S(\Theta)$, because if k is a nonzero vector in $\mathfrak{H}(\Theta)$ such that $h \cdot k^* \equiv 0 \bmod H_0^1$, then

$$\begin{aligned}(2.24)\qquad (S(\Theta)^n h, k) = (e^{int}h, k) &= \int e^{int}(h(e^{it}), k(e^{it}))_{\mathfrak{E}_* \oplus \mathfrak{E}}\, dm = \\ &= \int e^{int}(h \cdot k^*)(e^{it})\, dm = 0 \quad \text{for } n = 0, 1, \ldots.\end{aligned}$$

In case S_ϑ is dominant even for some $\vartheta \in [0, \frac{1}{2})$ then we have for every $h \in Z$ and a corresponding $k \neq 0$ such that $h \cdot k^* = 0$ a.e. on C, besides (2.24) also

$$\begin{aligned}(2.25)\qquad (S(\Theta)^{*n} h, k) = (h, S(\Theta)^n k) &= \int e^{-int}(h(e^{it}), k(e^{it}))_{\mathfrak{E}_* \oplus \mathfrak{E}}\, dm = \\ &= \int e^{-int}(h \cdot k^*)(e^{it})\, dm = 0 \quad \text{for } n = 0, 1, \ldots.\end{aligned}$$

Thus in the case $\vartheta\in[0, 1)$ the nonzero vector k is orthogonal to $\bigvee_0^\infty S(\Theta)^n h$, while in the case $\vartheta\in[0, \frac{1}{2})$, k is orthogonal both to $\bigvee_0^\infty S(\Theta)^n h$ and to $\bigvee_0^\infty S(\Theta)^{*n} h$.

Remark. The chains of equations (2.24) and (2.25) clearly hold, with the exception of the last members ("$=0$"), irrespective of any assumption on the set S_ϑ, and for any $h, k\in\mathfrak{H}(\Theta)$. They show that the function $h\cdot k^*\in L^1$ has the Fourier series $\sum c_n e^{int}$, with $c_n=(S(\Theta)^n h, k)$ and $c_{-n}=(S(\Theta)^{*n}h, k)$ for $n=0, 1, \ldots$. Note that this representation frees the definition of the product $h\cdot k^*$ from the model operator. For any (CNU) contraction on a Hilbert space $\mathfrak{H}$ we can define $h\cdot k^*$ $(h, k\in\mathfrak{H})$ as the function in L^1 with the Fourier series $\sum c_n e^{int}$ with $c_n=(T^n h, k)$ and $c_{-n}=(T^{*n}h, k)$ $(n=0, 1, \ldots)$; and this definition is clearly unitarily invariant.

3. Invariant subspaces

a) Let us formulate the above consequences of Corollaries A and A* in terms of a contraction operator T on the Hilbert space $\mathfrak{H}$, by using the model operator $S(\Theta_T)$, where $\{\mathfrak{D}_T, \mathfrak{D}_{T*}, \Theta_T(\lambda)\}$ is the characteristic function associated with T. As recalled in the Preliminaries of Section 2, $S(\Theta_T)$ is unitarily equivalent to T if T is (CNU); in the general case it is unitarily equivalent to the (CNU) part of T. As the unitary part (if any) of T does not effect Θ_T and the argumentations at the end of the first paragraph of section 2, we may disregard the assumption $T\in$(CNU).

Set, for $\vartheta\in[0, 1)$, in analogy to (2.19),

$$R_\vartheta = \{\mu\in D:\ \inf \sigma_e[(T_\mu T_\mu^*)^{1/2}] \leqq \vartheta\}. \tag{3.1}$$

Proposition 3.1. *Let T be a contraction acting on $\mathfrak{H}$. If R_ϑ is dominant for some $\vartheta<1$ then T has nontrivial invariant subspaces. Moreover, the set of non-cyclic vectors for T are dense in $\mathfrak{H}$.*

Proposition 3.2. *Let T be a contraction acting on $\mathfrak{H}$. If R_ϑ is dominant for some $\vartheta<\frac{1}{2}$ then the set of vectors $h\in\mathfrak{H}$ for which*

$$\bigvee_{n=0}^{\infty} \{T^n h, T^{*n} h\} \neq \mathfrak{H},$$

is dense in $\mathfrak{H}$.

Remark. The condition that the defect space $\mathfrak{D}_{T*}$ be infinite dimensional, is implicitly contained in the hypothesis that R_ϑ is dominant for some $\vartheta<1$, and hence non-void.

b) As a further application of Theorem A* we prove

Proposition 3.3. *Under the condition for T that the set R_ϑ is dominant for some $\vartheta<1$, there exists, for every inner function φ, a semi-invariant subspace $\mathfrak{L}$ for T such that the compression $T_\mathfrak{L}$ of T to $\mathfrak{L}$ be a C_0-class contraction with minimal function $m_\mathfrak{L}$ equal to φ, and with a cyclic vector; as a consequence $T_\mathfrak{L}$ has the Jordan model $S(\varphi)$.*

Proof. It sufficies to consider the model operator $T=S(\Theta)$; the assumption is then that the corresponding set S_ϑ (cf. (2.19)) be dominant for some $\vartheta<1$.

By Theorem A* this implies that there exist $h, k\in\mathfrak{H}(\Theta)$ such that

$$\bar{\varphi} \equiv h\cdot k^* \bmod H_0^1. \tag{3.2}$$

Consider the cyclic subspaces

$$\mathfrak{H}_1 = \bigvee_{n=0}^{\infty} T^n h \quad \text{and} \quad \mathfrak{H}_2 = \bigvee_{n=0}^{\infty} T^n \varphi(T) h \quad (= \varphi(T)\mathfrak{H}_1)^-$$

for T; clearly, $\mathfrak{H}_1\supset\mathfrak{H}_2$. Hence $\mathfrak{L}=\mathfrak{H}_1\ominus\mathfrak{H}_2$ is semi-invariant for T and the compression $T_\mathfrak{L}=P_\mathfrak{L}T|\mathfrak{L}$ (where $P_\mathfrak{L}$ denotes orthogonal projection from $\mathfrak{H}_1$ onto $\mathfrak{L}$) satisfies

$$v(T_\mathfrak{L}) = P_\mathfrak{L} v(T)|\mathfrak{L} \quad \text{for every} \quad v\in H^\infty. \tag{3.3}$$

So we have, in particular,

$$\varphi(T_\mathfrak{L})\mathfrak{L} = P_\mathfrak{L}\varphi(T)\mathfrak{L} \subset P_\mathfrak{L}\varphi(T)\mathfrak{H}_1 \subset P_\mathfrak{L}\mathfrak{H}_2 = \{0\}, \quad \varphi(T_\mathfrak{L}) = 0.$$

Hence, $T_\mathfrak{L}$ is of class C_0 and its minimal (inner) function $m_\mathfrak{L}$ is a divisor of φ: $\varphi=qm_\mathfrak{L}$, q inner. Thus, by (3.2), $\bar{q}=m_\mathfrak{L}\bar{\varphi}\equiv m_\mathfrak{L}\cdot(h\cdot k^*) \bmod H_0^1$, and hence, for every $v\in H^\infty$,

$$\int_0^{2\pi} v(e^{it})\overline{q(e^{it})}\, dm = \int_0^{2\pi} v(e^{it}) m_\mathfrak{L}(e^{it})(h\cdot k^*)(e^{it})\, dm = (v(T)m_\mathfrak{L}(T)h, k). \tag{3.4}$$

Next observe that, for any $v\in H^\infty$, we have

$$v(T)(h-P_\mathfrak{L}h)\in v(T)\mathfrak{H}_2 \subset \mathfrak{H}_2, \quad P_\mathfrak{L}v(T)(h-P_\mathfrak{L}h)\in P_\mathfrak{L}\mathfrak{H}_2 = \{0\},$$

and hence, by (3.3),

$$P_\mathfrak{L}v(T)h = P_\mathfrak{L}v(T)P_\mathfrak{L}h = v(T_\mathfrak{L})P_\mathfrak{L}h. \tag{3.5}$$

For $v=m_\mathfrak{L}$ this yields $P_\mathfrak{L}m_\mathfrak{L}(T)h=0$, and this in turn gives that $m_\mathfrak{L}(T)h\in\mathfrak{H}_2$. Therefore, there exists a sequence $\{p_j\}$ of polynomials such that $m_\mathfrak{L}(T)h= = \lim_{j\to\infty} p_j(T)\varphi(T)h$. Recalling (3.2) we obtain

$$(v(T)m_\mathfrak{L}(T)h, k) = \lim_{j\to\infty}((vp_j\varphi)(T)h, h) = \lim_{j\to\infty}\int vp_j\varphi\cdot(h\cdot k^*)\, dm = \\ = \lim_{j\to\infty}\int vp_j\, dm = 0$$

for every $v \in H_0^\infty$. In particular, take $v = q - q(0)$. Comparing with (3.4) we conclude that $\int (1 - q(0)\bar{q})\, dm = \int (q - q(0))\bar{q}\, dm = 0$, $|q(0)|^2 = 1$, and hence q is a constant, i.e., φ coincides with $m_{\mathfrak{L}}$.

It only remains to show that $T_{\mathfrak{L}}$ has a cyclic vector. Indeed $h_{\mathfrak{L}} = P_{\mathfrak{L}} h$ is such, because (3.5) implies for $v(\lambda) = \lambda^n$ $(n = 0, 1, \ldots)$.

$$\bigvee_{n=0}^{\infty} T_{\mathfrak{L}}^n h_{\mathfrak{L}} = \bigvee_{n=0}^{\infty} P_{\mathfrak{L}} T^n h = P_{\mathfrak{L}} \mathfrak{H}_1 = \mathfrak{L}.$$

This concludes the proof.

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(H. B. and C. M. P.)
UNIVERSITY OF MICHIGAN
ANN ARBOR, MI 48 109, USA

(C. F.)
INDIANA UNIVERSITY
BLOOMINGTON, IN 47401, USA
and
UNIVERSITÉ DE PARIS-SUD
91405 ORSAY, FRANCE

(B. SZ.-N.)
BOLYAI INSTITUTE,
UNIVERSITY SZEGED
6720 SZEGED, HUNGARY