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On invariant subspace lattices of C_{11}-contractions

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We say that a Hilbert space operator T belongs to the class $\mathcal{P}$, if it has the following property:

(P) every injection X from the commutant $\{T\}'$ of T is a quasi-affinity.

The class of C_0-contractions with property (P) was studied in [13], [15] and [3], while in [10] we characterized the class $C_{11}\cap\mathcal{P}$. It turned out that classes $C_0\cap\mathcal{P}$ and $C_{11}\cap\mathcal{P}$ are good generalizations of the corresponding cases of finite defect indices. In fact, both in C_0 and in C_{11}, property (P) is a quasi-similarity invariant. Moreover, as it was proved in [3], in the class $C_0\cap\mathcal{P}$ quasi-similarity induces isomorphism between the invariant subspace lattices. In the present paper we prove an analogous statement, concerning the C_{11}-parts of invariant subspace lattices of contractions belonging to $C_{11}\cap\mathcal{P}$. Moreover, we examine behaviour, under quasi-similarities, of hyperinvariant and invariant subspaces of $C_{11}\cap\mathcal{P}$-contractions, and we prove the reflexivity of bicommutant.

Throughout the paper bounded linear operators on complex separable Hilbert spaces will be considered. We follow the terminology and notation used in [10] and [12].

1. Preliminaries. It is well-known (cf. [12, Theorem I.3.2]) that for every contraction T of class C_{11} on the Hilbert space $\mathfrak{H}$ there exists a (unique) "canonical" decomposition $\mathfrak{H}=\mathfrak{H}_1\oplus\mathfrak{H}_2\oplus\mathfrak{H}_3$ of $\mathfrak{H}$ reducing T, such that $T_1=T|\mathfrak{H}_1$ is a completely non-unitary (c.n.u.) contraction of class C_{11}, $T_2=T|\mathfrak{H}_2$ is an absolutely continuous unitary (a.c.u.) operator and $T_3=T|\mathfrak{H}_3$ is a singular unitary (s.u.) operator. (We mean that the specral measures of T_2 and T_3 are absolutely continuous and singular, respectively, with respect to the Lebesgue measure.) The following two lemmas, concerning this decomposition, will play an important role reducing proofs to the c.n.u. case. We recall that for arbitrary operators, $T_1\in\mathcal{L}(\mathfrak{H}_1)$ and $T_2\in\mathcal{L}(\mathfrak{H}_2)$, $\mathcal{I}(T_1, T_2)$ denotes the set of intertwining operators, that is $\mathcal{I}(T_1, T_2)=$ $=\{X\in\mathcal{L}(\mathfrak{H}_1, \mathfrak{H}_2)|XT_1=T_2X\}$.

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Lemma 1. *Let T be a C_{11}-contraction and let $T=T_1\oplus T_2\oplus T_3$ be its canonical decomposition. Then we have:*

(i) $$\operatorname{Lat} T = \operatorname{Lat}(T_1\oplus T_2)\oplus \operatorname{Lat} T_3,$$
and
(ii) $$\{T\}' = \{T_1\oplus T_2\}'\oplus\{T_3\}'.$$

Proof. Let $\mathfrak{L}\in\operatorname{Lat} T$ be an arbitrary invariant subspace, and let us consider the decomposition $\mathfrak{L}=\mathfrak{L}_1\oplus\mathfrak{L}_2\oplus\mathfrak{L}_3$ of $\mathfrak{L}$ reducing for $T|\mathfrak{L}$, such that $T|\mathfrak{L}_1$ is a c.n.u. contraction, $T|\mathfrak{L}_2$ is a.c.u. and $T|\mathfrak{L}_3$ is s.u. operator. (The existence of such a decomposition follows by [12, Theorem I.3.2].)

Let X denote the operator $X=P_{1,2}|\mathfrak{L}_3\in\mathcal{I}(T|\mathfrak{L}_3, T_1\oplus T_2)$, where $P_{1,2}$ is the orthogonal projection of the space $\mathfrak{H}$ onto $\mathfrak{H}_1\oplus\mathfrak{H}_2$. Let M be an a.c.u. operator, quasi-similar to $T_1\oplus T_2$, and let $Z\in\mathcal{I}(T_1\oplus T_2, M)$ be a quasi-affinity. (Cf. [12, Prop. II.3.5 and Theorem II.6.4].) Now, because of $ZX\in\mathcal{I}(T|\mathfrak{L}_3, M)$, we infer by [5, Lemma 4.1] that the subspaces $(\ker(ZX))^\perp$ and $(\operatorname{ran}(ZX))^-$ reduce $T|\mathfrak{L}_3$ and M respectively, moreover $T|(\ker(ZX))^\perp$ is unitarily equivalent to $M|(\operatorname{ran}(ZX))^-$. Since $T|(\ker(ZX))^\perp$ is singular and $M|(\operatorname{ran}(ZX))^-$ is absolutely continuous unitary operator, it follows that $ZX=0$, and so $X=0$. Consequently, $\mathfrak{L}_3\subseteq\mathfrak{H}_3$.

Let Y denote the operator $Y=P_3|\mathfrak{L}_1\oplus\mathfrak{L}_2\in\mathcal{I}(T|\mathfrak{L}_1\oplus\mathfrak{L}_2, T_3)$, where P_3 is the orthogonal projection of the space $\mathfrak{H}$ onto $\mathfrak{H}_3$. Let $U_+\in\mathcal{L}(\mathfrak{K}_+)$ be the minimal isometric dilation of the contraction $T|\mathfrak{L}_1\oplus\mathfrak{L}_2$, and let $U\in\mathcal{L}(\mathfrak{K})$ be the minimal unitary dilation. We can extend the operator Y by the equation $Y'U_+^n h:=YT^n h$ $(h\in\mathfrak{L}_1\oplus\mathfrak{L}_2,\ n\geqq 0)$, and by taking bounded closure, to an operator $Y'\in\mathcal{I}(U_+, T_3)$, such that $Y'|\mathfrak{L}_1\oplus\mathfrak{L}_2=Y$. After that, the operator Y' can be extended by the equation: $Y''U^{-n}k:=T_3^{-n}Y'k$ $(k\in\mathfrak{K}_+,\ n\geqq 0)$, and by taking bounded closure, to an operator $Y''\in\mathcal{I}(U, T_3)$, such that $Y''|\mathfrak{K}_+=Y'$. Since U is an a.c.u. (cf. [12, Theorem II.6.4]) and T_3 is a s.u. operator, we infer as above, that $Y''=0$, and so $Y=0$. Therefore, we get that $\mathfrak{L}_1\oplus\mathfrak{L}_2\subseteq\mathfrak{H}_1\oplus\mathfrak{H}_2$, and property (i) is proved.

Property (ii) immediately follows by [5, Lemma 4.1].

Lemma 2. *Let U be an a.c.u. operator. Then there exists a c.n.u. C_{11}-contraction T, similar to U.*

Proof. Let $M=M_{E_1}\oplus M_{E_2}\oplus\ldots$ be the functional model of the a.c.u. operator U (cf. [10]). Moreover, for every n let ϑ_n be an outer function, such that

$$|\vartheta_n(e^{it})| = \begin{cases} 1, & \text{if } e^{it}\notin E_n \\ \dfrac{1}{2}, & \text{if } e^{it}\in E_n \end{cases}$$

holds a.e.. Then the function $\Theta=\begin{bmatrix}\vartheta_1 & & 0\\ & \vartheta_2 & \\ 0 & & \ddots\end{bmatrix}$ will be outer from both sides, and we have $\Theta(e^{it})^*\Theta(e^{it})\geqq\frac{1}{4}I$ a.e.. Now, on account of [10, Lemma 4], the c.n.u. C_{11}-contraction $T=S(\Theta)$ is similar to its Jordan model. But, in virtue of [10, Corollary 1], the Jordan model of T is exactly the operator M. Therefore, T is similar to M, and so to U also.

Let T be an arbitrary c.n.u. contraction, and let $d_T(e^{it})$ denote its *"defect function"*, that is $d_T(e^{it})=\operatorname{rank}[I-\Theta_T(e^{it})^*\Theta_T(e^{it})]^{1/2}$, where Θ_T is the characteristic function of T. We note that for the defect index d_T of T, introduced in [12], we have $d_T=\operatorname{rank}(I-T^*T)=\operatorname{rank}[I-\Theta_T(0)^*\Theta_T(0)]^{1/2}$. It was proved in [10] that a c.n.u. C_{11}-contraction T belongs to the class $\mathscr{P}$, if and only if T is a contraction of finite defect function, that is if its defect function, $d_T(e^{it})$, is finite a.e. on the unit circle.

2. The C_{11}-invariant subspace lattice. In the invariant subspace lattice of a C_{11}-contraction T the subspaces $\mathfrak{L}$, such that $T|\mathfrak{L}\in C_{11}$, have a particular interest. In this sections we examine this C_{11}-part of $\operatorname{Lat} T$.

Definition 1. For every C_{11}-contraction T, $\operatorname{Lat}_1 T$ denotes the C_{11}-*invariant subspace lattice of* T, that is $\operatorname{Lat}_1 T:=\{\mathfrak{L}\in\operatorname{Lat} T \,|\, T|\mathfrak{L}\in C_{11}\}$.

The following proposition shows that $\operatorname{Lat}_1 T$ is closed with respect to the span.

Proposition 1. *If T is a C_{11}-contraction and $\{\mathfrak{L}_\gamma\}_{\gamma\in\Gamma}\subseteq\operatorname{Lat}_1 T$, then* $\mathfrak{L}_\vee=\bigvee_{\gamma\in\Gamma}\mathfrak{L}_\gamma\in\operatorname{Lat}_1 T$.

Proof. Since $\mathfrak{L}_\vee$ is separable, there exists a countable subset $\{\gamma_j\}_{j=1}^\infty$ of Γ, such that $\bigvee_{j=1}^\infty\mathfrak{L}_{\gamma_j}=\mathfrak{L}_\vee$. For every j, let $U_j\in\mathscr{L}(\mathfrak{K}_j)$ be a unitary operator, quasi-similar to $T|\mathfrak{L}_{\gamma_j}$, and let $X_j\in\mathscr{I}(U_j, T|\mathfrak{L}_{\gamma_j})$ be a quasi-affinity, such that $\|X_j\|\leqq 2^{-j}$. Then the operator $X\colon \mathfrak{K}=\bigoplus_{j=1}^\infty\mathfrak{K}_j\to\mathfrak{L}_\vee$, $X\left(\bigoplus_{j=1}^\infty k_j\right)=\sum_{j=1}^\infty X_j k_j$ intertwines the unitary operator $U=\bigoplus_{j=1}^\infty U_j$ with $T|\mathfrak{L}_\vee$, $X\in\mathscr{I}(U, T|\mathfrak{L}_\vee)$. Taking into account that X has a dense range, we infer that $X^*\in\mathscr{I}((T|\mathfrak{L}_\vee)^*, U^*)$ is an injection, and so it follows that $(T|\mathfrak{L}_\vee)^*\in C_{1\cdot}$. Consequently, we have that $T|\mathfrak{L}_\vee\in C_{11}$, that is $\mathfrak{L}_\vee\in\operatorname{Lat}_1 T$.

In the sequel we show that, for every contraction T of class $C_{11}\cap\mathscr{P}$, $\operatorname{Lat}_1 T$ possesses the usual properties of invariant subspace lattices, if we replace intersection and orthogonal complement by suitable new operations. We need the following notion.

Definition 2. Let T be a C_{11}-contraction, and $\mathfrak{L}\in\operatorname{Lat} T$. By [12, Theorem II.4.1] there exists a unique decomposition $\mathfrak{L}=\mathfrak{L}'\oplus\mathfrak{L}''$ of $\mathfrak{L}$, such that

$\mathfrak{L}' \in \mathrm{Lat}(T|\mathfrak{L})$, $T|\mathfrak{L}' \in C_{11}$ and $T|_{\mathfrak{L}''} \in C_{\cdot 0}$. (We denote by $T|_{\mathfrak{L}''}$ the compression of T to the subspace $\mathfrak{L}'' \in \mathrm{Lat}_{\frac{1}{2}} T \setminus \mathrm{Lat}\, T$, that is $T|_{\mathfrak{L}''} = P_{\mathfrak{L}''} T|\mathfrak{L}''$. Cf. [2].) The subspace $\mathfrak{L}'$ is called as the *C_{11}-part of* $\mathfrak{L}$, and is denoted by $\mathfrak{L}' = \mathfrak{L}^{(1)} = \mathfrak{L}_T^{(1)}$.

The following lemma shows that $\mathfrak{L}^{(1)}$ is the greatest C_{11}-invariant subspace in $\mathfrak{L}$.

Lemma 3. *Let T be a C_{11}-contraction, and $\mathfrak{L} \in \mathrm{Lat}\, T$. If $\mathfrak{L}' \in \mathrm{Lat}_1 T$ and $\mathfrak{L}' \subseteq \mathfrak{L}$, then $\mathfrak{L}' \subseteq \mathfrak{L}^{(1)}$.*

Proof. Since $\mathfrak{L}', \mathfrak{L}^{(1)} \in \mathrm{Lat}_1 T$, we infer by Proposition 1 that $\mathfrak{L}'' = \mathfrak{L}' \vee \vee \mathfrak{L}^{(1)} \in \mathrm{Lat}_1 T$. Let us suppose that $\mathfrak{L}' \subsetneqq \mathfrak{L}^{(1)}$. Then there exists a non-zero vector $f \in \mathfrak{L}'' \ominus \mathfrak{L}^{(1)} \subseteq \mathfrak{L} \ominus \mathfrak{L}^{(1)}$. In virtue of $f \in \mathfrak{L}''$ and $\mathfrak{L}'' \in \mathrm{Lat}_1 T$ it follows that $\|P_{\mathfrak{L}} T^{*n} f\| \geqq \geqq \|P_{\mathfrak{L}''} T^{*n} f\| \not\to 0$. On the other hand $f \in \mathfrak{L} \ominus \mathfrak{L}^{(1)}$, and so $\|P_{\mathfrak{L}} T^{*n} f\| \to 0$. This being a contradiction, we infer that $\mathfrak{L}' \subseteq \mathfrak{L}^{(1)}$.

Definition 3. The *C_{11}-orthogonal complement* of a subspace $\mathfrak{L} \in \mathrm{Lat}_1 T$, C_{11}-invariant for T, is the subspace $\mathfrak{L}^{\perp_1} = \mathfrak{L}_T^{\perp_1}$, C_{11}-invariant for T^*, defined by $\mathfrak{L}^{\perp_1} = = \mathfrak{L}_T^{\perp_1} := (\mathfrak{L}^{\perp})_{T^*}^{(1)} \in \mathrm{Lat}_1 T^*$.

Proposition 2. *If T is a contraction of class $C_{11} \cap \mathscr{P}$ and $\mathfrak{L} \in \mathrm{Lat}_1 T$, then $(\mathfrak{L}^{\perp_1})^{\perp_1} = \mathfrak{L}$.*

Proof. In virtue of Lemmas 1 and 2 we may assume that T is a c.n.u. contraction. By the definition of $\mathfrak{L}^{\perp_1}$ it follows that $T|_{\mathfrak{L}^{\perp} \ominus \mathfrak{L}^{\perp_1}} \in C_{0\cdot}$, and so $d_{(T|_{\mathfrak{L}^{\perp} \ominus \mathfrak{L}^{\perp_1}})^*}(e^{it}) = 0$ a.e. (cf. [12, Prop. VI.3.5]). Taking into account that $d_{S^*}(e^{-it}) = = d_S(e^{it})$ for any c.n.u. C_{11}-contraction S (cf. [10, Cor. 1]), we infer that

$$d_{T|_{\mathfrak{L}^{\perp} \ominus \mathfrak{L}^{\perp_1}}}(e^{it}) = d_T(e^{it}) - d_{T|\mathfrak{L}}(e^{it}) - d_{T|_{\mathfrak{L}^{\perp_1}}}(e^{it}) =$$

$$= d_{T^*}(e^{-it}) - d_{(T|\mathfrak{L})^*}(e^{-it}) - d_{T^*|\mathfrak{L}^{\perp_1}}(e^{-it}) = d_{(T|_{\mathfrak{L}^{\perp} \ominus \mathfrak{L}^{\perp_1}})^*}(e^{-it}) = 0$$

a.e. (cf. also [12, Theorem VII.1.1 and Propositions VII.2.1, VII.3.3]). Therefore we have that $T|_{\mathfrak{L}^{\perp} \ominus \mathfrak{L}^{\perp_1}} = T|_{(\mathfrak{L}^{\perp_1})^{\perp} \ominus \mathfrak{L}} \in C_{00}$, and so $(\mathfrak{L}^{\perp_1})^{\perp_1} = ((\mathfrak{L}^{\perp_1})^{\perp})^{(1)} = \mathfrak{L}$.

Definition 4. The *C_{11}-intersection* $\overset{(1)}{\bigcap\limits_{\gamma \in \Gamma}} \mathfrak{L}_\gamma$ of a system of subspaces $\{\mathfrak{L}_\gamma\}_{\gamma \in \Gamma} \subseteq \subseteq \mathrm{Lat}_1 T$ is defined by $\overset{(1)}{\bigcap\limits_{\gamma \in \Gamma}} \mathfrak{L}_\gamma := (\bigcap\limits_{\gamma \in \Gamma} \mathfrak{L}_\gamma)^{(1)}$.

Proposition 3. *If T is a contraction of class $C_{11} \cap \mathscr{P}$ and $\{\mathfrak{L}_\gamma\}_{\gamma \in \Gamma} \subseteq \mathrm{Lat}_1 T$, then $\overset{(1)}{\bigcap\limits_{\gamma \in \Gamma}} \mathfrak{L}_\gamma = (\bigvee\limits_{\gamma \in \Gamma} \mathfrak{L}_\gamma^{\perp_1})^{\perp_1}$ and $\bigvee\limits_{\gamma \in \Gamma} \mathfrak{L}_\gamma = (\overset{(1)}{\bigcap\limits_{\gamma \in \Gamma}} \mathfrak{L}_\gamma^{\perp_1})^{\perp_1}$.*

Proof. In virtue of Proposition 2 it is enough to prove the first equality. Let $\mathfrak{L}'$ and $\mathfrak{L}''$ denote the subspaces $\overset{(1)}{\bigcap\limits_{\gamma \in \Gamma}} \mathfrak{L}_\gamma$ and $(\bigvee\limits_{\gamma \in \Gamma} \mathfrak{L}_\gamma^{\perp_1})^{\perp_1}$, respectively. Since

$(\bigvee_{\gamma\in\Gamma}\mathfrak{L}_\gamma^{\perp_1})^\perp\subseteqq(\mathfrak{L}_\gamma^{\perp_1})^\perp$, we infer by Lemma 3 and Proposition 2 that $\mathfrak{L}''=(\bigvee_{\gamma\in\Gamma}\mathfrak{L}_\gamma^{\perp_1})^{\perp_1}\subseteqq\subseteqq(\mathfrak{L}_\gamma^{\perp_1})^{\perp_1}=\mathfrak{L}_\gamma$. Therefore we have $\mathfrak{L}''\subseteqq\bigcap_{\gamma\in\Gamma}\mathfrak{L}_\gamma$, and so by Lemma 3 $\mathfrak{L}''\subseteqq\mathfrak{L}'$.

On the other hand, $\bigvee_{\gamma\in\Gamma}\mathfrak{L}_\gamma^{\perp_1}\subseteqq\bigvee_{\gamma\in\Gamma}\mathfrak{L}_\gamma^\perp$ implies $(\bigvee_{\gamma\in\Gamma}\mathfrak{L}_\gamma^{\perp_1})^\perp\supseteqq(\bigvee_{\gamma\in\Gamma}\mathfrak{L}_\gamma^\perp)^\perp=\bigcap_{\gamma\in\Gamma}\mathfrak{L}_\gamma\supseteqq\supseteqq\overset{(1)}{\bigcap_{\gamma\in\Gamma}}\mathfrak{L}_\gamma=\mathfrak{L}'$. Again by Lemma 3, it follows that $\mathfrak{L}'\subseteqq\mathfrak{L}''$.

As a consequence, we get the following:

Proposition 4. *Let T be a contraction of class $C_{11}\cap\mathscr{P}$, and let $\{\Gamma_\alpha\}_{\alpha\in L}$ be a system of sets of indices, $\Gamma=\bigcup_{\alpha\in L}\Gamma_\alpha$. If $\{\mathfrak{L}_\gamma\}_{\gamma\in\Gamma}\subseteqq\mathrm{Lat}_1T$, then*

$$\overset{(1)}{\bigcap}\{\overset{(1)}{\bigcap}\{\mathfrak{L}_\gamma|\gamma\in\Gamma_\alpha\}|\alpha\in L\}=\overset{(1)}{\bigcap}\{\mathfrak{L}_\gamma|\gamma\in\Gamma\}.$$

Finally we note that if U is a unitary operator, then $\mathrm{Lat}_1\,U$ coincides with the lattice of reducing subspaces.

3. Quasi-similarity invariance of Lat_1T. We show that, for contractions T of class $C_{11}\cap\mathscr{P}$, Lat_1T is a quasi-similarity invariant, and any quasi-affinity, intertwining such contractions, implements an isomorphism between the C_{11}-invariant subspace lattices. We need a lemma.

Lemma 4. *Let T_1 and T_2 be quasi-similar contractions of class $C_{11}\cap\mathscr{P}$, and let $X\in\mathscr{I}(T_1,T_2)$ be a quasi-affinity. Then, for every subspace $\mathfrak{L}\in\mathrm{Lat}_1T_1$, we have*

$$(X^*((X\mathfrak{L})^-)^{\perp_1})^-=\mathfrak{L}^{\perp_1}.$$

Proof. By Lemmas 1 and 2 we may assume that T_1 and T_2 are c.n.u. contractions. Let us denote by $\mathfrak{B}$ the subspace $\mathfrak{B}=(X\mathfrak{L})^-\in\mathrm{Lat}_1T_2$. In virtue of the proof of Proposition 2 we can write

$$d_{T_1^*|(X^*\mathfrak{B}^{\perp_1})^-}(e^{it})=d_{T_2^*|\mathfrak{B}^{\perp_1}}(e^{it})=d_{T_2^*}(e^{it})-d_{T_2^*|_{\mathfrak{B}}}(e^{it})=$$
$$=d_{T_1^*}(e^{it})-d_{T_1^*|_{\mathfrak{L}}}(e^{it})=d_{T_1^*|\mathfrak{L}^{\perp_1}}(e^{it})\quad\text{a.e. and so by [10, Cor. 1]}$$

$T_1^*|(X^*\mathfrak{B}^{\perp_1})^-$ is quasi-similar to $T_1^*|\mathfrak{L}^{\perp_1}$, $T_1^*|(X^*\mathfrak{B}^{\perp_1})^-\sim T_1^*|\mathfrak{L}^{\perp_1}$. On the other hand, since $\mathrm{Lat}_1T_1^*\ni(X^*\mathfrak{B}^{\perp_1})^-\subseteqq\mathfrak{L}^\perp$, we infer by Lemma 3 that $(X^*\mathfrak{B}^{\perp_1})^-\subseteqq\mathfrak{L}^{\perp_1}$. Now it follows by [10, Cor. 6] that $T_1^*|\mathfrak{L}^{\perp_1}\in\mathscr{P}$. Therefore we have $(X^*\mathfrak{B}^{\perp_1})^-==\mathfrak{L}^{\perp_1}$.

The following theorem is an analogue of [3, Prop. 4.8], concerning C_0-contractions, and it is a generalization of the corresponding part of [16, Theorem 2.2], concerning c.n.u. C_{11}-contractions with finite defect indices.

Theorem 1. *Let T_1 and T_2 be quasi-similar contractions of class $C_{11}\cap\mathscr{P}$. Then every injection $X\in\mathscr{I}(T_1,T_2)$ is a quasi-affinity, and the mapping φ_X: $\mathrm{Lat}_1T_1\to$*

$\to \mathrm{Lat}_1 T_2$, φ_X: $\mathfrak{L} \mapsto (X\mathfrak{L})^-$ *is a lattice-isomorphism. Moreover,* $T_1|\mathfrak{L}$ *and* $T_2|(X\mathfrak{L})^-$ *are quasi-similar, for every* $\mathfrak{L} \in \mathrm{Lat}_1 T_1$.

Proof. It is evident that, for every $\mathfrak{L} \in \mathrm{Lat}_1 T_1$, $(X\mathfrak{L})^- \in \mathrm{Lat}_1 T_2$, and $T_1|\mathfrak{L} \sim \sim T_2|(X\mathfrak{L})^-$. Since $T_2 \sim T_1 \sim T_2|(\mathrm{ran}\, X)^-$, and $T_2 \in \mathscr{P}$, it follows that X *is a quasi-affinity.*

Let us suppose that $\mathfrak{L}_1, \mathfrak{L}_2 \in \mathrm{Lat}_1 T_1$, and $\varphi_X(\mathfrak{L}_1) = \varphi_X(\mathfrak{L}_2) = \mathfrak{B}$. By Proposition 1 we infer that $\mathfrak{L} = \mathfrak{L}_1 \vee \mathfrak{L}_2 \in \mathrm{Lat}_1 T_1$, and so we have that $T_1|\mathfrak{L}_i \sim T_2|\mathfrak{B} \sim T_1|\mathfrak{L}$ $(i=1, 2)$. By [10, Corollary 6] it follows that $\mathfrak{L}_i = \mathfrak{L}$ $(i=1, 2)$. Therefore $\mathfrak{L}_1 = \mathfrak{L}_2$, and so φ_X *is an injection.*

Let $\mathfrak{B} \in \mathrm{Lat}_1 T_2$ be an arbitrary subspace. Then for the subspace

$$\mathfrak{L} = \left((X^* \mathfrak{B}^{\perp_1})^-\right)^{\perp_1} \in \mathrm{Lat}_1 T_1$$

we have by Lemma 4 and Proposition 2, that $(X\mathfrak{L})^- = (\mathfrak{B}^{\perp_1})^{\perp_1} = \mathfrak{B}$. Therefore φ_X *is surjective.*

Let $\{\mathfrak{L}_\gamma\}_{\gamma\in\Gamma} \subseteq \mathrm{Lat}_1 T_1$ be an arbitrary system of C_{11}-invariant subspaces. It is obvious that $\left(X\left(\bigvee_{\gamma\in\Gamma} \mathfrak{L}_\gamma\right)\right)^- = \bigvee_{\gamma\in\Gamma} (X\mathfrak{L}_\gamma)^-$. Let $\mathfrak{B}_1$ and $\mathfrak{B}_2$ denote the subspaces $\left(X\left(\overset{(1)}{\bigcap_{\gamma\in\Gamma}} \mathfrak{L}_\gamma\right)\right)^-$ and $\overset{(1)}{\bigcap_{\gamma\in\Gamma}} (X\mathfrak{L}_\gamma)^-$, respectively. On account of Lemma 4 and Proposition 3 we have

$$(X^*\mathfrak{B}_1^{\perp_1})^- = \left(\overset{(1)}{\bigcap_{\gamma\in\Gamma}} \mathfrak{L}_\gamma\right)^{\perp_1} = \bigvee_{\gamma\in\Gamma} \mathfrak{L}_\gamma^{\perp_1} = \bigvee_{\gamma\in\Gamma} \left(X^*((X\mathfrak{L}_\gamma)^-)^{\perp_1}\right)^- = \left(X^*\left(\bigvee_{\gamma\in\Gamma} ((X\mathfrak{L}_\gamma)^-)^{\perp_1}\right)\right)^- =$$

$$= \left(X^*\left(\overset{(1)}{\bigcap_{\gamma\in\Gamma}} (X\mathfrak{L}_\gamma)^-\right)^{\perp_1}\right)^- = (X^*\mathfrak{B}_2^{\perp_1})^-.$$

Since φ_{X^*} is an injection, we infer by Proposition 2 that $\mathfrak{B}_1 = \mathfrak{B}_2$.

4. Multiplicity-free C_{11}-contractions. In this section we prove two corollaries of Theorem 1, concerning multiplicity-free C_{11}-contractions. (An operator $T \in \mathscr{L}(\mathfrak{H})$ is called to be multiplicity-free, if it has a cyclic vector, that is if $\mathfrak{H} = \bigvee_{n\geq 0} T^n h$ for some vector $f \in \mathfrak{H}$. Cf. [14].) The first corollary is an analogue of [14, Theorem 2] about C_0-contractions.

Corollary 1. *For any C_{11}-contraction T, the following properties are equivalent:*

(i) *T is multiplicity-free*

(ii) *There are no different subspaces $\mathfrak{L}, \mathfrak{L}' \in \mathrm{Lat}_1 T$, such that $T|\mathfrak{L}$ is quasi-similar to $T|\mathfrak{L}'$.*

Proof. If T does not belong to the class $\mathcal{P}$, then T has neither property (i), nor property (ii). (Cf. [10, Corollary 1].) Therefore, we may assume that $T\in\mathcal{P}$. Let U be a unitary operator, quasi-similar to T. It can be easily seen that T has a cyclic vector if and only if U does. On the other hand, we infer by Theorem 1 that T and U have property (ii) in the same time. Since for unitary operators (i) and (ii) are obviously equivalent, the Corollary is proved.

Corollary 2. *If T is a multiplicity-free C_{11}-contraction, then* $\mathrm{Lat}_1 T\subseteq \subseteq \mathrm{Hyp\,lat}\, T$.

Proof. On account of Lemmas 1 and 2 we may assume that T is a c.n.u. contraction.

Let $\mathfrak{L}$ be a subspace from $\mathrm{Lat}_1 T$, and let us consider the set $\alpha=\{e^{it}|d_{T|\mathfrak{L}}(e^{it})=1\}$. Let $\mathfrak{L}'\in\mathrm{Lat}_1 T\cap\mathrm{Hyp\,lat}\,T$ denote the subspace corresponding to the set α by [12, Theorem VII.5.2]. It is evident that $d_{T|\mathfrak{L}}(e^{it})=d_{T|\mathfrak{L}'}(e^{it})$, and so $T|\mathfrak{L}\sim T|\mathfrak{L}'$. By Corollary 1 we infer $\mathfrak{L}=\mathfrak{L}'$. Therefore $\mathfrak{L}\in\mathrm{Hyp\,lat}\,T$ follows.

5. Quasi-similarity invariance of $\mathrm{Lat}_1 T\cap\mathrm{Hyp\,lat}\,T$. Now we study behaviour of the hyperinvariant subspaces in $\mathrm{Lat}_1 T$ under quasi-similarities. We need the following lemmas. (Lemmas 5 and 6 are analogues of corresponding statements about C_0-contractions, cf. [4].)

Lemma 5. *Let T be a C_{11}-contraction, and let $\mathfrak{L}\in\mathrm{Lat}\,T$ be an invariant subspace. Then $\mathfrak{L}\in\mathrm{Lat}_1 T$ if and only if $\mathfrak{L}$ is of the form $\mathfrak{L}=(\mathrm{ran}\,Y)^-$ for some $Y\in\{T\}'$.*

Proof.

a) If $\mathfrak{L}=(\mathrm{ran}\,Y)^-$ for some $Y\in\{T\}'$, then $(T|\mathfrak{L})^*\prec T^*|(\ker Y)^\perp$, where $T^*|(\ker Y)^\perp\in C_{1\cdot}$. ($T_1\prec T_2$ means that $\mathcal{I}(T_1, T_2)$ contains a quasi-affinity.) Therefore, it follows that $T|\mathfrak{L}\in C_{11}$.

b) Let us now suppose that $\mathfrak{L}\in\mathrm{Lat}_1 T$. There exist unitary operators U_1 and U, such that $U_1\sim T|\mathfrak{L}$ and $U\sim T$. Since $U_1\sim T|\mathfrak{L}\overset{i}{\prec} T\sim U$, it follows that $U_1\overset{i}{\prec} U$. ($T_1\overset{i}{\prec} T_2$ means that $\mathcal{I}(T_1, T_2)$ contains an injection.) Now we infer by [5, Lemma 4.1] that the subspace $(\mathrm{ran}\,X)^-$ is reducing for U, and U_1 is unitarely equivalent to $U|(\mathrm{ran}\,X)^-$, for any injection $X\in\mathcal{I}(U_1, U)$. Therefore U_1^* can be injected into U^*, and so $(T|\mathfrak{L})^*\sim U_1^*\overset{i}{\prec} U^*\sim T^*$. Let $Z\in\mathcal{I}((T|\mathfrak{L})^*, T^*)$ be an injection, and let $J\in\mathcal{I}(T|\mathfrak{L}, T)$ denote the inclusion of $\mathfrak{L}$ into $\mathfrak{H}$. Then we have that $Y=JZ^*\in\{T\}'$, and $\mathfrak{L}=(\mathrm{ran}\,Y)^-$.

Lemma 6. *If the C_{11}-contractions T_1, T_2 can be injected into each other, then they are quasi-similar.*

Proof. Let U_i be a unitary operator, quasi-similar to T_i $(i=1, 2)$. In virtue of [5, Lemma 4.1] it follows that U_1 and U_2 are unitarely equivalent to direct summands of each other. A simple application of the usual Cantor—Bernstein argument proves, that U_1 and U_2 are unitarely equivalent. (Cf. [9].) Therefore T_1 and T_2 are quasi-similar.

Lemma 7. *Let T_1 and T_2 be quasi-similar contractions of class $C_{11}\cap\mathcal{P}$. Then for any quasi-affinities $X\in\mathcal{I}(T_1,T_2)$, $Y\in\mathcal{I}(T_2,T_1)$ and for any subspace $\mathfrak{L}_1\in\text{Hyp lat } T_1\cap\cap\text{Lat}_1 T_1$ we have that $\mathfrak{L}_2=(X\mathfrak{L}_1)^-\in\text{Hyp lat } T_2\cap\text{Lat}_1 T_2$, and $(Y\mathfrak{L}_2)^-=\mathfrak{L}_1$.*

Proof. Let $\mathfrak{L}_2'\in\text{Hyp lat } T_2$ be the subspace defined by $\mathfrak{L}_2'=\bigvee_{B\in\{T_2\}'}(B\mathfrak{L}_2)^-$. Since $\mathfrak{L}_2\in\text{Lat}_1 T_2$, we infer by Lemma 5 that $\mathfrak{L}_2=(\text{ran } Z)^-$, for a $Z\in\{T_2\}'$. Now, for any $B\in\{T_2\}'$, we have $BZ\in\{T_2\}'$ and $(B\mathfrak{L}_2)^-=(\text{ran}(BZ))^-$, and so again by Lemma 5 $(B\mathfrak{L}_2)^-\in\text{Lat}_1 T_2$. Applying Proposition 1 it follows that $\mathfrak{L}_2'\in\text{Lat}_1 T_2$.

Let $\mathfrak{L}_1'$ denote the subspace $\mathfrak{L}_1'=(Y\mathfrak{L}_2')^-\in\text{Lat}_1 T_1$. Taking into account that $\mathfrak{L}_1\in\text{Hyp lat } T_1$ and, for any $B\in\{T_2\}'$, $YBX\in\{T_1\}'$, we infer that $\mathfrak{L}_1'=(Y\mathfrak{L}_2')^-==(Y(\bigvee_{B\in\{T_2\}'}(BX\mathfrak{L}_1)^-))^-=\bigvee_{B\in\{T_2\}'}(YBX\mathfrak{L}_1)^-\subseteqq\mathfrak{L}_1$. Summerizing these facts, we can write:

$$T_1|\mathfrak{L}_1 \prec T_2|\mathfrak{L}_2 \overset{i}{\prec} T_2|\mathfrak{L}_2' \prec T_1|\mathfrak{L}_1' \overset{i}{\prec} T_1|\mathfrak{L}_1,$$

and all operators occuring here are of class C_{11}. It follows by Lemma 6 that these operators are quasi-similar to each other. Taking into account that $T_2|\mathfrak{L}_2'\in\mathcal{P}$ and $T_1|\mathfrak{L}_1\in\mathcal{P}$ (cf. [10, Cor. 6]), we infer that $\mathfrak{L}_1=\mathfrak{L}_1'$ and $\mathfrak{L}_2=\mathfrak{L}_2'$. Therefore we have that $\mathfrak{L}_2\in\text{Hyp lat } T_2$ and $(Y\mathfrak{L}_2)^-=\mathfrak{L}_1$.

In virtue of the previous Lemma it follows immediately:

Theorem 2. *Let T_1 and T_2 be quasi-similar contractions of class $C_{11}\cap\mathcal{P}$. Then, for every quasi-affinity $X\in\mathcal{I}(T_1,T_2)$, the mapping φ_X: $\text{Hyp lat } T_1\cap\text{Lat}_1 T_1\to\to\text{Hyp lat } T_2\cap\text{Lat}_1 T_2$, φ_X: $\mathfrak{L}\mapsto(X\mathfrak{L})^-$ is a bijection, not depending on the particular choice of X.*

6. Reflexivity of the bicommutant. C. Apostol proved in [1], that if $T\in\mathcal{L}(\mathfrak{H})$ is an operator, quasi-similar to a normal operator, then there exists a basic system $\{\mathfrak{H}_n\}_{n\geqq1}$ of invariant subspaces of T such that $T|\mathfrak{H}_n$ is similar to a normal operator, for every n. We recall that a system $\{\mathfrak{H}_n\}_{n\geqq1}$ of subspaces of $\mathfrak{H}$ is called *basic*, if, for any n, the subspaces $\mathfrak{H}_n$, $(\bigvee_{k\neq n}\mathfrak{H}_k$ are complementary and $\bigcup_{n\geqq1}(\bigvee_{k\geqq n}\mathfrak{H}_k)=\{0\}$. We show that if T is a contraction of class $C_{11}\cap\mathcal{P}$, then the biinvariant subspaces $\mathfrak{H}_n$ can be chosen to be hyperinvariant.

Proposition 5. *Let $T\in\mathcal{L}(\mathfrak{H})$ be a contraction of class $C_{11}\cap\mathcal{P}$, and let $U\in\mathcal{L}(\mathfrak{K})$ be a unitary operator, quasi-similar to T. Then there exist a basic system $\{\mathfrak{H}\}_{n\geqq1}$ of subspaces of $\mathfrak{H}$, and a decomposition $\mathfrak{K}=\bigoplus_{n\geqq1}\mathfrak{K}_n$, such that $\mathfrak{H}_n\in\text{Hyp lat } T$, $\mathfrak{K}_n\in\text{Hyp lat } U$, and $T|\mathfrak{H}_n$ is similar to $U|\mathfrak{K}_n$, for every n.*

Proof. On account of Lemmas 1 and 2 we may assume that T is a c.n.u. contraction. By APOSTOL's theorem [1] there exist a basic system $\{\mathfrak{L}_k\}_{k=1}^{\infty}$ of invariant subspaces of T, and a decomposition $\mathfrak{K}=\bigoplus_{k\ge 1}\mathfrak{B}_k$ of $\mathfrak{K}$ reducing for U, such that, for every k, $T|\mathfrak{L}_k$ is similar to $U|\mathfrak{B}_k$. Let $C_k\in\mathscr{I}(U|\mathfrak{B}_k, T|\mathfrak{L}_k)$ be an affinity $(k=1, 2, \ldots)$.

Since $T\in\mathscr{P}$, we infer by [10, Corollary 2] that $U\in\mathscr{P}$ holds also. Now by [10, Lemma 7] it follows that $m\big(\bigcap_{n\ge 1} C\hat{\sigma}_n\big)=0$, where $\hat{\sigma}_n=C\sigma\big(\bigoplus_{k>n} U|\mathfrak{B}_k\big)$ $(n=1, 2, \ldots)$. (Here and in the sequel $\sigma(T)$ denotes the spectrum of T, and m denotes the normalized Lebesgue measure on the unit circle.) Let σ_n denote the set $\hat{\sigma}_1$, if $n=1$, and $\hat{\sigma}_n\setminus\hat{\sigma}_{n-1}$, if $n>1$. Then the sequence $\{\sigma_n\}_{n\ge 1}$ consists of pairwise disjoint sets, and we have $m\big(C\big(\bigcup_{n\ge 1}\sigma_n\big)\big)=0$. For every n, let $\mathfrak{K}_n$, $\mathfrak{K}'_n$, $\mathfrak{H}_n$, $\mathfrak{H}'_n$ be defined by $\mathfrak{K}_n=\chi_{\sigma_n}(U)\mathfrak{K}=\bigoplus_{k\ge 1}\mathfrak{K}_{n,k}$, $\mathfrak{K}'_n=\chi_{C\sigma_n}(U)\mathfrak{K}=\bigoplus_{k\ge 1}\mathfrak{K}'_{n,k}$, where $\mathfrak{K}_{n,k}=\chi_{\sigma_n}(U|\mathfrak{B}_k)\mathfrak{B}_k$, $\mathfrak{K}'_{n,k}=\chi_{C\sigma_n}(U|\mathfrak{B}_k)\mathfrak{B}_k$ $(k=1, 2, \ldots)$; and $\mathfrak{H}_n=\bigvee_{k\ge 1}\mathfrak{H}_{n,k}$, $\mathfrak{H}'_n=\bigvee_{k\ge 1}\mathfrak{H}'_{n,k}$, where $\mathfrak{H}_{n,k}=C_k\mathfrak{K}_{n,k}$, $\mathfrak{H}'_{n,k}=C_k\mathfrak{K}'_{n,k}$ $(k=1, 2, \ldots)$. It is clear that $\mathfrak{K}_{n,k}=\{0\}$ if $k>n$, and so $\mathfrak{K}_n=\bigoplus_{k=1}^{n}\mathfrak{K}_{n,k}$. It follows that $\mathfrak{H}_{n,k}=\{0\}$ if $k>n$, that is $\mathfrak{H}_n=\mathfrak{H}_{n,1}\dotplus\ldots\dotplus\mathfrak{H}_{n,n}$. It can be easily seen that the subspaces $\mathfrak{H}_n$ and $\mathfrak{H}'_n=\bigvee_{l\ne n}\mathfrak{H}_l$ are complementary, $\mathfrak{H}_n\dotplus\mathfrak{H}'_n=\mathfrak{H}$. Now let n_0 be an arbitrary natural number. It is obvious that, for every n, $\bigvee_{l\ge n}\mathfrak{H}_l\subseteq\big(\bigvee_{k>n_0}\mathfrak{L}_k\big)\dotplus\Big(\bigvee_{l\ge n}\Big(\bigvee_{k=1}^{n_0}\mathfrak{H}_{l,k}\Big)\Big)$, and so it follows: $\bigcap_{n\ge 1}\big(\bigvee_{l\ge n}\mathfrak{H}_l\big)\subseteq\big(\bigvee_{k>n_0}\mathfrak{L}_k\big)\dotplus\Big(\bigcap_{n\ge 1}\Big(\bigvee_{l\ge n}\Big(\bigvee_{k=1}^{n_0}\mathfrak{H}_{l,k}\Big)\Big)\Big)$. Since the mapping $C_1\oplus\ldots\oplus C_{n_0}$: $\mathfrak{B}_1\oplus\ldots\oplus\mathfrak{B}_{n_0}\to\mathfrak{L}_1\dotplus\ldots\dotplus\mathfrak{L}_{n_0}$ is an affinity, we infer that $\bigcap_{n\ge 1}\Big(\bigvee_{l\ge n}\Big(\bigvee_{k=1}^{n_0}\mathfrak{H}_{l,k}\Big)\Big)=\{0\}$. But this implies $\bigcap_{n\ge 1}\big(\bigvee_{l\ge n}\mathfrak{H}_l\big)\subseteq\bigvee_{k>n_0}\mathfrak{L}_k$. Taking into account that n_0 was chosen arbitrarily, it follows that $\bigcap_{n\ge 1}\big(\bigvee_{l\ge n}\mathfrak{H}_l\big)\subseteq\subseteq\bigcap_{n\ge 1}\big(\bigvee_{k\ge n}\mathfrak{L}_k\big)=\{0\}$. Therefore, we have shown that $\{\mathfrak{H}_n\}_{n\ge 1}$ is a *basic system.*

On the other hand, the operator $T|\mathfrak{H}_n$ is similar to $U|\mathfrak{K}_n$, and the operator $T|\mathfrak{H}'_n$ is quasi-similar to $U|\mathfrak{K}'_n$. Let $Y_n\in\mathscr{I}(U|\mathfrak{K}_n, T|\mathfrak{H}_n)$ and $Z_n\in\mathscr{I}(T|\mathfrak{H}'_n, U|\mathfrak{K}'_n)$ be quasi-affinities. Let $X\in\{T\}'$ be an arbitrary operator, and let us consider the matrix of X in the decomposition $\mathfrak{H}=\mathfrak{H}_n\dotplus\mathfrak{H}'_n$: $\begin{bmatrix}X_{11}^{(n)} & X_{12}^{(n)}\\ X_{21}^{(n)} & X_{22}^{(n)}\end{bmatrix}$. The relation $X\in\{T\}'$ implies that $X_{21}^{(n)}\in\mathscr{I}(T|\mathfrak{H}_n, T|\mathfrak{H}'_n)$, and so we have $Z_nX_{21}^{(n)}Y_n\in\mathscr{I}(U|\mathfrak{K}_n, U|\mathfrak{K}'_n)$. In virtue of the definition of subspaces $\mathfrak{K}_n$ and $\mathfrak{K}'_n$ it follows, using [5, Lemma 4.1], that $Z_nX_{21}^{(n)}Y_n=0$, and so we infer $X_{21}^{(n)}=0$. Consequently, the subspace $\mathfrak{H}_n$ is invariant for X. But $X\in\{T\}'$ was arbitrary. therefore we have $\mathfrak{H}_n\in\operatorname{Hyp\,lat} T$. The proof is completed.

Applying this Proposition we show that the bicommutant $\{T\}''$ of every contraction T of class $C_{11}\cap\mathscr{P}$ is a reflexive algebra (cf. [6, ch. 9]). This statement is a certain extension of the von Neumann double commutant theorem, which states that the bicommutant of every normal operator is reflexive (cf. [11, ch. 7]).

Theorem 3. *If T is a contraction of class $C_{11}\cap\mathscr{P}$, then*

$$\operatorname{Alg}\operatorname{Lat}'' T = \{T\}''.$$

(Here $\operatorname{Alg}\operatorname{Lat}'' T$ denotes the weakly closed algebra of operators which leave all the subspaces in $\operatorname{Lat}'' T$, the lattice of biinvariant subspaces of T, invariant.)

Proof. Let us consider the basic system $\{\mathfrak{H}_n\}_{n\ge 1}$ of hyperinvariant subspaces occurring in Proposition 5. Let $A\in\operatorname{Alg}\operatorname{Lat}'' T$ be an arbitrary operator. Since $\mathfrak{H}_n\in\operatorname{Lat}'' T$, we infer that $\mathfrak{H}_n\in\operatorname{Lat} A$, for every n. Let A_n, T_n denote the operators $A_n=A|\mathfrak{H}_n$, $T_n=T|\mathfrak{H}_n$ respectively. It can be easily seen that $\operatorname{Lat}'' T_n\subseteq\operatorname{Lat}'' T$. Therefore we have that $A_n\in\operatorname{Alg}\operatorname{Lat}'' T_n$. Taking into account that T_n is similar to a unitary operator, it follows that $A_n\in\{T_n\}''$, for every n. Since the subspaces $\mathfrak{H}_n$ $(n\ge 1)$ are hyperinvariant, we infer that $A\in\{T\}''$.

7. Behaviour of Lat T **under quasi-similarities.** Theorem 1 does not hold validity replacing $\operatorname{Lat}_1 T_i$ by $\operatorname{Lat} T_i$ $(i=1, 2)$. In fact, in the following example we have $(X\mathfrak{L})^-=\mathfrak{H}_2$, for every subspace $\mathfrak{L}\in\operatorname{Lat} T_1\setminus\operatorname{Lat}_1 T_1 (\neq\emptyset)$.

Example 1. Let U be the operator of multiplication by e^{it} on the space $L^2(C)$, where C denotes the unit circle on the complex field, and we consider the normalized Lebesgue measure on C. Let $\varphi\in L^\infty(C)$ be a function such that $\varphi(e^{it})\neq 0$ a.e., and $\int_C \log|\varphi|\,dm=-\infty$. Then X, the operator of multiplication by $\varphi(e^{it})$, will be a quasi-affinity belonging to $\{U\}'$. Let $\mathfrak{L}$ be an arbitrary non-reducing invariant subspace of U, $\mathfrak{L}\in\operatorname{Lat} U\setminus\operatorname{Lat}_1 U$. Then $\mathfrak{L}$ has the form $\mathfrak{L}=qH^2$, where $q\in L^\infty$ is a function such that $|q(e^{it})|=1$ a.e. (cf. [7, Theorem 3]). In virtue of Szegő's theorem (cf. [8, ch. 4]) it follows that $(\varphi H^2)^-=L^2$, and so we infer that $(X\mathfrak{L})^-=$ $=(\varphi(qH^2))^-=q(\varphi H^2)^-=qL^2=L^2$.

The following Propositions give some informations about the transfer of invariant subspaces, in the case $T_2=U$ is a unitary operator. We recall that an operator U is *completely unitary*, if U is unitary and $\operatorname{Lat}_1 U=\operatorname{Lat} U$, that is every invariant subspace of U is reducing. (Cf. [11, ch. 1.8].)

Proposition 6. *Let $T\in\mathscr{L}(\mathfrak{H})$ be a contraction of class $C_{11}\cap\mathscr{P}$, and let $U\in\mathscr{L}(\mathfrak{K})$ be a unitary operator, quasi-similar to T. Then there exist decompositions $\mathfrak{H}=\mathfrak{H}_1\dotplus\mathfrak{H}_2$ and $\mathfrak{K}=\mathfrak{K}_1\oplus\mathfrak{K}_2$ such that the following properties hold:*

(i) $\mathfrak{H}_i\in\operatorname{Hyp\,lat} T$, $\mathfrak{K}_i\in\operatorname{Hyp\,lat} U$, $T|\mathfrak{H}_i\sim U|\mathfrak{K}_i$ *and* $U|\mathfrak{K}_i$ *is completely unitary for* $i=1, 2$;

(ii) *every operator* $X \in \mathcal{I}(T, U)$ *has a diagonal matrix with respect to these decompositions;*

(iii) $$\mathrm{Lat}_1 T = \mathrm{Lat}_1 (T|\mathfrak{H}_1) \dotplus \mathrm{Lat}_1 (T|\mathfrak{H}_2).$$

Proof. As for the existence of decompositions possessing properties (i) and (ii), see the proof of Proposition 5. (We recall that U is completely unitary, if $\sigma(U) \neq C$. Cf. [11, Th. 1.23].) Let us prove now that property (iii) holds also. If $\mathfrak{L}_i \in \mathrm{Lat}_1 (T|\mathfrak{H}_i)$ $(i=1, 2)$, then we infer by Proposition 1 that $\mathfrak{L} = \mathfrak{L}_1 \dotplus \mathfrak{L}_2 \in \mathrm{Lat}_1 T$. Let us suppose contrary that $\mathfrak{L} \in \mathrm{Lat}_1 T$. It follows by Lemma 5, that there exists an operator $Y \in \{T\}'$ such that $\mathfrak{L} = (Y\mathfrak{H})^-$. Taking into account that $\mathfrak{H}_i \in \mathrm{Hyp\,lat}\, T$ $(i=1, 2)$, we see that $(Y\mathfrak{H})^- = (Y_1\mathfrak{H}_1)^- \dotplus (Y_2\mathfrak{H}_2)^-$, where $Y_i = Y|\mathfrak{H}_i \in \{T|\mathfrak{H}_i\}'$ $(i=1, 2)$. Therefore $(Y_i\mathfrak{H}_i)^- \in \mathrm{Lat}_1 (T|\mathfrak{H}_i)$ $(i=1, 2)$ again by Lemma 5.

Proposition 7. *Let us suppose that the contraction* T *of class* $C_{11} \cap \mathcal{P}$ *is quasi-similar to a completely unitary operator* U, *and* $X \in \mathcal{I}(T, U)$ *is a quasi-affinity. Then, for every subspace* $\mathfrak{L} \in \mathrm{Lat}_1 T$, $(X\mathfrak{L})^- = (X((\mathfrak{L}^{\perp_1})^{\perp}))^-$.

Proof. On account of Lemmas 1 and 2 we may assume that T is a c.n.u. contraction. Let $\mathfrak{L}'$ denote the subspace $\mathfrak{L}' = (\mathfrak{L}^{\perp_1})^{\perp} \in \mathrm{Lat}\, T$. In virtue of the proof of Proposition 2 we infer that $T|_{\mathfrak{L}_0} \in C_{00}$, where $\mathfrak{L}_0 = \mathfrak{L}' \ominus \mathfrak{L} \in \mathrm{Lat}_{\frac{1}{2}} T$. The matrix of the operator $(T|\mathfrak{L}')^n$ with respect to the decomposition $\mathfrak{L}' = \mathfrak{L} \oplus \mathfrak{L}_0$ is of the form

$$(T|\mathfrak{L}')^n = \begin{bmatrix} (T|\mathfrak{L})^n & N^{(n)} \\ 0 & (T|_{\mathfrak{L}_0})^n \end{bmatrix}.$$

Since $XT^n = U^n X$, it follows that $XN^{(n)} f_0 + X(T|_{\mathfrak{L}_0})^n f_0 = U^n X f_0$, for any $f_0 \in \mathfrak{L}_0$.

Let us suppose that $\mathfrak{B}' = (X\mathfrak{L}')^- \neq (X\mathfrak{L})^- = \mathfrak{B}$, and let P denote the orthogonal projection onto the subspace $\mathfrak{B}' \ominus \mathfrak{B}$. Since U is completely unitary, we have that $PU = UP$. The relation $\mathfrak{B}' \neq \mathfrak{B}$ implies, that there exists a vector $f_0 \in \mathfrak{L}_0$ such that $PXf_0 \neq 0$. Now we infer that $\|PU^n X f_0\| = \|U^n PXf_0\| = \|PXf_0\| > 0$, for every n. On the other hand $\|PU^n Xf_0\| = \|PXN^{(n)} f_0 + PX(T|_{\mathfrak{L}_0})^n f_0\| = \|PX(T|_{\mathfrak{L}_0})^n f_0\| \leq \leq \|X\| \, \|(T|_{\mathfrak{L}_0})^n f_0\| \to 0$, if $n \to \infty$. This is a contradiction, and so we get that $\mathfrak{B}' = \mathfrak{B}$.

8. A note on basic systems. Finally we give an example for a basic system $\{\mathfrak{H}_n\}_{n \geq 1}$ with the property that $f \notin \bigvee_{n \geq 1} P_n f$ for some vector f. Here P_n denotes the projection onto the subspace $\mathfrak{H}_n$, corresponding to the decomposition $\mathfrak{H} = \mathfrak{H}_n \dotplus \dotplus \left(\bigvee_{k \neq n} \mathfrak{H}_k\right)$. This fact strongly limits the usefulness of Proposition 5.

Example 2. Let $\{\varphi_n\}_{n=1}^{\infty} \cup \{\psi_n\}_{n=1}^{\infty} \cup \{f\}$ be an orthonormal basis in the Hilbert space $\mathfrak{H}$, and let $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ be sequences of positive real numbers. We define the vectors g_1, h_1, h_0' by

$$g_1=\alpha_1\varphi_1+\frac{1}{2}f,\quad h_1=-\alpha_1\varphi_1+\frac{1}{2}f,\quad h_0'=0,$$

and the subspaces $\mathfrak{H}_1$, $\mathfrak{H}_1'$ by

$$\mathfrak{H}_1=g_1\vee h_0',\quad \mathfrak{H}_1'=h_1\vee(\bigvee_{n\ge 2}\varphi_n)\vee(\bigvee_{n\ge 1}\psi_n).$$

Let $n\ge 1$ be an arbitrary integer, and let us assume that, for every natural number $k\le n$, the vectors g_k, h_k, h_{k-1}', and the subspaces $\mathfrak{H}_k$, $\mathfrak{H}_k'$ have been already introduced. Then the vectors g_{n+1}, h_{n+1}, h_n', and the subspaces $\mathfrak{H}_{n+1}$, $\mathfrak{H}_{n+1}'$ will be defined by the following equalities:

$$g_{n+1}=\alpha_{n+1}\varphi_{n+1}+\frac{1}{2}h_n,\quad h_{n+1}=-\alpha_{n+1}\varphi_{n+1}+\frac{1}{2}h_n,\quad h_n'=h_n+\beta_n\psi_n,$$

$$\mathfrak{H}_{n+1}=g_{n+1}\vee h_n',\quad \mathfrak{H}_{n+1}'=h_{n+1}\vee(\bigvee_{k\ge n+2}\varphi_k)\vee(\bigvee_{k\ge n+1}\psi_k).$$

A straightforward computation shows that $\bigvee_{k\ge 1}\mathfrak{H}_k=\mathfrak{H}$, and $\bigvee_{k\ge n+1}\mathfrak{H}_k=\mathfrak{H}_n'$ for every n, provided the sequence $\{\beta_n\}_{n=1}^{\infty}$ tends to zero. This implies that, for every n, $\mathfrak{H}=\mathfrak{H}_n\dot{+}(\bigvee_{k\ne n}\mathfrak{H}_k)$. We can easily verify also that $\bigcap_{n\ge 1}(\bigvee_{k\ge n+1}\mathfrak{H}_k)=\bigcap_{n\ge 1}\mathfrak{H}_n'=\{0\}$, if the series $\sum_{n=1}^{\infty}4^n\alpha_n^2$ is not convergent.

Let $\{\varepsilon_n\}_{n=1}^{\infty}$ be a sequence of positive numbers, such that $\sum_{n=1}^{\infty}\varepsilon_n<\frac{1}{2}$. Let us now define the sequence $\{\alpha_n\}_{n=1}^{\infty}$ such that the following inequalities hold: $\alpha_1^2>\frac{1}{4\varepsilon_1}$, and

$$\alpha_n^2>\frac{1}{\varepsilon_n}\left(\frac{1}{4^n}+\frac{\alpha_1^2}{4^{n-1}}+\ldots+\frac{\alpha_{n-1}^2}{4}\right),$$

for every $n>1$. It is evident that in this case $\sum_{n=1}^{\infty}4^n\alpha_n^2=\infty$. Let us assume that the sequence $\{\beta_n\}_{n=1}^{\infty}$ tends to zero. Then the system $\{\mathfrak{H}_n\}_{n=1}^{\infty}$ will be basic, and $P_nf=g_n$, for every n.

Let g_n' and χ_n be the vectors, defined by

$$g_n'=\frac{g_n}{\|g_n\|}=\varphi_n+\chi_n\quad (n=1,2,\ldots).$$

After a short computation we conclude that $\|\chi_n\|^2<2\varepsilon_n$ for every n, and so $\sum_{n=1}^{\infty}\|\chi_n\|^2<1$.

Now let $a_1, \ldots, a_n$ be arbitrary complex numbers, where $n \geqq 1$ is an arbitrary natural number. Then we have

$$\left\|f - \sum_{i=1}^{n} a_i g_i'\right\| = \left\|f - \sum_{i=1}^{n} a_i \varphi_i - \sum_{i=1}^{n} a_i \chi_i\right\| \geqq$$

$$\geqq \left\|f - \sum_{i=1}^{n} a_i \varphi_i\right\| - \left\|\sum_{i=1}^{n} a_i \chi_i\right\| \geqq \left\|f - \sum_{i=1}^{n} a_i \varphi_i\right\| -$$

$$- \sum_{i=1}^{n} |a_i| \|\chi_i\| \geqq \left[1 + \sum_{i=1}^{n} |a_i|^2\right]^{\frac{1}{2}} - \left[\sum_{i=1}^{n} |a_i|^2\right]^{\frac{1}{2}} \left[\sum_{i=1}^{\infty} \|\chi_i\|^2\right]^{\frac{1}{2}}.$$

Taking into account that $\inf\{(1+x)^{\frac{1}{2}} - \varepsilon x^{\frac{1}{2}} | x \geqq 0\} > 0$, if $0 < \varepsilon < 1$, we infer that $\left\|f - \sum_{i=1}^{n} a_i g_i'\right\| \geqq \delta$ for some $\delta > 0$, independent on n, and on the numbers $a_1, \ldots, a_n$. Therefore $f \notin \bigvee_{n \geqq 1} g_n' = \bigvee_{n \geqq 1} g_n$, that is $f \notin \bigvee_{n \geqq 1} P_n f$.

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