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A Mal'cev type condition for the semi-distributivity of congruence lattices

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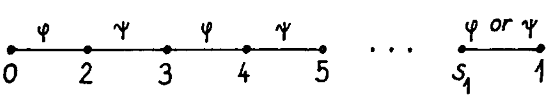
1. Introduction. A variety of algebras is said to be *congruence semi-distributive* if in the congruence lattices of its algebras the *semi-distributive law*,

$$(\forall\varphi)(\forall\psi)(\forall\eta)(\varphi\vee\psi=\varphi\vee\eta\Rightarrow\varphi\vee\psi=\varphi\vee(\psi\wedge\eta)),$$

holds. JÓNSSON [4, Problem 2.18] and GUMM [3] ask whether there exists a weak Mal'cev condition that characterizes congruence semi-distributivity of varieties. Now, to characterize congruence semi-distributivity of varieties, we intend to present a Mal'cev type condition, which is somewhat weaker than a weak Mal'cev condition in the sense of JÓNSSON [4].

2. A Mal'cev type condition. First, for any integers $n\geqq 1$ and $s_1, \ldots, s_n>1$ we define a graph $G(s_1, \ldots, s_n)$ whose vertices are the integers $0, 1, \ldots, k(s_1, \ldots, s_n)$. The edges of $G(s_1, \ldots, s_n)$ will be denoted by ordered pairs (i,j) with $i<j$, and will be coloured by the elements of $\Gamma=\{\varphi, \psi, \eta\}$. (The pair (i,j) without providing $i<j$ can mean the edge (j,i) for $j<i$.)

Let $k(s_1)=s_1$ and define $G(s_1)$ as follows:



(the colours φ and ψ alternate).

Suppose $G(s_1, \ldots, s_n)$ is already defined and consider the following linear ordering of the edges of $G(s_1, \ldots, s_n)$:

$$(i_1,j_1)<(i_2,j_2) \quad \text{iff either} \quad i_1<i_2 \quad \text{or} \quad i_1=i_2 \quad \text{and} \quad j_1<j_2.$$

Suppose n is odd (even, respectively). Let

$$(i_0,j_0)<(i_1,j_1)<\ldots<(i_t,j_t)$$

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be the ψ-coloured (η-coloured, resp.) edges of $G(s_1, \ldots, s_n)$ whose endpoints cannot be connected by a path which consists of edges coloured by the elements of $\Gamma \setminus \{\psi\}$ ($\Gamma \setminus \{\eta\}$, resp.). Now we construct the graph $G(s_1, \ldots, s_n, s_{n+1})$ by adding new vertices and new edges as follows:

(i) we add $(t+1)(s_{n+1}-1)$ new vertices, i.e.

$$k(s_1, \ldots, s_n, s_{n+1}) = k(s_1, \ldots, s_n)+(t+1)(s_{n+1}-1),$$

and for any r, $0 \leqq r \leqq t$,

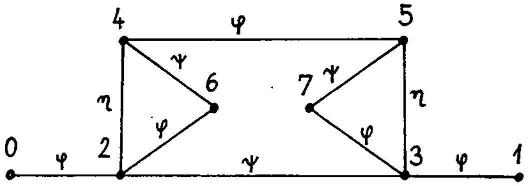
(ii) denoting $k(s_1, \ldots, s_n)$ by k we add the edges

$$\big(i_r, k+r(s_{n+1}-1)+1\big);$$
$$\big(k+r(s_{n+1}-1)+q, k+r(s_{n+1}-1)+q+1\big), \quad 1 \leqq q \leqq s_{n+1}-2;$$
$$\big(k+r(s_{n+1}-1)+s_{n+1}-1, j_r\big),$$

among which

$$\big(i_r, k+r(s_{n+1}-1)+1\big);$$
$$\big(k+r(s_{n+1}-1)+q, k+r(s_{n+1}-1)+q+1\big), \quad 1 \leqq q \leqq s_{n+1}-2, \quad q \text{ even};$$
$$\big(k+r(s_{n+1}-1)+s_{n+1}-1, j_r\big), \quad \text{provided } s_{n+1} \text{ is odd},$$

are coloured by η (φ, resp.) and the others are coloured by φ (ψ, resp.). For example, $G(3, 3, 2)$ is the following graph.



For $\pi \in \Gamma$ define $\pi(s_1, \ldots, s_n)$ to be the equivalence relation on the vertex set of $G(s_1, \ldots, s_n)$ generated by $\{(i, j)\colon (i, j)$ is an edge of $G(s_1, \ldots, s_n)$ coloured by $\pi\}$.

For $m \geqq 1$ let $U(m, s_1, \ldots, s_n)$ denote the following strong Mal'cev condition: There exist $k(s_1, \ldots, s_n)+1$-ary terms $f_0, f_1, \ldots, f_m$ such that the identities

$$f_0(x_i\colon i \leqq k) = x_0, \quad f_m(x_i\colon i \leqq k) = x_1,$$
$$f_j(x_{i\varphi}\colon i \leqq k) = f_{j+1}(x_{i\varphi}\colon i \leqq k) \quad \text{for } j \text{ even}, \quad 0 \leqq j \leqq m-1,$$
$$f_j(x_{i\psi}\colon i \leqq k) = f_{j+1}(x_{i\psi}\colon i \leqq k) \quad \text{for } j \text{ odd}, \quad 0 \leqq j \leqq m-1, \quad \text{and}$$
$$f_j(x_{i\eta}\colon i \leqq k) = f_{j+1}(x_{i\eta}\colon i \leqq k) \quad \text{for } j \text{ odd}, \quad 0 \leqq j \leqq m-1;$$

hold, where $k = k(s_1, \ldots, s_n)$ and $f_j(x_i\colon i \leqq k)$ stands for $f_j(x_0, x_1, \ldots, x_k)$. Here, for $\pi \in \Gamma$ and $i \leqq k(s_1, \ldots, s_n)$, $i\pi$ denotes the smallest integer j $(0 \leqq j \leqq k(s_1, \ldots, s_n))$ for which $(i, j) \in \pi(s_1, \ldots, s_n)$.

Now we can formulate the following

Theorem. *For any variety* **V** *of algebras the following two conditions are equivalent:*

(i) **V** *is congruence semi-distributive;*

(ii) *For any infinite sequence* $\mathbf{s}=(s_1, s_2, s_3, \ldots)$ *of integers* $(s_i>1, i=1, 2, 3, \ldots)$ *there exist integers* $m, n\geqq 1$ *such that* $U(m, s_1, \ldots, s_n)$ *holds in* **V**.

3. The proof of the Theorem. In order to prove our theorem, we need several statements.

(i) *implies* (ii). Let **V** be a congruence semi-distributive variety of similarity type τ. Suppose $\mathbf{s}=(s_1, s_2, \ldots)$ is an infinite sequence of integers $s_i>1$ $(i=1, 2, \ldots)$. Note that for $n<t$, $G(s_1, \ldots, s_n)$ is a subgraph of $G(s_1, \ldots, s_t)$, i.e., for $i, j\leqq \leqq k(s_1, \ldots, s_n)$ (i, j) is a π-coloured edge in $G(s_1, \ldots, s_n)$ iff it is a π-coloured edge in $G(s_1, \ldots, s_t)$. Let $G(\mathbf{s})$ be the direct union of the graphs $G(s_1, \ldots, s_n)$ $(n\geqq 1)$ and let $X=X(\mathbf{s})=\{0, 1, 2, \ldots\}$ denote the vertex set of $G(\mathbf{s})$. For $\pi\in\Gamma$ let $\pi(\mathbf{s})= =\bigcup_{n=1}^{\infty} \pi(s_1, \ldots, s_n)$.

Claim 1. *For* $n\leqq t$ *and* $\pi\in\Gamma$, *both* $\pi(s_1, \ldots, s_t)$ *and* $\pi(\mathbf{s})$ *restricted to* $\{0, 1, \ldots, k(s_1, \ldots, s_n)\}$ *are* $\pi(s_1, \ldots, s_n)$.

This claim is an easy consequence of the definitions. By Claim 1, for any $\pi\in\Gamma$, $\pi(\mathbf{s})$ is an equivalence relation. Since

$$\varphi(s_1, \ldots, s_n)\vee\psi(s_1, \ldots, s_n)\subseteq\varphi(s_1, \ldots, s_{n+1})\vee\eta(s_1, \ldots, s_{n+1})\subseteq$$
$$\subseteq\varphi(s_1, \ldots, s_{n+2})\vee\psi(s_1, \ldots, s_{n+2})$$

is also obvious from our definitions, we have

Claim 2. $\varphi(\mathbf{s})\vee\psi(\mathbf{s})=\varphi(\mathbf{s})\vee\eta(\mathbf{s})$ *in the lattice of equivalence relations on* X.

Now consider $F(X)$, the free algebra in **V** generated by X. For any $\pi\in\Gamma$ let $\hat{\pi}$ denote the congruence of $F(X)$ generated by the relation $\pi(\mathbf{s})$. Claim 2 together with the well-known descriptions of the join of congruences and the congruence generated by a relation (cf. GRÄRZER [2], Lemma 2 and Theorem 4 in § 10, Chapter 1) immediately imply

Claim 3. $\hat{\varphi}\vee\hat{\psi}=\hat{\varphi}\vee\hat{\eta}$ *in the congruence lattice of* $F(X)$.

Since $0\equiv 1$ $(\hat{\varphi}\vee\hat{\psi})$ in $F(X)$, from Claim 3 and from the assumption made on **V** we obtain $0\equiv 1$ $(\hat{\varphi}\vee(\hat{\psi}\vee\hat{\eta}))$. Therefore there are elements $a_0, \ldots, a_m$ in $F(X)$ such that

$$\begin{aligned} & a_0=0, \quad a_m=1 \\ & a_j\equiv a_{j+1}(\hat{\varphi}) \quad \text{for } j \text{ even} \\ & a_j\equiv a_{j+1}(\hat{\psi}) \quad \text{for } j \text{ odd} \\ & a_j\equiv a_{j+1}(\hat{\eta}) \quad \text{for } j \text{ odd.} \end{aligned} \tag{1}$$

Since X generates $F(X)$, there is a finite subset of X that generates a subalgebra containing all the a_j $(0 \leqq j \leqq m)$. Hence there are $n \geqq 1$ and $(k(s_1, \ldots, s_n)+1)$-ary terms in $\mathbf{V}$ such that

$$a_j = f_j(i\colon i \leqq k(s_1, \ldots, s_n)) \tag{2}$$

holds for all $j \leqq m$ in $\mathbf{V}$. For $\pi \in \Gamma$ let

$$X_\pi = \{x_i\colon i \in X \text{ and } (i, j) \in \pi(\mathrm{s}) \text{ implies } i \leqq j\}$$

and define an onto mapping $g_\pi\colon X \to X_\pi$ by $ig_\pi = x_j$ iff $j = \min\{t\colon (t, i) \in \pi(\mathrm{s})\}$. Let us denote by $W(Y)$ and $W(Y_\pi)$ the absolutely free algebras of type τ generated by $Y = \{y_i\colon i \in X\}$ and $Y_\pi = \{y_i\colon x_i \in X_\pi\}$, respectively. Let $F(X_\pi)$ be the free $\mathbf{V}$-algebra generated by X_π. We consider the natural homomorphisms $u\colon W(Y) \to F(X)$ and $v\colon W(Y_\pi) \to F(X_\pi)$ defined by $y_i u = i$ $(i \in X)$ and $y_i v = x_i$ $(x_i \in X_\pi)$. Let $p\colon F(X) \to F(X_\pi)$ and $q\colon W(Y) \to W(Y_\pi)$ be the unique homomorphisms for which $ip = ig_\pi$ $(i \in X)$ and $y_i q = y_{ig_\pi}$ $(i \in X)$. Then the diagram

$$\begin{tikzcd} W(Y) \arrow[r, "q"] \arrow[d, "u"] & W(Y_\pi) \arrow[d, "v"] \\ F(X) \arrow[r, "p"] & F(X_\pi) \end{tikzcd}$$

commutes. Furthermore, we have $\hat{\pi} \subseteq \operatorname{Ker} p$ since $\pi(\mathrm{s}) = \operatorname{Ker} g_\pi$.

Now, by (1) and (2), the identities $f_0(x_i\colon i \leqq k(s_1, \ldots, s_n)) = x_0$ and

$$f_m(x_i\colon i \leqq k(s_1, \ldots, s_n)) = x_1$$

are evidently satisfied in $\mathbf{V}$. For the rest of the identities in $U(m, s_1, \ldots, s_n)$, let $\pi \in \Gamma$ and let $a_j \equiv a_{j+1}(\hat{\pi})$ be one of the formulae listed in (1). Denoting $k(s_1, \ldots, s_n)$ by k, we can compute:

$$\begin{aligned}
f_j(x_{i\pi}\colon i \leqq k) &= f_j(y_{i\pi}v\colon i \leqq k) = f_j(y_{ig_\pi}v\colon i \leqq k), \quad \text{by Claim 1,} \\
&= f_j(y_i qv\colon i \leqq k) \\
&= f_j(y_i up\colon i \leqq k), \quad \text{by the commutativity of the diagram,} \\
&= f_j(ip\colon i \leqq k) = f_j(i\colon i \leqq k)p = a_j p = a_{j+1}p, \quad \text{since } \hat{\pi} \subseteq \operatorname{Ker} p, \\
&= f_{j+1}(i\colon i \leqq k)p = f_{j+1}(ip\colon i \leqq k) = f_{j+1}(y_i up\colon i \leqq k) \\
&= f_{j+1}(y_i qv\colon i \leqq k), \quad \text{by the commutativity of the diagram,} \\
&= f_{j+1}(y_{ig_\pi}v\colon i \leqq k) \\
&= f_{j+1}(y_{i\pi}v\colon i \leqq k), \quad \text{by Claim 1,} \\
&= f_{j+1}(x_{i\pi}\colon i \leqq k).
\end{aligned}$$

Therefore the identity $f_j(x_{i\pi}\colon i\leqq k)=f_{j+1}(x_{i\pi}\colon i\leqq k)$ holds in $F(X_\pi)$, whence it holds in **V** as well. Hence **V** satisfies (ii).

To prove the converse, let **V** be a variety satisfying (ii). Let $\hat{\varphi}$, $\hat{\psi}$, $\hat{\eta}$ be congruences of an algebra A in **V** such that $\hat{\varphi}\vee\hat{\psi}=\hat{\varphi}\vee\hat{\eta}$. We have to show that $\hat{\varphi}\vee\hat{\psi}\subseteq\hat{\varphi}\vee(\hat{\psi}\wedge\hat{\eta})$, which is clearly equivalent to $\hat{\varphi}\vee\hat{\psi}=\hat{\varphi}\vee(\hat{\psi}\wedge\hat{\eta})$. Let a_0, a_1 be arbitrary elements of A that are congruent modulo $\hat{\varphi}\vee\hat{\psi}$. We define an infinite sequence s and assign an element a_i in A to each vertex i of $G(\mathrm{s})$ by means of induction. Let $s_1\geqq 2$ be the smallest integer for which $(a_0, a_1)\in\hat{\varphi}\circ\hat{\psi}\circ\hat{\varphi}\circ\hat{\psi}\circ\dots$ (s_1 factors) and let us choose elements a_j $(2\leqq j\leqq k(s_1)=s_1)$ from A such that

$$
\begin{aligned}
&(a_0, a_2)\in\hat{\varphi},\\
&(a_j, a_{j+1})\in\hat{\varphi} \quad \text{for } j \text{ odd}, \quad 3\leqq j<s_1,\\
&(a_{s_1}, a_1)\in\hat{\varphi} \quad \text{provided } s_1 \text{ is odd},\\
&(a_j, a_{j+1})\in\hat{\psi} \quad \text{for } j \text{ even}, \quad 2\leqq j<s_1,\\
&(a_{s_1}, a_1)\in\hat{\psi}, \quad \text{provided } s_1 \text{ is even}.
\end{aligned}
$$

Then $G(s_1)$ and the elements chosen from A have the following property:

(3) if the graph has an edge (i, j) coloured by π then $(a_i, a_j)\in\hat{\pi}$.

Suppose $s_1, \dots, s_n$ and a_j $(j\leqq k(s_1, \dots, s_n))$ are already defined. Then let $s_{n+1}\geqq 2$ be the smallest integer such that $G(s_1, \dots, s_n, s_{n+1})$ has property (3) with appropriate further elements $a_i\in A$ $(k(s_1, \dots, s_n)<i\leqq k(s_1, \dots, s_n, s_{n+1}))$ associated with the new vertices. There exist such an integer s_{n+1} and such elements $a_i\in A$, since whenever we have elements b, c, d and e in A with $(b, c)\in\hat{\psi}$ and $(d, e)\in\hat{\eta}$ then, by $\hat{\psi}\subseteq\hat{\varphi}\vee\hat{\eta}$ and $\hat{\eta}\subseteq\hat{\varphi}\vee\hat{\psi}$, there are integers t, t' such that $(b, c)\in\hat{\varphi}\circ\hat{\eta}\circ\hat{\varphi}\circ\hat{\eta}\circ\dots$ (t factors) and $(d, e)\in\hat{\varphi}\circ\hat{\psi}\circ\hat{\varphi}\circ\hat{\psi}\circ\dots$ (t' factors).

Let m and n be the integers that exist by (ii) for the sequence $\mathrm{s}=(s_1, s_2, s_3, \dots)$ constructed above and let $f_0, f_1, \dots, f_m$ be $(k(s_1, \dots, s_n)+1)$-ary terms satisfying the identities of $U(m, s_1, \dots, s_n)$ throughout **V**. Let k stand for $k(s_1, \dots, s_n)$. It remains to show that

$$
(4)\qquad
\begin{aligned}
&f_j(a_i\colon i\leqq k)\equiv f_{j+1}(a_i\colon i\leqq k)\ (\hat{\varphi}) \quad \text{for } j \text{ even},\\
&f_j(a_i\colon i\leqq k)\equiv f_{j+1}(a_i\colon i\leqq k)\ (\hat{\psi}) \quad \text{for } j \text{ odd and}\\
&f_j(a_i\colon i\leqq k)\equiv f_{j+1}(a_i\colon i\leqq k)\ (\hat{\eta}) \quad \text{for } j \text{ odd}.
\end{aligned}
$$

Indeed, then $(a_0, a_1)=(f_0(a_i\colon i\leqq k),\ f_m(a_i\colon i\leqq k))\in\hat{\varphi}\circ(\hat{\psi}\wedge\hat{\eta})\circ\hat{\varphi}\circ(\hat{\psi}\wedge\hat{\eta})\circ\dots$ (m factors) $\subseteq\hat{\varphi}\vee(\hat{\psi}\wedge\hat{\eta})$, completing the proof. Since $(a_i, a_{i\pi})\in\hat{\pi}$ $(\pi\in\Gamma)$ follows from (3), for j even we can compute:

$$f_j(a_i\colon i\leqq k)\,\hat{\varphi}\,f_j(a_{i\varphi}\colon i\leqq k)=f_{j+1}(a_{i\varphi}\colon i\leqq k)\,\hat{\varphi}\,f_{j+1}(a_i\colon i\leqq k).$$

Hence $f_j(a_i: i \leqq k) \equiv f_{j+1}(a_i: i \leqq k)$ (ϕ) holds for j even and the rest of (4) follows similarly. The proof of the Theorem is complete.

4. Concluding remarks. In this section we mention some statements concerning congruence semi-distributivity. The proofs are omitted because they are easy but most of them would require a long formulation.

A variety $\mathbf{V}$ is said to be *n-permutable* $(n \geqq 2)$ if $\varphi \vee \psi = \varphi \circ \psi \circ \varphi \circ \psi \circ \ldots$ (n factors) holds for any congruences φ and ψ of any algebra in $\mathbf{V}$. It is easy to see that the method we used yields the following result, too.

Proposition 1. *An m-permutable variety* $\mathbf{V}$ *is congruence semi-distributive if and only if* $U(m, m, \ldots, m)$ *(where m occurs* $n+1$ *times) holds in V for some* $n \geqq 1$.

Making use of Claim 1 it can be shown that whenever $U(m, s_1, \ldots, s_n)$ holds in a variety $\mathbf{V}$ then $U(m+1, s_1, \ldots, s_n)$ and $U(m, s_1, \ldots, s_n, s_{n+1})$ hold in $\mathbf{V}$ as well. Therefore, condition (ii) in the Theorem is equivalent to:

(iii) For any infinite sequence $\mathbf{s}=(s_1, s_2, s_3, \ldots)$ of integers $s_i \geqq 2$ $(i=1, 2, 3, \ldots)$ there exists an integer $n \geqq 1$ such that $U(n, s_1, \ldots, s_n)$ holds in $\mathbf{V}$.

In some varieties the terms and identities are easy to handle. For example, it is not hard to check that there are no $m, n \geqq 2$ for which $U\left(m, 3, 2, \ldots, \frac{1}{2}(5-(-1)^n)\right)$ holds in the variety of semilattices. Therefore the variety of semilattices is not congruence semi-distributive. However, as it was shown by Papert [5], it is congruence dually semi-distributive.

As a non-trivial example of varieties satisfying the conditions of the Theorem we can mention Polin's variety $\mathbf{P}$. Indeed, as it was shown by Day and Freese [1], $\mathbf{P}$ is congruence semi-distributive, but it is even not congruence modular.

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