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The algebraic representation of semigroups and lattices; representing lattice extensions

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Introduction

A monoid S and lattice L are *jointly algebraic* if there is a universal algebra $\mathfrak{A}$ with $S\cong\operatorname{End}\mathfrak{A}$ and $L\cong\operatorname{Su}\mathfrak{A}$. For S and L jointly algebraic *submonoids* of S which are also jointly algebraic with L are studied in [3]. Here we consider certain *lattice extensions* of L which are also jointly algebraic with S. Concrete representations are again used to derive abstract results.

1. Concrete representations

As in [3] we say a partial unary algebra $\mathfrak{B}=\langle B;f\rangle_{f\in S\cup L}$ *represents* S (a monoid) and L (a compactly generated lattice) on B provided: (i) the operations $f\in S$ form a transformation monoid on B with $fg(b)=f(g(b))$ and $\operatorname{id}(b)=b$, for $f,g\in S$, $b\in B$ and, (ii) the operations $p,q\in L$ are partial identity maps on B with range $p\cap$ $\cap$range $q=$range $p\wedge q$, and the map denoted by $1\in L$ is the total identity map on B. The representation is *faithful* if for any $f,g\in S$ with $f\neq g$ there is a $b\in B$ with $f(b)\neq g(b)$, and for any $p,q\in L$, $p\neq q$, we have range $p\neq$range q. We use $\mathfrak{B}^n$ to denote the usual n-fold direct power of $\mathfrak{B}$.

We shall use systems of equations, Σ, of the form $fx=g$, with coefficients $f,g\in S\cup L$, as defined in [2]. Spt Σ is the *support* of Σ, i.e. the set of points on which Σ has a solution (cf. [2]). Observe that for a homomorphism $\alpha\colon\mathfrak{A}\to\mathfrak{B}$ between partial unary algebras each of which faithfully represents S and L we have that $a\in\operatorname{Spt}\Sigma$ on $\mathfrak{A}$ implies $\alpha(a)\in\operatorname{Spt}\Sigma$ on $\mathfrak{B}$.

Let $\mathfrak{B}$ be a faithful representation of S and L on B.

Definition 1.1. For $C\subseteq B$ the rank of C in $\mathfrak{B}$ is $R(C)=\bigwedge\{f\mid f\in L, \operatorname{id}\restriction C\subseteq f\}$.

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The *rank function* R maps subsets of B into the lattice L. For convenience we denote $R(\{b\})$ by $R(b)$, for $b \in B$. For a sequence $\mathbf{D} \in B^n$ we use the lattice join to define the rank of $\mathbf{D}$ by

$$R(\mathbf{D}) = \bigvee_{i=1}^{n} R(\mathbf{D}_i).$$

Note that for D finite, $D \subseteqq B$, $R(D) = R(\mathbf{D})$ for any $\mathbf{D} \in B^n$ with range $\mathbf{D} = D$.

We shall need a form of the concrete representation theorem for endomorphisms and subalgebras found in [2]. The letter n will denote a positive integer, and $D \subset_f B$ will abbreviate "D is a finite subset of B".

Definition 1.2. We say Statement 3 holds for $\mathfrak{B}$, or more briefly $\mathrm{St}_3\, B$ provided given any $b \in B$ and $\mathbf{D} \in B^n$ with $R(\mathbf{D}) \geqq R(b)$ there is a homomorphism $\alpha\colon \mathfrak{B}^n \to \mathfrak{B}$ with $\alpha(\mathbf{D}) = b$.

Recall from [3] that we write (B, S, L) as a triple to denote a faithful representation of S as a transformation monoid on B and of L as an intersection structure on B. Clearly $\mathfrak{B}$ is a faithful representation of S and L on B if and only if $(B, S, \{f(B) \mid f \in L\})$ holds. The work in [2] made use of the following *Statement* 2 concerning (B, S, L): we say $\mathrm{St}_2\,(B, S, L)$ holds provided

$$\forall C \subseteqq B\Big[C = \bigcup_{D \subset_f C}\ \bigcap_{D \subset \mathrm{Spt}\,\Sigma} \mathrm{Spt}\,\Sigma \Rightarrow C \in L\Big].$$

Clearly $\mathrm{St}_2\,(B, S, \{f(B) \mid f \in L\})$ is equivalent to Statement 2′ concerning $\mathfrak{B}$, viz

$$\mathrm{St}_2'\,\mathfrak{B}\colon \forall C \subseteqq B\Big[C = \bigcup_{D \subset_f C}\ \bigcap_{D \subseteqq \mathrm{Spt}\,\Sigma} \mathrm{Spt}\,\Sigma \Rightarrow \mathrm{id}\restriction C \in L\Big]$$

by virtue of the natural correspondence between subsets of B and their respective partial identities. The form of the representation theorem we need follows from:

Theorem 1.1. $\mathrm{St}_3\,\mathfrak{B} \Leftrightarrow \mathrm{St}_2'\,\mathfrak{B}$.

Proof. Assume $\mathrm{St}_3\,\mathfrak{B}$ holds for $\mathfrak{B}$ and let C satisfy the hypotheses of $\mathrm{St}_2'\,\mathfrak{B}$. Note $\mathrm{id}\restriction C \in L$ iff $\mathrm{id}\restriction C = \bigvee_{D \subset_f C} R(D)$ iff $C = \bigcup_{D \subset_f C} \mathrm{range}\, R(D)$. Thus to show $\mathrm{id}\restriction C \in L$ it suffices to prove that $\mathrm{range}\, R(D) = \bigcap_{D \subseteqq \mathrm{Spt}\,\Sigma} \mathrm{Spt}\,\Sigma$ for D finite. We have $\mathrm{range}\; R(D) = \mathrm{range} \bigwedge \{Q \mid Q \in L,\ \mathrm{id}\restriction D \subseteqq Q\} = \bigcap_{\mathrm{id}\restriction D \subseteqq Q \in L} \mathrm{range}\, Q$ and for $Q \in L$, $D \subseteqq \mathrm{range}\, Q = \mathrm{dom}\, Q = \mathrm{Spt}\,\{Qx^{\Sigma} = Q\}$ and hence $\mathrm{range}\, R(D) \supseteqq \bigcap_{D \subseteqq \mathrm{Spt}\,\Sigma} \mathrm{Spt}\,\Sigma$. To show the opposite inclusion, fix $b \in \mathrm{range}\, R(D)$ and let $\mathbf{D} \in B^n$ with $\mathrm{range}\,\mathbf{D} = D$. Since $b \in \mathrm{range}\, R(D)$ we have $R(b) \leqq R(\mathbf{D})$, thus by $\mathrm{St}_3\,\mathfrak{B}$ there is a homomorphism $\alpha\colon \mathfrak{B}^n \to \mathfrak{B}$ with $\alpha(\mathbf{D}) = b$. Now $D \subseteqq \mathrm{Spt}\,\Sigma$ on $\mathfrak{B}$ implies $\mathbf{D} \in \mathrm{Spt}\,\Sigma$ on $\mathfrak{B}^n$ and applying α we have $\alpha(\mathbf{D}) = b \in \mathrm{Spt}\,\Sigma$ on $\mathfrak{B}$. Thus $D \subseteqq \mathrm{Spt}\,\Sigma \Rightarrow b \in \mathrm{Spt}\,\Sigma$, and $\mathrm{range}\, R(D) \subseteqq \bigcap_{D \subseteqq \mathrm{Spt}\,\Sigma} \mathrm{Spt}\,\Sigma$.

Conversely assume $\mathrm{St}_2' \mathfrak{B}$. Let $b \in B$, $\mathbf{D} \in B^n$ with $R(b) \leqq R(\mathbf{D})$; we show for any system of equations Σ over S and L that $\mathbf{D} \in \mathrm{Spt}\,\Sigma$ on $B^n \Rightarrow b \in \mathrm{Spt}\,\Sigma$ on B. To see this observe that St_2 says for $\bar{C} = \bigcup_{D \subset_f C} \bigcap_{D \subseteq \mathrm{Spt}\,\Sigma} \mathrm{Spt}\,\Sigma$, that $\bar{C} = C$ implies $C \in \{f(B) | f \in L\}$. Also $\bar{\bar{C}} = \bar{C}$ since the indicated bar operation is a closure operator (cf. Lemma 5 of [2]). Thus $\bar{C} \in \{f(B) | f \in L\}$ and hence $\mathrm{id} \restriction \bar{C} \in L$. Now since $R(b) \leqq R(D)$, where $D = \mathrm{range}\,\mathbf{D}$, we have $b \in \mathrm{range}\, R(D) = \bigcap_{\mathrm{id} \restriction D \subseteq Q \in L} \mathrm{range}\, Q$. But $\mathrm{id} \restriction D \subseteq \mathrm{id} \restriction \bar{D} \in L$, therefore $b \in \mathrm{range}\, \mathrm{id} \restriction \bar{D}$, i.e. $b \in \bar{D} = \bigcup_{E \subset_f D} \bigcap_{D \subseteq \mathrm{Spt}\,\Sigma} \mathrm{Spt}\,\Sigma = \bigcup_{D \subseteq \mathrm{Spt}\,\Sigma} \mathrm{Spt}\,\Sigma$ since D is finite. Thus $b \in \mathrm{Spt}\,\Sigma$ whenever $D \subseteq \mathrm{Spt}\,\Sigma$ and the assertion $[\mathbf{D} \in \mathrm{Spt}\,\Sigma$ on $B^n \Rightarrow b \in \mathrm{Spt}\,\Sigma$ on $B]$ follows. To show St_3 holds we obtain the required homomorphism as follows: consider the system of equations whose variables are indexed by B^n, and let $f x_{\mathbf{h}} = x_{\mathbf{k}} \in \Sigma$ iff $f(\mathbf{h}) = \mathbf{k}$, where $f \in S \cup L$. Thus Σ is the full diagram of S and L on $\mathfrak{B}^n$. Let $\Gamma = \Sigma \cup \{\mathrm{id}\, x_{\mathbf{D}} = \mathrm{id}\}$. Choose $\boldsymbol{\beta}$, an assignment of the variables of Σ to be $\boldsymbol{\beta}(x_{\mathbf{e}}) = \mathbf{e}$, i.e. every variable is assigned to a constant map. Clearly $\boldsymbol{\beta}$ satisfies Γ at $\mathbf{D}$, hence $\mathbf{D} \in \mathrm{Spt}\,\Gamma$ on $\mathfrak{B}^n$. Then by the above argument $b \in \mathrm{Spt}\,\Gamma$ on $\mathfrak{B}$. Let $\hat{\beta}$ be an assignment which satisfies Σ on $\mathfrak{B}$. Then $\hat{\beta}(x_{\mathbf{D}})(b) = b$ since $\mathrm{id}\, x_{\mathbf{D}} = \mathrm{id} \in \Gamma$. Let $\alpha\colon \mathfrak{B}^n \to \mathfrak{B}$ be given by $\alpha(\mathbf{e}) = \hat{\beta}(x_{\mathbf{e}})(b)$, thus $\alpha(\mathbf{D}) = b$. It is easy to verify that α is a homomorphism. □

Corollary 1.1. *For a monoid S and a compactly generated lattice L, S and L are jointly algebraic iff there is a faithful representation $\mathfrak{B} = \langle B, f \rangle_{f \in S \cup L}$ in which S is locally closed, and each compact $t \in L$ is singly generated* (viz $t = \bigwedge_{\alpha \in p \in L} p$ *for some* $\alpha \in B$) *and $\mathfrak{B}$ satisfies the mapping condition* St_3

Proof. Let S, L be jointly algebraic. By Theorem 2 of [3] there is an algebra $\mathscr{L}$ with each compact subalgebra singleton generated and $\mathrm{End}\,\mathscr{L} \cong S$, $\mathrm{Su}\,\mathscr{L} \cong L$. Thus there is a representation of the required sort. Conversely if $\mathfrak{B}$ is a faithful representation of S and L satisfying the three conditions above, observe using the *proof* of Theorem 1 of [3] that the representation on the foliation $(\mathfrak{F}(B)$, l.c. $S, L)$ is algebraic. We need for that proof, besides the explicitly given conditions, only the fact that $\mathrm{St}_2\, \mathfrak{B}$ holds; but from Theorem 1 above we have $\mathrm{St}_3\, \mathfrak{B} \Rightarrow \mathrm{St}_2\, \mathfrak{B}$. Hence our hypothesis regarding the mapping condition can be used to replace the (stronger) assumption in the earlier paper that $\mathfrak{B}$ itself was algebraic. Finally S is locally closed in $\mathfrak{F}(\mathfrak{B})$ whenever S is locally closed in $\mathfrak{B}$ (see [3] for $\mathfrak{F}(\mathfrak{B})$, the foliation of $\mathfrak{B}$). Hence $(\mathfrak{F}(B), S, L)$ is itself algebraic, and S, L are jointly algebraic. □

Note that the representation $\mathfrak{B}$ itself need *not* be a concrete realization of S and L as $\mathrm{End}\,\mathfrak{A}$ and $\mathrm{Su}\,\mathfrak{A}$ for any algebra $\mathfrak{A}$: the assertion merely guarantees the existence of some such representation.

2. Algebraic lattice extensions

If H is a lattice we denote by H_k the compact elements of H. An ideal $J \subseteq H$ is *compactly embedded* in H provided the map $\pi\colon H \to J$ given by $\pi x = \bigvee_{\substack{j \in J \\ j \leq x}} j$ preserves joins and compactness.

Theorem 2.1. *If S and L are jointly algebraic and $L \cong J$ for some ideal $J \subseteq H$ which is compactly embedded in H, then S and H are jointly algebraic.*

Proof. We may assume that $S = \operatorname{End} \mathfrak{A}$ and $L = J = \operatorname{Su} \mathfrak{A}$ for some algebra $\mathfrak{A} = \langle A; P \rangle$, and further that each $p \in L_K$ is singleton generated (see [3]). Let $\hat{\mathfrak{A}}$ be the partial unary algebra $\hat{\mathfrak{A}} = \langle A; f \rangle_{f \in S \cup L}$ of the faithful algebraic representation (A, S, L). For each $p \in L_K$ fix $p^* \in A$ so that the subalgebra of $\mathfrak{A}$ generated by p^*, $[p^*] = p$. We represent S and H (faithfully) on the disjoint union $A \mathbin{\dot{\cup}} H_K$ and verify that the representation is locally closed and satisfies St_3 and each compact $t \in H$ is singly generated. From Corollary 1.1 it follows that S and H are jointly algebraic.

Definition 2.1. Let $B = A \mathbin{\dot{\cup}} H_K$ and let r map B to H as follows: for $b \in B$

$$r(b) = \begin{cases} b & (b \in H_k) \\ \bigwedge_{\substack{a \in p \\ p \in L}} p & (b = a \in A). \end{cases}$$

Further define for $q \in H$, $B_q = \{b \in B \mid r(b) \leq q\}$ and for $f \in S$, $f \neq \mathrm{id}$ let

$$f(x) = \begin{cases} f(x) & (x \in A) \\ f((\pi x)^*) & (x \in H_k). \end{cases}$$

Lemma 2.1. *The partial unary algebra $\mathfrak{B} = \langle B; f \rangle_{f \in S \cup H}$ corresponding to $(B, S, \{B_q \mid q \in H\})$ for $q \in H$ and $f \in S$, as given in Definition 2.1, is a faithful representation of S and H and each compact $t \in H$ is singly generated.*

Proof. Immediate. □

Lemma 2.2. *The function $r(b)$ of Definition 2.1 assigns to each $b \in B$ the rank $\{b\}$ in the representation $\mathfrak{B}$, i.e. $r(b) = R(b)$.*

Proof. Easy. □

In the following lemma $[A]^{\mathfrak{B}}$ is the *subalgebra of $\mathfrak{B}$ generated by A.*

Lemma 2.3. *The map $\varepsilon\colon \mathfrak{B}^n \to ([A]^{\mathfrak{B}})^n$ defined by $(\varepsilon \mathbf{D})_i = \varepsilon(\mathbf{D}_i)$ where $\varepsilon(x) = \begin{cases} x & (x \in A) \\ (\pi x)^* & (x \in H_k) \end{cases}$ is a homomorphism.*

Proof. Clearly $R(\varepsilon\mathbf{D}_i)\leqq R(\mathbf{D}_i)$ and thus $R(\varepsilon\mathbf{D})\leqq R(\mathbf{D})$, so ε preserves partial identity maps. Furthermore if $f\in S$, $f\neq\mathrm{id}$ and $\mathbf{D}=(p_1, \ldots, p_r, a_1, \ldots, a_t)$ then $f(\varepsilon(p_1, \ldots, a_t))=f(\varepsilon p_1, \ldots, a_t)=(f(\varepsilon p_1)^*, \ldots, f(a_t))=(f(p_1), \ldots, f(a_t))=\varepsilon f(p_1, \ldots, a_t)$. Hence ε is substitutive over f. Clearly ε is substitutive over $f=\mathrm{id}$, hence ε is a homomorphism. □

Lemma 2.4. *If $b\in B\cap A$ and $\mathbf{D}\in B^n$ with $R(\mathbf{D})\geqq R(b)$ then there is a homomorphism Γ: $\mathfrak{B}^n\to\mathfrak{B}$ with $\Gamma(\mathbf{D})=b$.*

Proof. First we prove that for $\mathbf{D}=(p_1, \ldots, p_r, a_1, \ldots, a_t)$

$$[R(\mathbf{D})\geqq R(b)\Rightarrow R(\varepsilon\mathbf{D})\geqq R(b)].$$

To see this note $R(\mathbf{D})\geqq R(b)\Rightarrow\pi R(\mathbf{D})\geqq R(b)$ since π is fixed on J, and π join preserving implies $\pi R(\mathbf{D})\geqq\pi R(b)=R(b)$. But $\pi R(\mathbf{D})=R(\varepsilon\mathbf{D})$, as follows: $R(\varepsilon\mathbf{D})=$ $=R(\varepsilon p_1, \ldots, \varepsilon p_r, \varepsilon a_1, \ldots, \varepsilon a_t)=R(\pi p_1^*, \ldots, \pi p_r^*, a_1, \ldots, a_t)=\left(\bigvee_{i=1}^{r} R(\pi p_1^*)\right)\vee\left(\bigvee_{i=1}^{t} R(a_i)\right)=$ $=\left(\bigvee_{i=1}^{r}\pi p_i\right)\vee\left(\bigvee_{i=1}^{t}\pi R(a_i)\right)=\pi\left\{\left(\bigvee_{i=1}^{r} p_i\right)\vee\left(\bigvee_{i=1}^{t} R(a_i)\right)\right\}=\pi R(\mathbf{D})$. Hence

$$[R(\mathbf{D})\geqq R(b)\Rightarrow R(\varepsilon\mathbf{D})=\pi R(\mathbf{D})\geqq R(b)].$$

To complete the proof of Lemma 2.4 we use the fact that $\hat{\mathfrak{A}}$ is jointly algebraic concrete representation of S and L and hence satisfies $\mathrm{St}_2\,\hat{\mathfrak{A}}$ (cf. Theorem 3 of [2]), and thus by Theorem 1.1 $\hat{\mathfrak{A}}$ satisfies $\mathrm{St}_3\,\hat{\mathfrak{A}}$. So there is a homomorphism γ: $\hat{\mathfrak{A}}^n\to\hat{\mathfrak{A}}$ with $\gamma(\varepsilon\mathbf{D})=b$. Note the map γ is in fact a homomorphism γ: $([A]^{\mathfrak{B}^n})\to[A]^{\mathfrak{B}}$ since clearly $A\in\mathrm{Su}\,\mathfrak{B}$ and γ admits each $f\in S\cup L$; moreover $\forall a\in A\; R(a)\in L$ thus $R(\gamma(a))\leqq R(a)$ so γ admits partial identities $f\in H-L$ as well.

Finally let $\Gamma=\gamma\circ\varepsilon$. Clearly Γ has the required properties, and this completes the proof of Lemma 2.4. □

Lemma 2.5. *If $b\in B-A$ and $R(\mathbf{D})\geqq R(b)$ then there is a homomorphism ν: $\mathfrak{B}^n\to\mathfrak{B}$ with $\nu(\mathbf{D})=b$.*

Proof. Let $R(\mathbf{D})\geqq R(\mathbf{b})$ for some $\mathbf{D}\in B^n$ and some $b\in B-A$. We may assume $\mathbf{D}\notin A^n$ (since $\mathbf{D}\in A^n\Rightarrow R(\mathbf{D})\in J$ and thus $R(b)\in J$ (J is an ideal of H), in which case $b\in A$). Thus $\mathbf{D}\neq f\mathbf{E}$ for any $\mathbf{E}\in B^n$ (unless $\mathbf{E}=\mathbf{D}$ H), and $f=\mathrm{id}$). Observe that $R(\mathbf{D})\geqq R(b)=r(b)=b\geqq\pi b=R((\pi b)^*)$, so by Lemma 2.4 there is a homomorphism Γ: $\mathfrak{B}^n\to\mathfrak{B}$ with $\Gamma(\mathbf{D})=(\pi b)^*$. Define ν: $\mathfrak{B}^n\to\mathfrak{B}$ as follows for $\mathbf{E}\in B^n$: $\nu(\mathbf{E})=\begin{cases}\Gamma(\mathbf{E}) & (\mathbf{E}\neq\mathbf{D})\\ b & (\mathbf{E}=\mathbf{D})\end{cases}$. Clearly $R(\nu\mathbf{E})\leqq R(\mathbf{E})$, so ν preserves the partial identity operations $\mathrm{id}\restriction B_q$. To see that ν is a homomorphism it remains only to check that $\nu(f\mathbf{E})=f(\nu\mathbf{E})$ for $f\in S$. This is clearly so for $f=\mathrm{id}$;

so assume $f \neq \mathrm{id}$. Then

$$v(f\mathbf{E}) = \Gamma(f\mathbf{E}) = f(\Gamma\mathbf{E}) = \begin{cases} f(v\mathbf{E}) & \text{if } \mathbf{E} \neq \mathbf{D} \\ f((\pi b)^*) & \text{if } \mathbf{E} = \mathbf{D} \end{cases} = \begin{cases} f(v\mathbf{E}) & \text{if } \mathbf{E} \neq \mathbf{D} \\ f(b) & \text{if } \mathbf{E} = \mathbf{D} \end{cases} = f(v\mathbf{E}). \square$$

Combining Lemmas 2.4 and 2.5 we see that St_3 holds for $\mathfrak{B}$. It remains to show that the representation of S is locally closed.

Let h be in the local closure of S as represented in $\mathfrak{B}$. Since the representation of S in $\mathfrak{A}$ is algebraic it must be locally closed, and hence $h \upharpoonright A = f \upharpoonright A$ for some $f \in S$. Now suppose $b \in H_K$. $h \upharpoonright \{b, (\pi b)^*\} = g \upharpoonright \{b, (\pi b)^*\}$ for some $g \in S$ and thus $h(b) = g(b) = g((\pi b)^*) = h((\pi b)^*) = f((\pi b)^*) = f(b)$. Consequently $h = f$, and S is locally closed, which completes the proof of Theorem 2.1. $\square$

3. Representation of ordinal sums

Given two lattices L, T we identify the 0 of L with the 1 of T to obtain a new lattice $T \dotplus L$ the *ordinal sum* of T and L, in the usual way. Thus the new lattice has as elements $T \cup L$ with the identification $\{0_L\} = \{1_T\}$ and the ordering given by $t \leqq l \ \forall t \in T, \forall l \in L$ and $t_1 \leqq t_2 (l_1 \leqq l_2)$ iff $t_1 \leqq t_2$ in T ($l_1 \leqq l_2$ in L).

Corollary 3.1. *If S and L are jointly algebraic and the 1 in L is compact (or in particular if L is finite), then S and $L \dotplus T$ are jointly algebraic, for any compactly generated lattice T.*

Proof. In Theorem 2.1 let $J = L \subseteqq L \dotplus T$. $\square$

Corollary 3.1 says roughly that one can add on above an algebraic lattice. The following theorem will allow us to add on below as well.

Theorem 3.2. *If S and L are jointly algebraic then S and $T \dotplus L$ are also jointly algebraic for any compactly generated lattice T.*

Proof. We may assume that $S = \mathrm{End}\,\mathfrak{A}$, $L = \mathrm{Su}\,\mathfrak{A}$ for some algebra $\mathfrak{A} = \langle A; \mathfrak{P} \rangle$ where the minimal subalgebra of $\mathrm{Su}\,\mathfrak{A}$ is non-empty (otherwise we may use the following argument (due to M. Gould). Let $B = A \cup \{a, b\}$, and let Q be the unary operation defined by $Q(x) = x$ for $x \in A$, $Q(a) = b$, $Q(b) = a$. For $P \in \mathfrak{P}$ and x with range $x \cap \{a, b\} \neq \emptyset$ let $P(x) = \ddot{a}$. Clearly $\mathrm{End}\,\mathfrak{A} \cong \mathrm{End}\,\mathfrak{B}$, $\mathrm{Su}\,\mathfrak{A} \cong \mathrm{Su}\,\mathfrak{B}$, where $\mathfrak{B} = \langle B, \mathfrak{P} \cup \{Q, a, b\rangle\}$). We present S and $T \dotplus L$ on the disjoint union $B = A \dot{\cup} T$ and apply Corollary 1.1 to conclude that S and $T \dotplus L$ are jointly algebraic. Let $\mathfrak{B} = \langle B, f \rangle_{f \in S \cup (T \dotplus L)}$ with $f \in S \cup L$ given by $f(a) = \begin{cases} f(a) & (x = a \in A \\ t & (x = t \in T) \end{cases}$

and $f\in T-\{1_T\}$ given by $f(x)=\begin{cases} t & (x=t\leqq f \text{ in } T) \\ \text{undefined} & (x=t\ \ t\not\leqq f) \\ \text{undefined} & (x=a\in A) \end{cases}$ and $f=1_T$ given by $f(x)=\begin{cases} t\ (x=t\in T) \\ x\ (x\in[\emptyset)^{\mathfrak{A}}) \\ \text{undefined otherwise} \end{cases}$. It is routine to verify that $\mathfrak{B}$ is a faithful representation of S and $T\dotplus L$ on B. We shall show $\mathrm{St}_3\,\mathfrak{B}$ holds. First let $b\in T$ and $D\in B^n$ with $R(D)\geqq R(b)$. Define $\gamma\colon B^n\to B$ as follows for $\mathbf{E}\in B^n$.

$$\gamma(\mathbf{E})=b \quad \text{if} \quad R(\mathbf{E})\geqq R(b), \quad \gamma(\mathbf{E})=R(\mathbf{E}) \quad \text{if} \quad R(\mathbf{E})\not\geqq R(b).$$

Clearly γ preserves $f\in T\dotplus L$. Furthermore for $f\in S$ since $\gamma(\mathbf{E})\in T$ we have $f\gamma(\mathbf{E})=$ $=\gamma(\mathbf{E})=\gamma f(\mathbf{E})$ (if $\mathbf{E}\in T^n$ we have $f(\mathbf{E})=\mathbf{E}$, if $\mathbf{E}\notin T^n$, say $\mathbf{E}_i\in A$ then $R(\mathbf{E})\geqq$ $\geqq R(b)$ so $\gamma(\mathbf{E})=b$ and $\gamma(f\mathbf{E})=b$ as well since $f(\mathbf{E})\notin T^n$ either). Now let $b\in A$ with $R(\mathbf{D})\geqq R(b)$. Thus $\mathbf{D}$ meets A^n, that is $I=\{i\,|\,1\leqq i\leqq n,\ \mathbf{D}_i\in A\}\neq\emptyset$. Let $m=|I|$. Let $\mathbf{D}_{n_1},\ldots,\mathbf{D}_{n_m}$ be the coordinate projections of $\mathbf{D}$ which are in A. The map $\sigma\colon\mathfrak{B}^n\to\mathfrak{B}^m$ with $(\sigma\mathbf{E})_i=\mathbf{E}_{n_i}$ is a homomorphism. Now $\mathrm{St}_3\,\hat{\mathfrak{A}}$ holds for $\hat{\mathfrak{A}}=\langle A,f\rangle_{f\in S\cup L}$ with $S=\mathrm{End}\,\mathfrak{A}$ and $L=\{\mathrm{id}{\restriction}C\,|\,C\in\mathrm{Su}\,\mathfrak{A}\}$ since $L=\mathrm{Su}\,\mathfrak{A}$ Hence there is a homomorphism $\varepsilon\colon\hat{\mathfrak{A}}^m\to\hat{\mathfrak{B}}$ with $\varepsilon(\sigma(\mathbf{D}))=b$ (clearly $R(\mathbf{D})\geqq R(b)$ in $\mathfrak{B}$ implies $R(\sigma(\mathbf{D}))\geqq R(b)$ in $\hat{\mathfrak{A}}$). Now let $\nu\colon\mathfrak{B}^m\to\mathfrak{B}$ be as follows:

$$\nu(\mathbf{E})=\begin{cases}\varepsilon(\mathbf{E}) & \text{if} \quad \mathbf{E}\in A^m \\ 0_T & \text{otherwise}\end{cases}$$

The map ν is a homomorphism and the composition $\nu\circ\sigma\colon\mathfrak{B}^n\to\mathfrak{B}$ with $\nu\circ\sigma(D)=b$ is the required homomorphism.

Once again all that remains is to show that the representation of S is locally closed. This follows immediately from the fact that S is locally closed in $\hat{\mathfrak{A}}$. Furthermore without loss of generality each compact $t\in\mathrm{Su}\,\mathfrak{A}$ is singly generated, and it follows that each compact $t\in H$ is singly generated. This completes the proof of Theorem 3.2.

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