Acta Sci. Math., 43 (1981), 295—299

Classes of universal algebras, their non-factors and periodic rings

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Certain classes of algebras are determined by a family of their "non-factors": a lattice is modular iff it does not contain a copy of the pentagon (the five element non-modular lattice), a lattice is distributive iff it does not contain a copy of either the pentagon or the diamond (the five element modular non-distributive lattice). We characterize such classes as the classes of algebras closed under the formation of subalgebras, homomorphic images and direct limits. We also specify this characterization for the class of rings whose multiplicative semigroups are periodic. We follow the notations and terminology of G. GRÄTZER [1].

Definition 1 (GRÄTZER [1, p. 129]). A *direct family* of algebras $\mathcal{A}$ is defined to be a triplet of the following objects:

(i) a directed partially ordered set $\langle I; \leqq \rangle$;

(ii) algebras $A_i = \langle A_i, F \rangle$, $i \in I$ of some fixed type;

(iii) homomorphisms ψ_{ij} of A_i into A_j for all $i \leqq j$ such that $\psi_{ij}\psi_{jk} = \psi_{ik}$ if $i \leqq j \leqq k$ and ψ_{ii} is the identity mapping for all $i \in I$.

$x \equiv y$ iff $x \in A_i$, $y \in A_j$ and there is $k \geqq i, j$ and $x\psi_{ik} = y\psi_{jk}$ is an equivalence relation on $A = \bigcup \{A_i | i \in I\}$. $A/\equiv$ is denoted by A_∞. The operation f_γ on A_∞ are defined as follows: Let $x_j \in A_{i_j}$; $0 \leqq j < n_\gamma$, and let $m \geqq i_j$ for all $0 \leqq j < n$. Then $x_j \equiv x'_j \in A_m$ where $x'_j = x_j \psi_{i_j m}$. $f_\gamma(\hat{x}_0, \ldots, \hat{x}_{n-1}) = \hat{f}_\gamma(x'_0, \ldots, x'_{n_\gamma - 1})$, where $\hat{x} = [x]\equiv$. The definition of f_γ does not depend on m.

The algebra $\mathfrak{A}_\infty = \langle A_\infty, F \rangle$ is called the direct limit of the direct family of algebras $\mathcal{A}$ and is denoted by $\varinjlim \mathcal{A}$.

Definition 2. An algebra $\mathfrak{A}$ is said to be a *factor* of an algebra $\mathfrak{B}$ if $\mathfrak{A}$ is a homomorphic image of a subalgebra of $\mathfrak{B}$.

Received February 8, 1980, and in revised form April 27, 1981.

Definition 3. Let $\mathbf{V}$ be a class of algebras of a fixed type and let $\mathbf{L}$ be a subclass of $\mathbf{V}$. $\mathbf{N}(\mathbf{V}, \mathbf{L})$ is the class of all algebras of $\mathbf{V}$ no factor of which belongs to $\mathbf{L}$.

Theorem 1. *Let* $\mathbf{V}$ *be a variety of algebras of a fixed type and let* $\mathbf{K} \subseteqq \mathbf{V}$. $\mathbf{K}$ *is closed under the formation of subalgebras, homomorphic images and direct limits iff* $\mathbf{K} = \mathbf{N}(\mathbf{V}, \mathbf{L})$ *for some class* $\mathbf{L}$ *of finitely generated algebras of* $\mathbf{V}$.

The proof will be based on two lemmas.

Lemma 2. If $\mathfrak{A}_\infty = \varinjlim \mathscr{A}$, *where* $\mathscr{A}$ *is a direct family of algebras as in Definition* 1 *and* $\mathfrak{B}$ *is a subalgebra of* $\mathfrak{A}_\infty$, *then* $\mathfrak{B} = \varinjlim \mathscr{B}$ *where* $\mathscr{B}$ *is the direct family of subalgebras* $\mathfrak{B}_i$ *of* $\mathfrak{A}_i$, $i \in I$.

Proof. Let $\mathfrak{B}_i = \{x | x \in A_i, \hat{x} \in B\}$, $i \in I$. Then $\mathfrak{B}_i$ is a subalgebra of $\mathfrak{A}_i$ and $\mathfrak{B}$ is the direct limit of $\mathfrak{B}_i$, $i \in I$ where the homomorphisms $\bar{\psi}_{ij}$ are the restrictions of ψ_{ij} to $\mathfrak{B}_i$.

Lemma 3. *If* $\mathfrak{A}_\infty = \varinjlim \mathscr{A}$, *where* $\mathscr{A}$ *is a direct family of algebras as in Definition* 1 *and* $\mathfrak{B}$ *is a finitely generated homomorphic image of* $\mathfrak{A}_\infty$, *then* $\mathfrak{B}$ *is a homomorphic image of* $\mathfrak{A}_i$ *for some* $i \in I$.

Proof. Let $\mathfrak{B}$ be generated by $\{b_k | 0 \leqq k < n\}$ and let α be the homomorphism of $\mathfrak{A}_\infty$ onto $\mathfrak{B}$. Then there are $\hat{a}_k \in \mathfrak{A}_\infty$ such that $\bar{a}_k \alpha = b_k$, $0 \leqq k < n$. Let $a_k \in A_{i_k}$ and let $m \geqq i_k$, $0 \leqq k < n$. The composition of the natural homomorphism of $\mathfrak{A}_m$ into $\mathfrak{A}_\infty$ and α is a homomorphism of $\mathfrak{A}_m$ onto $\mathfrak{B}$.

Proof of Theorem 1. Let $\mathbf{K} \subseteqq \mathbf{V}$ be closed under the formation of subalgebras, homomorphic images and direct limits. Let $\mathbf{L}$ be the class of all finitely generated algebras of $\mathbf{V}$ not belonging to $\mathbf{K}$. It is clear that $\mathbf{K} \subseteqq \mathbf{N}(\mathbf{V}, \mathbf{L})$. Let $\mathfrak{A} \in \mathbf{N}(\mathbf{V}, \mathbf{L})$. Then every finitely generated subalgebra of $\mathfrak{A}$ belongs to $\mathbf{K}$ (since no factor of $\mathfrak{A}$ is in $\mathbf{L}$). But $\mathfrak{A}$ is the direct limit of its finitely generated subalgebras (cf. [1 p. 130]). Since $\mathbf{K}$ is closed under direct limits $\mathfrak{A} \in \mathbf{K}$.

Conversely let $\mathbf{L}$ be class of finitely generated algebras of $\mathbf{V}$ and $\mathbf{K} = \mathbf{N}(\mathbf{V}, \mathbf{L})$. From the definition of $\mathbf{N}(\mathbf{V}, \mathbf{L})$ it is clear that $\mathbf{K}$ is closed under the formation of subalgebras and homomorphic images. Let $\mathfrak{A}_\infty = \varinjlim \mathscr{A}$, where $\mathscr{A}$ is a direct family of algebras $\mathfrak{A}_i \in \mathbf{K}$, $i \in I$. Let $\mathfrak{C}$ be a finitely generated factor of $\mathfrak{A}_\infty$. Then $\mathfrak{C}$ is a homomorphic image of subalgebra $\mathfrak{B}$ of $\mathfrak{A}_\infty$. By Lemma 2, $\mathfrak{B}$ is the direct limit of subalgebras $\mathfrak{B}_i$ of $\mathfrak{A}_i$, $i \in I$, and by Lemma 3, $\mathfrak{C}$ is a homomorphic image of $\mathfrak{B}_i$ for some $i \in I$. Thus $\mathfrak{C}$ is a factor of $\mathfrak{A}_i \in \mathbf{K}$. Hence $\mathfrak{C} \notin \mathbf{L}$. Thus $\mathfrak{A}_\infty \in \mathbf{K}$.

If $\mathbf{S}$ denotes the variety of all semigroups and $\mathbf{G}$ denotes the variety of all groups, then $\mathbf{N}(\mathbf{G}, \{\mathfrak{C}\})$ is the class of all periodic groups and $\mathbf{N}(\mathbf{S}, \{\mathfrak{N}\})$ is the class of all periodic semigroups. $\mathfrak{C}$ is an infinite cyclic group and $\mathfrak{N}$ is the additive semigroup of positive integers.

Lemma 4. *Let* $\mathbf{V}$ *be a variety of algebras of a fixed type. Suppose* $\mathbf{L}_1 \subseteq \mathbf{L}_2 \subseteq \mathbf{V}$ *and* $\mathbf{L}_2$ *is a class of finitely generated algebras such that every member of* $\mathbf{L}_2$ *has a factor in* $\mathbf{L}_1$. *Then* $\mathbf{N}(\mathbf{V}, \mathbf{L}_1) = \mathbf{N}(\mathbf{V}, \mathbf{L}_2)$.

In fact $\mathbf{L} \to \mathbf{N}(\mathbf{V}, \mathbf{L})$ gives a Galois connection between classes of finitely generated algebras of $\mathbf{V}$ and classes of algebras of $\mathbf{V}$ closed under the formation of subalgebras, homomorphic images and direct limits. Thus $\mathbf{N}(\mathbf{V}, \mathbf{L}_2) \subseteq \mathbf{N}(\mathbf{V}, \mathbf{L}_1)$. If $\mathfrak{A} \in \mathbf{N}(\mathbf{V}, \mathbf{L}_1)$, no factor of $\mathfrak{A}$ belongs to $\mathbf{L}_2$, as no factor of $\mathfrak{A}$ belongs to $\mathbf{L}_1$ and every member of $\mathbf{L}_2$ has a factor in $\mathbf{L}_1$.

The following result was proved in [3].

Theorem 5. *The following conditions on an associative ring* $\mathfrak{A}$ *are equivalent:*
(i) *for every* $a \in A$, *there is a positive integer* r *and a polynomial* $h(t)$ *with integral coefficients such that* $a^r + a^{r+1} h(a) = 0$.
(ii) *every element* $a \in A$ *generates a finite semigroup under multiplication.*

A ring satisfying the conditions of Theorem 5 is called periodic [3]. Thus a periodic ring is a ring whose multiplicative semigroup is periodic.

The following result establishes a characterization of the class of all periodic rings similar to that of periodic groups and semigroups given in the comments before Lemma 4.

Theorem 6. *Let* $\mathbf{L}$ *be the set of all quotient rings of* $xZ[x]$ *by* $xh(x)Z[x]$ *where* $h(x)$ *is an irreducible polynomial of* $Z[x]$ *and* $|h(0)| > 1$. *Then* $\mathbf{N}(\mathbf{R}, \mathbf{L})$ *is the class of all periodic rings, where* $\mathbf{R}$ *is the variety of all associative rings.*

The proof depends on a number of lemmas.

Lemma 7. *A ring* $\mathfrak{A}$ *is periodic iff both* $T(\mathfrak{A})$ *and* $\mathfrak{A}/T(\mathfrak{A})$ *are periodic;* $T(\mathfrak{A})$ *is the torsion ideal of* $\mathfrak{A}$.

Proof. If $\mathfrak{A}$ is periodic, then every factor of $\mathfrak{A}$ is periodic. Let $T(\mathfrak{A})$ and $\mathfrak{A}/T(\mathfrak{A})$ be periodic and $a \in \mathfrak{A}$. Then there is a positive integer r and a polynomial $h(t) \in Z[t]$ such that $b = a^r + a^{r+1} h(a) \in T(\mathfrak{A})$. There is $s > 0$ and $g(t) \in Z[t]$ such that $b^s + b^{s+1} g(b) = 0$. I.e. $(a^r + a^{r+1} h(a))^s + (a^r + a^{r+1} h(a))^{s+1} g(a^r + a^{r+1} h(a)) = 0$. Hence $a^{rs} + a^{rs+1} H(a) = 0$ for some $H(t) \in Z[t]$.

Lemma 8 (Rédei [5]). *All rings generated by one element are (to within an isomorphism)* $xZ[x]/xd(x)B$, *where* $d(x)$ *runs through the polynomials from* $Z[x]$ *with positive leading coefficient and* $B = Z[x]$ *or* B *runs through the primitive ideals of* $Z[x]$; B *is primitive if* B *is not the product of a principal proper ideal of* $Z[x]$ *and another ideal of* $Z[x]$. *Every primitive ideal of* $Z[x]$ *contains positive integers.*

Lemma 9 (Lewin [4]). *A subring of finite index in a finitely generated ring is finitely generated.*

Lemma 10. *The class of periodic rings is* $\mathbf{N}(\mathbf{R}, \mathbf{M})$ *where* $\mathbf{M}$ *is the set of all quotients of* $xZ[x]$ *by* $x^n h(x)$, n *is a positive integer*, x *does not divide* $h(x)$ *and* $h(x)$ *does not divide* $1-x^t$ *for any* t.

Proof. If $\mathfrak{A}$ is periodic, then no factor of $\mathfrak{A}$ belongs to $\mathbf{M}$, since the element $x+x^n h(x)Z[x]$ generates an infinite semigroup in $xZ[x]/x^n h(x)Z[x]$; otherwise $x^r - x^{r+s} \in x^r h(x)Z[x]$, implying $h(x)|x^r - x^{r+s}$ for some $r, s>0$, i.e. $h(x)|1-x^s$. Conversely if $\mathfrak{A} \in \mathbf{N}(\mathbf{R}, \mathbf{M})$, then $\mathfrak{A}$ is periodic, otherwise $\mathfrak{A}/T(\mathfrak{A})$ or $T(\mathfrak{A})$ is not periodic, by Lemma 7.

Case 1. $\mathfrak{C} = \mathfrak{A}/T(\mathfrak{A})$ is not periodic. By Theorem 5, there is $b \in \mathfrak{C}$ generating an infinite semigroup. The subring $\mathfrak{D}$ of $\mathfrak{C}$ generated by b is isomorphic to $xZ[x]/x^n h(x)B$ where $h(0) \neq 0$ and $B = Z[x]$ or B is a primitive ideal of $Z[x]$ (Lemma 8). As B contains positive integers, let $m \in Z$, $m>1$, $m \in B$ but $1 \notin B$. Then $mb^n h(b)=0$ in $\mathfrak{C}$ but $b^n h(b) \neq 0$ in $\mathfrak{C}$. I.e. $b^n h(b) \neq 0$ is a torsion element in $\mathfrak{A}/T(\mathfrak{A})$. Hence $1 \in B$ and $B = Z[x]$. Thus $\mathfrak{D} \cong xZ[x]/x^n h(x)Z[x]$. Since b generates an infinite semigroup $x^s - x^{s+t} \notin x^n h(x)Z[x]$ for any $s, t>0$. Thus $h(x)$ does not divide $1-x^t$ for any $t>0$, i.e. $\mathfrak{D}$ is isomorphic to a member of $\mathbf{M}$. Hence a factor of $\mathfrak{A}$ is in $\mathbf{M}$.

Case 2. $T(\mathfrak{A})$ is not periodic. There is $b \in T(\mathfrak{A})$ generating an infinite semigroup $\mathfrak{S}$. If m is the characteristic of b and $\mathfrak{C}$ is the subring of $T(\mathfrak{A})$ generated by b, then m is the characteristic of $\mathfrak{C}$ and $\mathfrak{C} = \mathfrak{C}_1 \oplus \ldots \oplus \mathfrak{C}_k$ where $m = p_1^{n_1} \ldots p_k^{n_k}$ is the prime factorization of m (cf. McCoy [2]). If $b = b_1 + \ldots + b_k (b_i \in \mathfrak{C}_i)$ and b_i generates a semigroup $\mathfrak{S}_i$ under multiplication, then $\mathfrak{C}_i$ is generated by b_i and $\mathfrak{S} \subseteqq \mathfrak{S}_1 \times \ldots \times \mathfrak{S}_k$. Hence at least one of $\mathfrak{S}_1, \ldots, \mathfrak{S}_k$ is infinite. Thus there is $d \in T(\mathfrak{A})$ of characteristic p^n where p is a prime and $n>0$ such that d generates an infinite semigroup. Let $\mathfrak{D}$ be the subring of $T(\mathfrak{A})$ generated by d. Then $p^n \mathfrak{D} = 0$. We claim that $\mathfrak{D}/p\mathfrak{D}$ is infinite. If $\mathfrak{D}/p\mathfrak{D}$ is finite, then $p\mathfrak{D}$ is of finite index in the finitely generated ring $\mathfrak{D}$. By Lemma 9 $p\mathfrak{D}$ is finitely generated. But $p\mathfrak{D}$ is nilpotent and of characteristic p^{n-1}. Hence $p\mathfrak{D}$ is finite and so $\mathfrak{D}$ is a finite ring ($|\mathfrak{D}| = |p\mathfrak{D}||\mathfrak{D}/p\mathfrak{D}|$) contradicting the assumption that d generates an infinite semigroup. Hence $\mathfrak{D}/p\mathfrak{D}$ is an infinite ring of prime characteristic p and is generated by one element. Hence $\mathfrak{D}/p\mathfrak{D} \cong$ $\cong xZ_p[x]/J$ where J is an ideal of $xZ_p[x]$. But all ideals of $Z_p[x]$ are principal. Hence $J = x^n h(x) Z_p[x]$. If $h(x) \neq 0$, then every element in $xZ_p[x]/J$ can be written in the form $a_1 x + a_2 x^2 + \ldots + a_r x^r$ where $r = n+k-1$, k=degree of $h(x)$ and $a_1, \ldots, a_r \in Z_p$. I.e., if $h(x) \neq 0$ $\mathfrak{D}/p\mathfrak{D}$ is finite. Thus $\mathfrak{D}/p\mathfrak{D} \cong xZ_p[x] \cong xZ[x]/pxZ[x]$. This shows that a factor of $\mathfrak{A}$ is in $\mathbf{M}$.

Lemma 11. *Let* $h(x) \in Z[x]$, $h(x) \neq x$, $h(x) \neq -x$ *and let* $h(x)$ *be irreducible.* $h(x)$ *does not divide* $1-(xq(x))^t$ *for any* $q(x) \in Z[x]$ *and any positive integer* t *iff* $|h(0)|>1$.

Proof. If $|h(0)|>1$, then $h(x)=m+xg(x)$ where $m\in Z$, $|m|>1$ and $g(x)\in Z[x]$. If $h(x)|1-(xq(x))^t$, then $1-(xq(x))^t=h(x)f(x)$. Thus $1-0=mf(0)$ which is impossible if $|m|>1$. Conversely, if $h(x)\neq\pm x$ and $h(x)$ is irreducible, then $h(0)=m\neq 0$. If $|m|=1$, then $\pm h(x)=1+xg(x)$. Hence $h(x)|1-(-xg(x))$.

Proof of Theorem 6. By Lemma 11, $\mathbf{L}\subseteq\mathbf{M}$. By Lemma 4 we need to show that every member of $\mathbf{M}$ has a factor in $\mathbf{L}$. Let $h(x)$ be an irreducible divisor of $g(x)$ where $g(x)$ is not divisible by x, and $g(x)$ does not divide $1-x^r$ for any $r>0$. If $h(x)\nmid 1-(xq(x))^t$ for any $t>0$ and $q(x)\in Z[x]$, then the ring $xZ[x]/xh(x)Z[x]$ is a homomorphic image of $xZ[x]/x^n g(x)Z[x]$.

If $h(x)|1-(xq(x))^t$ for some $t>0$ and $q(x)\in Z[x]$, then $xq(x)-(xq(x))^{t+1}=$ $=h(x)xq(x)f(x)$. Set $I=xh(x)Z[x]$ and $a=xq(x)+I$. Then $a=a^{t+1}$ in $xZ[x]/I=$ $=\mathfrak{A}$. Hence $a^t=e$ is an idempotent in $\mathfrak{A}$. Further, e is of characteristic 0. If $me=0$ for $m>0$, then $mxq(x)\in I$. Hence $h(x)|mq(x)$. But $h(x)\nmid q(x)(1-(xq(x))^t$ is divisible by $h(x)$. Hence $h(x)|m$. This is in contradiction with $h(x)\nmid 1-x^r$ for any $r>0$ and $h(x)|1-(xq(x))^t$ for some $t>0$ and $q(x)\in Z[x]$. Hence e generates a subring of $\mathfrak{A}$ isomorphic to Z. Thuss $2e$ generates a subring of $\mathfrak{A}$ isomorphic to $xZ[x]/x(x-2)Z[x]$. Now $x-2$ is irreducible, $|-2|>1$. Thus a factor of $xZ[x]/x^n g(x)Z[x]$ is in $\mathbf{L}$.

Corollary 12. *A ring $\mathfrak{A}$ is either periodic or a factor $\mathfrak{B}$ of $\mathfrak{A}$ is such that every nonzero member generates an infinite semigroup.*

This follows from Theorem 6. If $b\in\mathfrak{B}=xZ[x]/I$ where $I=xh(x)Z[x]$ and $h(x)$ is irreducible and $|h(0)|>1$, $b\neq 0$. Thus, b generates an infinite semigroup. Since $b^r-b^{r+s}=0$ iff $b=xq(x)+I$ and $(xq(x))^r-(xq(x))^{r+s}\in I$. Hence

$$h(x)|(xq(x))^r[1-x(q(x))^s],$$

$h(x)\nmid(xq(x))^r$ since $b\neq 0$. Thus $h(x)|1-(xq(x))^s$ contradicting Lemma 1.1

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