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Jordan models and diagonalization of the characteristic function

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Introduction

The study of Jordan models of some classes of operators on an infinite-dimensional Hilbert space was started by B. SZ.-NAGY and C. FOIAŞ in [8], where the existence of Jordan model was proved for C_0 contractions with finite defect indices. This result was generalized in [1] for general C_0 contractions.

Another approach to these questions is to use some sort of diagonalization of the characteristic function. This method which was developed by E. A. NORDGREN and B. MOORE III in [4], [10] has the advantage that it gives also some description of the functions appearing in Jordan models. Extensions and further applications of this approach were given in [9], [6] and [5].

The aim of this paper is to continue these investigations. In the first section we deal with $C_{\cdot 0}$ contractions (i.e. $T^{*n}\to 0$ strongly) and show what remains valid from the Jordan model in the general case.

In the second section we give a new proof of the existence of Jordan models for general C_0 contractions (see [1]). In the same time we prove again relations for the functions appearing in the Jordan model (see [5]).

We use the usual notation (see [6], [7]). By E_n $(0\leqq n\leqq\infty)$ we denote the n-dimensional complex Hilbert space. $\mathcal{M}(m, n)$ $(1\leqq m, n\leqq\infty)$ means the set of all $m\times n$ matrices $A=(a_{ij})$ over H^∞ for which the corresponding analytic operator valued function (E_n, E_m, A) is bounded, i.e. $\|A(\lambda)\|\leqq K$ for some constant K independent of λ on the open unit disc D. Instead of $\mathcal{M}(n, n)$ we also write shortly $\mathcal{M}(n)$.

For $A\in\mathcal{M}(m, n)$ and a natural number $r\leqq\min(m, n)$ we define $\mathcal{D}_r(A)$ as the largest common inner divisor of all minors of A of order r. The invariant factors $\mathcal{E}_r(A)$ are defined by $\mathcal{E}_1(A)=\mathcal{D}_1(A)$ and $\mathcal{E}_r(A)=\mathcal{D}_r(A)/\mathcal{D}_{r-1}(A)$ for $r\geqq 2$ (we put $\mathcal{E}_r(A)=0$ if $\mathcal{D}_r(A)=0$).

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For $A\in\mathcal{M}(m,n)$ inner we define the operator $S(A)$ on the Hilbert space $\mathfrak{H}(A)= =H^2(E_n)\ominus AH^2(E_n)$ by $S(A)u=P_{\mathfrak{H}(A)}(\lambda u)$.

If T is an operator on $\mathfrak{H}$ and T' is an operator on $\mathfrak{H}'$ we write $T\overset{i}{\prec}T'$ if there exists an injective operator $X\colon \mathfrak{H}\to\mathfrak{H}'$ such that $XT=T'X$. If X can be chosen such that $\overline{X\mathfrak{H}}=\mathfrak{H}'$ we write $T\prec T'$. T and T' are called quasisimilar ($T\sim T'$) if $T\prec T'$ and $T'\prec T$.

I.

We start with the following version of a lemma of M. SHERMAN (the proof is the same as in [6]).

Lemma 1. *Let $h\in H^\infty$, $\omega_1,\omega_2,\ldots\in H^\infty$ inner, $\varepsilon>0$. Then there exists a complex number x, $|x|<\varepsilon$ such that $(h+x)\wedge\omega_j=1$ $(j=1,2,\ldots)$.*

Lemma 2 is an easy modification of the Main Lemma of [6]:

Lemma 2. *Let $f_{ik}\in H^\infty$, $i,k=1,2,\ldots$, $|f_{ik}|\leqq M$ for some constant M. Let $h_1,h_2,\ldots\in H^\infty$ satisfy $\sum_{i=1}^\infty |h_i(\lambda)|\leqq M'$ where M' is a constant independent of $\lambda\in D$. Let $\omega_1,\omega_2,\ldots\in H^\infty$ be inner and $\varepsilon>0$. Then there exist complex numbers $x_1,x_2,\ldots$ such that $\sum_{i=1}^\infty |x_i|<\varepsilon$, $(h_1+x_1)\wedge\omega_j=1$ $(j=1,2,\ldots)$ and $\sum_{k=1}^\infty (h_k+x_k)f_{ik}=\left(\bigwedge_{k=1}^\infty f_{ik}\right)r_i$, where $r_i\wedge\omega_j=1$ for $i,j=1,2,\ldots$.*

Proof. By Lemma 1 we can find an $x_1\in\mathbf{C}$, $|x_1|<\varepsilon/2$ such that $(h_1+x_1)\wedge\omega_j=1$ for $j=1,2,\ldots$, $h_1+x_1\not\equiv 0$ and $(h_1+x_1)\wedge f_{i,2}=1$ $(i=1,2,\ldots)$. For $i=1,2,\ldots$ denote $g_i=(h_1+x_1)f_{i,1}+\sum_{k=2}^\infty h_k f_{ik}$. Obviously $g_i\in H^\infty$ and $g_i\wedge\bigwedge_{k=2}^\infty f_{ik}=\bigwedge_{k=1}^\infty f_{ik}$. By the Main Lemma of [6] applied to the functions g_i/d_i, f_{ik}/d_i where $d_i=\bigwedge_{k=1}^\infty f_{ik}$ there exists a sequence of complex numbers $x_2,x_3,\ldots$ such that $\sum_{i=2}^\infty |x_i|<\varepsilon/2$ and $g_i+\sum_{k=2}^\infty x_k f_{ik}=\left(g_i\wedge\bigwedge_{k=2}^\infty f_{ik}\right)r_i=\left(\bigwedge_{k=1}^\infty f_{ik}\right)r_i$ where $r_i\wedge\omega_j=1$ $(i,j=1,2,\ldots)$. At the same time

$$g_i+\sum_{k=2}^\infty x_k f_{ik}=\sum_{k=1}^\infty (h_k+x_k)f_{ik}.$$

Lemma 3 is a modification of Theorem 1, [6].

Lemma 3. *Let $A\in\mathcal{M}(m,n)$, $1\leqq m,n\leqq\infty$. Let $\omega_1,\omega_2,\ldots\in H^\infty$ be inner. Then there exists $\varphi\in H^\infty$, $\varphi\wedge\omega_j=1$ $(j=1,2,\ldots)$ and $\Delta\in\mathcal{M}(m)$, $\Lambda\in\mathcal{M}(n)$ having the scalar multiple φ such that $A\Lambda=\Delta B$, where B has the form $B=\operatorname{diag}(\mathscr{E}_1(A),A_1)$, $A_1\in\mathcal{M}(m-1,n-1)$ and $\mathscr{E}_1(A)|A_1$.*

Moreover if $\{h_i\}_{i=1}^n$ *is a sequence of functions from* H^∞ *satisfying* $\sum_{i=1}^n |h_i(\lambda)| \leqq M$ *for some constant* M *independent on* $\lambda \in D$ *and* $\varepsilon > 0$, *then we can choose* Δ, Λ *and* B *in such a way that the first column* $\{g_i\}_{i=1}^n$ *of the matrix* A *satisfies the relation*

$$\sum_{i=1}^n |h_i(\lambda) - g_i(\lambda)| < \varepsilon \quad (\lambda \in D).$$

Lemma 3 differs from Theorem 1, [6] only in the last statement. The proof proceeds in the same way as in [6] using Lemma 2 instead of the Main lemma of [6]. Therefore we omit it.

The last statement will be used in the second section only.

Lemma 4. *Let* $A, B \in \mathcal{M}(m, n)$, $1 \leqq m, n \leqq \infty$, *let* $\Delta \in \mathcal{M}(m)$ *and* $\Lambda \in \mathcal{M}(n)$ *have a scalar multiple* $\varphi \in H^\infty$ *i.e.* $\Delta\Delta^a = \Delta^a\Delta = \varphi I_m$, $\Lambda\Lambda^a = \Lambda^a\Lambda = \varphi I_n$ *for some* $\Delta^a \in \mathcal{M}(m)$, $\Lambda^a \in \mathcal{M}(n)$. *Let* $A\Lambda = \Delta B$ *and let* B *have the form* $B = \operatorname{diag}(\mathcal{D}_1(A), A_1)$, $A_1 \in \mathcal{M}(m-1, n-1)$, $\mathcal{D}_1(A) | A_1$ *(i.e. we have the situation from the previous lemma). Then for every integer* k, $2 \leqq k \leqq \min(m, n)$ *we have*

$$\mathcal{E}_k(A) | \mathcal{E}_{k-1}(A_1) \cdot \varphi^{2k-1} \quad \text{and} \quad \mathcal{E}_{k-1}(A_1) | \mathcal{E}_k(A) \cdot \varphi^{2k-1}.$$

Proof. It holds $\varphi A = \Delta B \Lambda^a$, $\varphi B = \Delta^a A \Lambda$. The Cauchy—Binet multiplication rule implies $\mathcal{D}_k(A) = 0$ if and only if $\mathcal{D}_k(B) = 0$. Further, if $\mathcal{D}_k(A), \mathcal{D}_k(B) \neq 0$ it holds

$$\mathcal{D}_k(A) | \varphi^k \mathcal{D}_k(B), \; \mathcal{D}_k(B) | \varphi^k \mathcal{D}_k(A).$$

Clearly $\mathcal{D}_k(B) = \mathcal{D}_1(A) \cdot \mathcal{D}_{k-1}(A_1)$ for $k \geqq 2$. Hence

$$\mathcal{E}_k(A) = \mathcal{D}_k(A)/\mathcal{D}_{k-1}(A) | \mathcal{D}_k(B)\, \varphi^k/\mathcal{D}_{k-1}(A) =$$

$$= \mathcal{D}_1(A)\, \mathcal{D}_{k-1}(A_1)\, \varphi^{2k-1}/(\mathcal{D}_{k-1}(A)\, \varphi^{k-1}) | \mathcal{D}_1(A) \cdot$$

$$\cdot \mathcal{D}_{k-1}(A_1)\, \varphi^{2k-1}/\mathcal{D}_{k-1}(B) | \mathcal{D}_{k-1}(A_1)\, \varphi^{2k-1}/\mathcal{D}_{k-2}(A_1) = \mathcal{E}_{k-1}(A_1)\, \varphi^{2k-1}.$$ Similarly,

$$\mathcal{E}_{k-1}(A_1) = \mathcal{D}_{k-1}(A_1)/\mathcal{D}_{k-2}(A_1) =$$

$$= \mathcal{D}_k(B)/\mathcal{D}_{k-1}(B) | \mathcal{D}_k(A)\, \varphi^k \varphi^{k-1}/\mathcal{D}_{k-1}(A) = \mathcal{E}_k(A)\, \varphi^{2k-1}.$$

Lemma 5. *Let* $A, B \in \mathcal{M}(m, n)$, $1 \leqq m, n \leqq \infty$, $\Delta \in \mathcal{M}(m)$, $\Lambda \in \mathcal{M}(n)$, $A\Lambda = \Delta B$. *Let* Δ, Λ *have a scalar multiple* $\varphi \in H^\infty$. *Let* $A = A_i A_e$, $B = B_i B_e$ *be the canonical inner-outer factorizations of* A *and* B, *i.e.* $A_e \in \mathcal{M}(k, n)$, $B_e \in \mathcal{M}(k', n)$ *are outer,* $A_i \in \mathcal{M}(m, k)$, $B_i \in \mathcal{M}(m, k')$ *inner functions. Define the operator* $X \colon \mathfrak{H}(B_i) \to \mathfrak{H}(A_i)$ *by* $Xu = P_{\mathfrak{H}(A_i)} \Delta u$ $(u \in \mathfrak{H}(B_i))$. *Then*

1) $S(A_i)X = XS(B_i)$, and 2) $\varphi(S(B_i)|\mathfrak{N}) = 0$ *where* $\mathfrak{N} = \operatorname{Ker} X$.

Remark. Let $\varphi\equiv 1$. Then conditions 1, 2 mean $S(B_i)\overset{i}{\prec} S(A_i)$. For a general function $\varphi\in H^\infty$ these conditions are a weakening of that relation.

Proof of Lemma 5. First we prove $P_{\mathfrak{H}(A_i)}\Delta B_i w=0$ for every $w\in H^2(E_k)$. As B_e is an outer function the set $B_e H^2(E_n)$ is dense in $H^2(E_k)$. So we can suppose $w=B_e w'$ for some $w'\in H^2(E_n)$ and use the continuity of the mapping $P_{\mathfrak{H}(A_i)}\Delta B_i$. Then

$$P_{\mathfrak{H}(A_i)}\Delta B_i w = P_{\mathfrak{H}(A_i)}\Delta B w' = P_{\mathfrak{H}(A_i)} A \Lambda w' = P_{\mathfrak{H}(A_i)} A_i A_e \Lambda w' = 0.$$

Let now $u\in\mathfrak{H}(B_i)$. Then we have (for some $w\in H^2(E_k)$, $w'\in H^2(E_{k'})$)

$$S(A_i)Xu = P_{\mathfrak{H}(A_i)}U_+ P_{\mathfrak{H}(A_i)}\Delta u = P_{\mathfrak{H}(A_i)}U_+\Delta u + P_{\mathfrak{H}(A_i)}U_+ A_i w =$$

$$= P_{\mathfrak{H}(A_i)}\Delta U_+ u = P_{\mathfrak{H}(A_i)}\Delta P_{\mathfrak{H}(B_i)}U_+ u + P_{\mathfrak{H}(A_i)}\Delta B_i w' = XS(B_i)u$$

(where U_+ is the operator of multiplication by the identical function in the spaces $H^2(E_m)$ and $H^2(E_k)$, respectively).

Let $u\in\mathfrak{H}(B_i)$, $Xu=0$ i.e. $u\in A_i H^2(E_k)$. Then $\varphi u=\Delta^a\Delta u\in\Delta^a A_i H^2(E_k)$. It holds $\varphi(S(B_i))u=P_{\mathfrak{H}(B_i)}(\varphi u)\in P_{\mathfrak{H}(B_i)}\Delta^a A_i H^2(E_k)$. So it is sufficient to prove $P_{\mathfrak{H}(B_i)}\Delta^a A_i w=0$ for each $w\in H^2(E_k)$. As $A_e H^2(E_n)$ is dense in $H^2(E_k)$ we may assume $w=A_e w'$ for some $w'\in H^2(E_n)$. Then

$$P_{\mathfrak{H}(B_i)}\Delta^a A_i w = P_{\mathfrak{H}(B_i)}\Delta^a A w' = P_{\mathfrak{H}(B_i)} B \Lambda^a w' = P_{\mathfrak{H}(B_i)} B_i B_e \Lambda^a w' = 0.$$

Lemma 6. *Let $1\leqq m, n\leqq\infty$, let $A_r\in\mathcal{M}(m-r, n-r)$ for $0\leqq r<n+1$, $A_0=A$ inner. Let $B_r\in\mathcal{M}(m-r, n-r)$ $(0\leqq r<n)$ such that $B_r=\operatorname{diag}(s_r, A_{r+1})$ where $s_r\in H^\infty$ is inner and $A_r\Lambda_r=\Delta_r B_r$ for some $\Delta_r\in\mathcal{M}(m-r)$, $\Lambda_r\in\mathcal{M}(n-r)$ having a scalar multiple $\varphi_r\in H^\infty$. Let further $t_r\in H^\infty$ $(1\leqq r<n+1)$ satisfy $t_r|s_{r-1}$ and $t_r\wedge\varphi_i=1$ $(1\leqq r<n+1, 0\leqq i<n)$. Then*

$$\bigoplus_{j=1}^{n} S(t_j)\overset{i}{\prec} S(A).$$

Proof. As A_0 is inner it holds $m\geqq n$. Let $A_r=A_{ri}A_{re}$, $B_r=B_{ri}B_{re}$ be the canonical inner-outer factorizations of A_r and B_r, respectively. Then $B_{ri}=\operatorname{diag}(s_r, A_{r+1,i})$, $B_{re}=\operatorname{diag}(1, A_{r+1,e})$. Define the operators X_r: $\mathfrak{H}(B_{ri})\to\mathfrak{H}(A_{ri})$ by $X_r=P_{\mathfrak{H}(A_{ri})}\Delta_r|\mathfrak{H}(B_{ri})$ $(0\leqq r<n)$. We have $X_r S(B_{ri})=S(A_{ri})X_r$ by Lemma 5. For $0\leqq r<n$ define the operators Z_r: $\mathfrak{H}(t_{r+1})\to\mathfrak{H}(s_r)$ by $Z_r u=P_{\mathfrak{H}(s_r)}\left(\frac{s_r}{t_{r+1}}u\right)=\frac{s_r}{t_{r+1}}u$ $(u\in\mathfrak{H}(t_{r+1}))$. It is easy to see that $Z_r S(t_{r+1})=S(s_r)Z_r$ and Z_r is an injective operator (in fact it is an isometry). The situation is shown in the following

diagram:

$$\begin{array}{lll}
\mathfrak{H}(t_{r+1}) & \mathfrak{H}(t_2) & \mathfrak{H}(t_1) \\
\quad \Big\downarrow z_r & \quad\Big\downarrow z_1 & \quad \Big\downarrow z_0 \\
\left.\begin{array}{l}\mathfrak{H}(s_r) \\ \ldots \oplus \\ \mathfrak{H}(A_{r+1,i})\end{array}\right\} = \mathfrak{H}(B_{ri}) \xrightarrow{X_r} \mathfrak{H}(A_{ri}) \ldots \mathfrak{H}(A_{2i}) & \left.\begin{array}{l}\mathfrak{H}(s_1) \\ \oplus \\ \end{array}\right\} = \mathfrak{H}(B_{1i}) \xrightarrow{X_1} \mathfrak{H}(A_{1i}) & \left.\begin{array}{l}\mathfrak{H}(s_0) \\ \oplus \\ \end{array}\right\} = \mathfrak{H}(B_{0i}) \xrightarrow{X_0} \mathfrak{H}(A)
\end{array}$$

Define further the operators W_r: $\mathfrak{H}(t_{r+1}) \to \mathfrak{H}(A)$ by $W_r = X_0 X_1 \ldots X_r Z_r$ (we consider the spaces $\mathfrak{H}(s_r)$ and $\mathfrak{H}(A_{r+1,i})$ as subspaces of $\mathfrak{H}(B_{ri})$). Obviously $W_r S(t_{r+1}) = S(A) W_r$. Let $T \in \mathcal{M}(n)$, $T = \operatorname{diag}(t_1, t_2, \ldots)$, $\mathfrak{H}(T) = \bigoplus_{j=1}^{n} \mathfrak{H}(t_j)$, $S(T) = \bigoplus_{j=1}^{n} S(t_j)$. Define the operator W: $\mathfrak{H}(T) \to \mathfrak{H}(A)$ by $W\left(\bigoplus_{j=1}^{n} u_j\right) = \sum_{j=1}^{n} j^{-1} a_{j-1}^{-1} W_{j-1} u_j$ where $a_{j-1} = \max\left\{1, \max\{\|X_k \ldots X_{j-1} Z_{j-1}\| \; |k = 0, \ldots, j-1\}\right\}$. As

$$\sum_{j=1}^{n} (j^{-1} a_{j-1}^{-1} \|W_{j-1}\|)^2 \leqq \sum_{j=1}^{\infty} j^{-2} < \infty,$$

the definition of W is correct and W is a bounded operator. Further $WS(T) = S(A) W$.

It suffices to prove that W is injective. Suppose on the contrary that $Wu = 0$ for some $0 \neq u \in \mathfrak{H}(T)$, $u = \bigoplus_{j=1}^{n} u_j$, $u_j \in \mathfrak{H}(t_j)$. Let k be the minimal integer with $u_k \neq 0$. To simplify the notation denote $u_j' = j^{-1} a_{j-1}^{-1} Z_{j-1} u_j$, $u_j' \in \mathfrak{H}(s_{j-1})$. We have $0 = \sum_{j=k}^{n} X_0 X_1 \ldots X_{j-1} u_j' = X_0 X_1 \ldots X_{k-1} u_k' + \sum_{j=k+1}^{n} (X_0 X_1 \ldots X_{k-1}) X_k \ldots X_{j-1} u_j' = (X_0 \ldots X_{k-1})(u_k' + w)$ where $u_k' \in \mathfrak{H}(s_{k-1})$, $w = \sum_{j=k+1}^{n} X_k \ldots X_{j-1} u_j'$, $w \in \mathfrak{H}(A_{ki})$. So $X_1 \ldots X_{k-1}(u_k' + w) \in \operatorname{Ker} X_0$ and by Lemma 5

$$\begin{aligned}0 &= \varphi_0(S(B_{0i})) X_1 \ldots X_{k-1}(u_k' + w) = \\ &= X_1 \varphi_0(S(B_{1i})) X_2 \ldots X_{k-1}(u_k' + w) = \ldots = \\ &= X_1 \ldots X_{k-1} \varphi_0(S(B_{k-1,i}))(u_k' + w).\end{aligned}$$

Hence $X_2 \ldots X_{k-1} \varphi_0(S(B_{k-1,i}))(u_k' + w) \in \operatorname{Ker} X_1$ and repeating the same argument as before we get finally

$$(\varphi_0 \ldots \varphi_{k-1})(S(B_{k-1,i}))(u_k' + w) = 0.$$

As $u_k' \in \mathfrak{H}(s_{k-1})$, $w \in \mathfrak{H}(A_{ki})$ and both these subspaces are reducing with respect to $S(B_{k-1,i})$ we have also

$$(\varphi_0 \ldots \varphi_{k-1})(S(B_{k-1,i})) u_k' = 0.$$

On the other hand

$$t_k(S(B_{k-1,l}))u_k' = t_k(S(s_{k-1}))Z_{k-1}k^{-1}\|W_{k-1}\|^{-1}u_k = k^{-1}\|W_{k-1}\|^{-1}Z_{k-1}t_k(S(t_k))u_k = 0.$$

As $t_k \wedge (\varphi_0 \dots \varphi_{k-1}) = 1$, necessarily $u_k' = 0$. The operator Z_{k-1} being injective we conclude $u_k = 0$, a contradiction.

Theorem 7. *Let $T \in C_{.0}$ (i.e. $T^{*n} \to 0$ strongly) and $n = \delta_T$, $m = \delta_{T^*}$ be the defect indices of T. Then*

$$\bigoplus_{j=1}^{n} S(\mathscr{E}_j(A)) \prec T$$

where $A \in \mathscr{M}(m, n)$ is the characteristic function of T.

Proof. It is well known ([7]) that $A = A_0$ is inner and T is unitarily equivalent to $S(A)$. Therefore $\mathscr{D}_j(A) \neq 0$ and $\mathscr{E}_j(A) \neq 0$ for each j.

By Lemma 3 there exist matrices $\Delta_0 \in \mathscr{M}(m)$, $\Lambda_0 \in \mathscr{M}(n)$ having a scalar multiple $\varphi_0 \in H^\infty$, $\varphi_0 \wedge \mathscr{E}_j(A) = 1$ $(1 \leqq j < n+1)$ and a matrix $B_0 \in \mathscr{M}(m, n)$, $B_0 = \operatorname{diag}(\mathscr{D}_1(A), A_1)$, $A_1 \in \mathscr{M}(m-1, n-1)$ such that $A\Lambda_0 = \Delta_0 B_0$.

Analogously, for $r < n$ we can find inductively matrices $\Delta_r \in \mathscr{M}(m-r)$ $\Lambda_r \in \mathscr{M}(n-r)$ having a scalar multiple $\varphi_r \in H^\infty$, $\varphi_r \wedge \mathscr{E}_j(A) = 1$ $(1 \leqq j < n+1)$ and a matrix $B_r \in \mathscr{M}(m-r, n-r)$, $B_r = \operatorname{diag}(\mathscr{D}_1(A_r), A_{r+1})$ and $A_r \Lambda_r = \Delta_r B_r$.

Put $t_j = \mathscr{E}_j(A)$ $(1 \leqq j < n+1)$, $s_j = \mathscr{D}_1(A_j)$ $(0 \leqq j < n)$. By Lemma 4 it is $\mathscr{E}_k(A_r) | \mathscr{E}_{k-1}(A_{r+1}) \varphi_r^{2k-1}$, $\mathscr{E}_{k-1}(A_{r+1}) | \mathscr{E}_k(A_r) \varphi_r^{2k-1}$. Hence

$$t_k = \mathscr{E}_k(A_0) | \mathscr{E}_{k-1}(A_1)\, \varphi_0^{2k-1} | \mathscr{E}_{k-2}(A_2)\, \varphi_0^{2k-1} \varphi_1^{2k-3} | \dots | \mathscr{E}_1(A_{k-1})\, \varphi_0^{2k-1} \varphi_1^{2k-3} \dots \varphi_{k-2}^3.$$

As $(\varphi_0^{2k-1} \dots \varphi_{k-2}^3) \wedge t_k = 1$, necessarily $t_k | \mathscr{E}_1(A_{k-1}) = s_{k-1}$.

The required result follows now immediately from the previous lemma.

Remark. For $n < \infty$ the statement of Theorem 7 follows from [9]: If we denote $\mathfrak{H}(J) = \mathfrak{H}(T) \oplus H^2(E_{m-n})$, $S(J) = S(T) \oplus \bigoplus_1^{m-n} S(0)$ (where $S(0)$ is the unilateral shift; multiplication by the identical function) then there exist two injective operators W_1, W_2: $\mathfrak{H}(J) \to \mathfrak{H}(A)$ intertwining the operators $S(J)$ and $S(A)$ such that $W_1 \mathfrak{H}(J) \vee W_2 \mathfrak{H}(J) = \mathfrak{H}(A)$.

In the case $m = n = \infty$ this cannot be true. It may happen that $\mathscr{E}_i(A) = 1$ for each i. Then $S(J)$ is the trivial operator and no sort of density of the images can hold.

II.

The aim of this section is to give a new proof of the existence of the Jordan model of C_0-contractions. In the same time we prove the formulas for the functions appearing in the Jordan model (see [5]).

We start with the modification of Lemmas 3 and 4 for matrices having a scalar multiple.

Lemma 8. *Let* $A\in\mathcal{M}(n)$, $1\leqq n\leqq\infty$ *be an inner function having a scalar multiple* $\psi\in H^\infty$, ψ *inner, let* $\Omega\in\mathcal{M}(n)$ *satisfies* $A\Omega=\Omega A=\psi I_n$. *Then there exists a function* $\chi\in H^\infty$, $\chi\wedge\psi=1$ *and matrices* Δ, $\Lambda\in\mathcal{M}(n)$ *with the scalar multiple* χ *(i.e.* $\Delta\Delta^a=\Delta^a\Delta=\Lambda^a\Lambda=\Lambda\Lambda^a=\chi I_n$ *for some* $\Delta^a, \Lambda^a\in\mathcal{M}(n)$*) such that* $A\Lambda=\Delta B$ *where* $B\in\mathcal{M}(n)$ *has the form* $B=\operatorname{diag}(\psi/\mathscr{E}_1(\Omega), A_1)$, B *is inner and* $A_1\in\mathcal{M}(n-1)$ *has a scalar multiple* $\psi_1|\psi\chi/\mathscr{E}_1(\Omega)$, ψ_1 *inner.*

Further, for every integer k, $1\leqq k<n$, $\psi/\mathscr{E}_{k+1}(\Omega)|\psi_1/\mathscr{E}_k(\Omega_1)$, $\psi_1/\mathscr{E}_k(\Omega_1)|\psi\chi/\mathscr{E}_{k+1}(\Omega)$ *where* $\Omega_1\in\mathcal{M}(n-1)$ *satisfies* $A_1\Omega_1=\Omega_1A_1=\psi_1I_{n-1}$.

Moreover if $\varepsilon>0$ *and* $\{h_i\}_{i=1}^n$ *is a sequence of* H^∞*-functions satisfying* $\sum_{i=1}^n|h_i(\lambda)|\leqq K$ *for some constant* K *independent on* $\lambda\in D$, *then* Δ, Λ *and* A_1 *can be chosen in such a way that* $\sum_{i=1}^n|h_i(\lambda)-g_i(\lambda)|<\varepsilon$ $(\lambda\in D)$ *where* $\{g_i\}_{i=1}^n$ *is the first column of* Δ.

Proof. By Lemma 3 there exist matrices $M, N\in\mathcal{M}(n)$ with a scalar multiple $\varphi\in H^\infty, \varphi\wedge\psi=1$ such that $\Omega N=M\Omega'$, where $\Omega'\in\mathcal{M}(n)$, $\Omega'=\operatorname{diag}(\mathscr{E}_1(\Omega), \Omega_1')$, $\Omega_1'\in\mathcal{M}(n-1)$ and $\mathscr{E}_1(\Omega)|\Omega_1'$. Multiplying the equation $\Omega N=M\Omega'$ from left by N^aA we get $\psi\varphi I_n=N^aAM\Omega'$. If C denotes the matrix $C=N^aAM$, then $\Omega'C=(\Omega'N^a)AM=M^a(\Omega A)M=M^a\psi M=\varphi\psi I_n$ also holds, and so C has the scalar multiple $\varphi\psi$.

The matrix C is necessarily of the form $C=\operatorname{diag}(\varphi\psi/\mathscr{E}_1(\Omega), C_1)$ where $C_1\in\mathcal{M}(n-1)$ and $C_1(\Omega_1'/\mathscr{E}_1(\Omega))=(\Omega_1'/\mathscr{E}_1(\Omega))C_1=(\varphi\psi/\mathscr{E}_1(\Omega))I_{n-1}$. From the equation $C=N^aAM$ we infer that $CM^a=(\varphi N^a)A$. Taking the canonical inner-outer factorization $C_1=C_{1i}C_{1e}$ of C_1 we have that $C=\operatorname{diag}(\psi/\mathscr{E}_1(\Omega), C_{1i})\operatorname{diag}(\varphi, C_{1e})$, and so $\operatorname{diag}(\psi/\mathscr{E}_1(\Omega), C_{1i})\operatorname{diag}(\varphi, C_{1e})M^a=(\varphi N^a)A$. Since C_1 has the scalar multiple $(\varphi_i\psi/\mathscr{E}_1(\Omega))\varphi_e$ where $\varphi=\varphi_i\varphi_e$ is the canonical inner-outer factorization of φ, C_{1e} has the scalar multiple φ_e, and so φ also. Now it is obvious that the matrices $\Lambda^a=\operatorname{diag}(\varphi, C_{1e})M^a$ and $\Delta^a=\varphi N^a$ have the scalar multiple $\chi=\varphi^2$, that is $\Delta\Delta^a=\Delta^a\Delta=\Lambda\Lambda^a=\Lambda^a\Lambda=\chi I_n$ with some matrices $\Delta, \Lambda\in\mathcal{M}(n)$, particularly $\Delta=N$. Defining the matrix B by $B=\operatorname{diag}(\psi/\mathscr{E}_1(\Omega), C_{1i})$, we infer that $B\Lambda^a=\Delta^aA$, and so $\Delta B=A\Lambda$. On the other hand, $\chi\wedge\psi=1$ and the matrix $A_1=C_{1i}$ has an inner scalar multiple ψ_1 such that $\psi_1|\psi\chi/\mathscr{E}_1(\Omega)$.

Let now $\varepsilon>0$ and $h_1, h_2, \ldots \in H^\infty$, $\sum_{i=1}^{n} |h_i(\lambda)| \leqq K$. By Lemma 3 it was possible to choose the matrix N such that $\sum_{i=1}^{n} |h_i(\lambda)-g_i(\lambda)|<\varepsilon$ $(\lambda \in D)$, where $\{g_i\}_{i=1}^n$ is the first column of N. The same of course holds for the matrix $\Delta = N$.

It suffices to prove the statement about invariant factors of Ω and Ω_1. The matrix $A_1 = C_{1i}$ has the scalar multiple $\psi_1 = (\varphi\psi/\mathscr{E}_1(\Omega))_i = (\psi/\mathscr{E}_1(\Omega)) \cdot \varphi_i$ and $A_1(C_{1e}\,\Omega_1'/\mathscr{E}_1(\Omega)) = (\varphi\psi/\mathscr{E}_1(\Omega))I_{n-1} = \psi_1 \varphi_e I_{n-1}$. Hence $\Omega_1 \varphi_e = C_{1e}\,\Omega_1'/\mathscr{E}_1(\Omega)$. Then the Cauchy—Binet rule implies $d_k = \mathscr{D}_k(\Omega_1'/\mathscr{E}_1(\Omega)) | \mathscr{D}_k(\Omega_1)\varphi_e^k$; hence $d_k | \mathscr{D}_k(\Omega_1)$. Similarly $(C_{1e})^a\,\Omega_1 = \Omega_1'/\mathscr{E}_1(\Omega)$ (where $C_{1e}(C_{1e})^a = (C_{1e})^a C_{1e} = \varphi_e I_{n-1}$) and so $\mathscr{D}_k(\Omega_1)|d_k$. This gives $\mathscr{D}_k(\Omega_1) = d_k$ and $\mathscr{E}_k(\Omega_1) = \mathscr{D}_k(\Omega_1)/\mathscr{D}_{k-1}(\Omega_1) = d_k/d_{k-1} = \mathscr{E}_k(\Omega_1')/\mathscr{E}_1(\Omega)$. It holds (by Lemma 4) $\mathscr{E}_{k-1}(\Omega_1') | \mathscr{E}_k(\Omega)\varphi^{2k-1}$, $\mathscr{E}_k(\Omega) | \mathscr{E}_{k-1}(\Omega_1')\varphi^{2k-1}$. Hence

$$\psi/\mathscr{E}_{k+1}(\Omega) | \psi\varphi^{2k+1}/\mathscr{E}_k(\Omega_1') | \psi_1 \varphi^{2k+2}\mathscr{E}_1(\Omega)/\mathscr{E}_k(\Omega_1') = \psi_1 \varphi^{2k+2}/\mathscr{E}_k(\Omega_1).$$

As $\varphi \wedge \psi = 1$ we conclude $\psi/\mathscr{E}_{k+1}(\Omega) | \psi_1/\mathscr{E}_k(\Omega_1)$.

The relation $\psi_1/\mathscr{E}_k(\Omega_1) | \chi\psi/\mathscr{E}_{k+1}(\Omega)$ may be proved similarly.

Theorem 9. *Let T be a C_0-contraction, A the characteristic function of T and n the defect index of T $(1 \leqq n \leqq \infty)$. Let $\Omega \in \mathscr{M}(n)$ satisfies $A\Omega = \Omega A = \psi I_n$, where $\psi \in H^\infty$ is inner (such an Ω exists by [7]). Then*

$$\bigoplus_{i=1}^{n} S(\psi/\mathscr{E}_i(\Omega)) \overset{i}{\prec} T.$$

Proof. The operator T is unitarily equivalent to the operator $S(A)$ so it is sufficient to deal with $S(A)$. We use again Lemma 6.

By Lemma 8 there exist $\varphi_0 \in H^\infty$, $\varphi_0 \wedge \psi = 1$ and $\Delta_0, \Lambda_0, B_0 \in \mathscr{M}(n)$, where $A\Lambda_0 = \Delta_0 B_0$, Δ_0 and Λ_0 have the scalar multiple φ_0 and B_0 has the form $B_0 = \operatorname{diag}(s_0, A_1)$, $s_0 = \psi/\mathscr{D}_1(\Omega)$, $A_1 \in \mathscr{M}(n-1)$. Further A_1 is inner and has a scalar multiple $\psi_1 | \varphi_0\psi/\mathscr{E}_1(\Omega)$, ψ_1 inner. Denote $\Omega_1 \in \mathscr{M}(n-1)$ the matrix satisfying $A_1\Omega_1 = \Omega_1 A_1 = \psi_1 I_{n-1}$.

In the same way we can find for $r<n$ inductively matrices $\Delta_r, \Lambda_r, B_r \in \mathscr{M}(n-r)$ and a function $\varphi_r \in \mathfrak{H}^\infty$, $\varphi_r \wedge (\psi\varphi_0 \ldots \varphi_{r-1}) = 1$ such that $A_r\Lambda_r = \Delta_r B_r$, Δ_r and Λ_r have the scalar multiple φ_r, B_r has the form $B_r = \operatorname{diag}(s_r, A_{r+1})$, $s_r = \psi_r/\mathscr{D}_1(\Omega_r)$ and $A_{r+1} \in \mathscr{M}(n-r)$ has a scalar multiple ψ_{r+1}, $\psi_{r+1} | \varphi_r s_r | \varphi_r \psi_r$, ψ_{r+1} inner. Note that $\psi_r \wedge \varphi_r = 1$. Let $\Omega_{r+1} \in \mathscr{M}(n-r-1)$ satisfies $A_{r+1}\Omega_{r+1} = \Omega_{r+1}A_{r+1} = \psi_{r+1}I_{n-r-1}$.

Denote $t_j = \psi/\mathscr{E}_j(\Omega)$. Then $t_j = \psi/\mathscr{E}_j(\Omega) | \psi_1/\mathscr{E}_{j-1}(\Omega_1) | \ldots | \psi_{j-1}/\mathscr{E}_1(\Omega_{j-1}) = s_{j-1}$ (by Lemma 8). Further $t_j | \psi$ and $\varphi_r \wedge \psi = 1$ implies $t_j \wedge \varphi_r = 1$ for every j, r.

Now application of Lemma 6 completes the proof.

Remark. By Lemma 8 we infer also that

$$s_j = \psi_j/\mathcal{E}_1(\Omega_j)|\psi_{j-1}\varphi_{j-1}/\mathcal{E}_2(\Omega_{j-1})|\dots|t_{j+1}\varphi_{j-1}\dots\varphi_0.$$

Therefore, we have $g_j=s_j/t_{j+1}|\varphi_0\dots\varphi_{j-1}$. This fact will be used later.

Our goal will be now to show that we can choose matrices Δ_r, Λ_r and B_r in such a way that the range of the operator W (see Lemma 6) is a dense subspace of $\mathfrak{H}(A)$ i.e. $\bigoplus_{i=1}^{n} S(\psi/\mathcal{E}_i(\Omega))\prec S(A)$.

Lemma 10. *Let $A, B, \Delta, \Lambda\in\mathcal{M}(n)$, $1\leqq n\leqq\infty$, let A and B be inner and $A\Lambda=\Delta B$. Let Δ, Λ have a scalar multiple $\varphi\in H^\infty$ and A a scalar multiple $\psi\in H^\infty$, $\psi\wedge\varphi=1$, ψ inner. Let the operator X: $\mathfrak{H}(B)\to\mathfrak{H}(A)$ be defined by $X=P_{\mathfrak{H}(A)}\Delta|\mathfrak{H}(B)$. Then $\overline{Xf(S(B))\mathfrak{H}(B)}=\mathfrak{H}(A)$ for every function $f\in H^\infty$, $f\wedge\psi=1$.*

Proof. Let $v\in\mathfrak{H}(A)$, $v\perp\overline{Xf(S(B))\mathfrak{H}(B)}$. First we prove $v\perp\varphi fH^2(E_n)$. Let $w\in H^2(E_n)$. Then (for suitable $w', w''\in H^2(E_n)$)

$$\begin{aligned}(v, \varphi fw) &= (v, P_{\mathfrak{H}(A)}\varphi fw) = (v, P_{\mathfrak{H}(A)}\Delta\Delta^a fw) =\\ &= (v, P_{\mathfrak{H}(A)}\Delta P_{\mathfrak{H}(B)}\Delta^a fw)+(v, P_{\mathfrak{H}(A)}\Delta Bw') =\\ &= (v, XP_{\mathfrak{H}(B)}f\Delta^a w)+(v, P_{\mathfrak{H}(A)}A\Lambda w') =\\ &= (v, XP_{\mathfrak{H}(B)}fP_{\mathfrak{H}(B)}\Delta^a w)+(v, XP_{\mathfrak{H}(B)}fBw'') =\\ &= (v, Xf(S(B))\,P_{\mathfrak{H}(B)}\Delta^a w) = 0.\end{aligned}$$

Further $\psi H^2(E_n)\subset AH^2(E_n)$ because A has the scalar multiple ψ. As $v\perp AH^2(E_n)$ we infer $v\perp\psi H^2(E_n)$. Now $v\perp\varphi fH^2(E_n)$, $v\perp\psi H^2(E_n)$ and $\varphi f\wedge\psi=1$ implies $v=0$ (see [3]).

Lemma 11. *If the assumptions of Theorem 9 hold, then using the notation of Theorem 9 and Lemma 6, we have*

$$\mathfrak{H}(A) = \bigvee_{j=0}^{r} X_0\dots X_j(\varphi_0\dots\varphi_j)(S(s_j))\,\mathfrak{H}(s_j)\vee X_0\dots X_r(\varphi_0\dots\varphi_r)(S(A_{r+1}))\,\mathfrak{H}(A_{r+1})$$

for each integer r, $0\leqq r<n$.

Proof. We proceed by induction on r. For $r=0$ the statement

$$\mathfrak{H}(A) = X_0\varphi_0(S(s_0))\,\mathfrak{H}(s_0)\vee X_0\varphi_0(S(A_1))\,\mathfrak{H}(A_1) = \overline{X_0\varphi_0(S(B_0))\,\mathfrak{H}(B_0)}$$

follows from the previous lemma.

Suppose the statement is true for $r-1$. Then

$$\mathfrak{H}(A) = \bigvee_{j=0}^{r-1} X_0\dots X_j(\varphi_0\dots\varphi_j)(S(s_j))\,\mathfrak{H}(s_j)\vee X_0\dots X_{r-1}(\varphi_0\dots\varphi_{r-1})(S(A_r))\,\mathfrak{H}(A_r).$$

Further (by Lemma 10)

$$\mathfrak{H}(A_r) = \overline{X_r\varphi_r(S(B_r))\,\mathfrak{H}(B_r)}. \quad \text{So}$$

$$X_0\dots X_{r-1}(\varphi_0\dots\varphi_{r-1})(S(A_r))\,\mathfrak{H}(A_r)\subset$$
$$\subset \overline{X_0\dots X_{r-1}(\varphi_0\dots\varphi_{r-1})(S(A_r))X_r\varphi_r(S(B_r))\,\mathfrak{H}(B_r)} =$$
$$= \overline{X_0\dots X_r(\varphi_0\dots\varphi_r)(S(B_r))\,\mathfrak{H}(B_r)} =$$
$$= X_0\dots X_r(\varphi_0\dots\varphi_r)(S(s_r))\,\mathfrak{H}(s_r)\vee$$
$$\vee X_0\dots X_r(\varphi_0\dots\varphi_r)(S(A_{r+1}))\,\mathfrak{H}(A_{r+1}).$$

Together with the induction assumption this gives the statement of the lemma for r.

Lemma 12. (We use again the notation of Theorem 9 and Lemma 6.)

$$\mathfrak{H}(A) = \bigvee_{j=0}^{r} W_j\mathfrak{H}(t_{j+1})\vee X_0\dots X_r(\varphi_0\dots\varphi_r)(S(A_{r+1}))\,\mathfrak{H}(A_{r+1})$$

for every integer r, $0\leqq r<n$.

Proof. Clearly it is sufficient to prove

$$W_j\mathfrak{H}(t_{j+1})\supset X_0\dots X_j(\varphi_0\dots\varphi_j)S(s_j)\mathfrak{H}(s_j) \quad (0\leqq j<n).$$

As $W_j=X_0\dots X_jZ_j$ it is sufficient to show $Z_jH(t_{j+1})\supset(\varphi_0\dots\varphi_j)S(s_j)\mathfrak{H}(s_j)$. Let us recall (see the Remark after Theorem 9) that $g_j|(\varphi_0\dots\varphi_{j-1})$, g_j is inner and $Z_ju=P_{\mathfrak{H}(s_j)}(g_ju)=g_ju$ for $u\in\mathfrak{H}(t_{j+1})$ and for $g_j=s_j/t_{j+1}$. Then $(\varphi_0\dots\varphi_j)(S(s_j))\mathfrak{H}(s_j)\subset\subset g_j(S(s_j))\,\mathfrak{H}(s_j)=P_{\mathfrak{H}(s_j)}g_j\,\mathfrak{H}(s_j)\subset P_{\mathfrak{H}(s_j)}g_jH^2=P_{H(s_j)}g_j\,t_{j+1}H^2\vee P_{\mathfrak{H}(s_j)}g_j\,\mathfrak{H}(t_{j+1})==Z_j\mathfrak{H}(t_{j+1})$.

Theorem 13. *Let T be a C_0-contraction, A the characteristic function of T, let n be the defect index of T, $1\leqq n\leqq\infty$. Let $\Omega\in\mathcal{M}(n)$ satisfies $A\Omega=\Omega A=\psi I_n$, $\psi\in H^\infty$ inner. Then*

$$\bigoplus_{j=1}^{n} S(\psi/{}_j\mathcal{E}_j(\Omega))\prec T.$$

Proof. We use again the notation of Theorem 9 and Lemma 6. We show that the matrices Δ_r, A_r and B_r $(0\leqq r<n)$ in the proof of Theorem 9 can be chosen such that $\overline{W\mathfrak{H}(T)}=\mathfrak{H}(A)$.

If $\mathfrak{H}(A_k)=\{0\}$ for some k (particularly if $n<\infty$) then the statement follows from the previous lemma.

Suppose in the sequel that $n=\infty$ and $\mathfrak{H}(A_k)\neq\{0\}$ for every k. Let $a_1, a_2, \dots$ be a countable set dense in $\mathfrak{H}(A)$. Let $\{b_j\}_{j=1}^\infty$ be a sequence of elements of this set in which every element a_i $(1\leqq i<\infty)$ occurs infinitely many times. It suffices to prove that having chosen matrices Δ_j, Λ_j and B_j for $j<k$ (k fixed), we can find matrices Δ_k, Λ_k and B_k such that $\operatorname{dist}\left(b_k, \bigvee_{j=0}^{k} W_j \mathfrak{H}(t_{j+1})\right) < k^{-1}$. Having done such a selection for every k the space $\overline{W\mathfrak{H}(T)} = \bigvee_{j=0}^{\infty} W_j \mathfrak{H}(t_{j+1})$ would contain all elements a_j ($j=1, 2, \ldots$) which form a dense subset of $\mathfrak{H}(A)$.

By the previous lemma there exists an element $c \in (\varphi_0 \ldots \varphi_{k-1})(S(A_k))\mathfrak{H}(A_k)$ such that

$$\operatorname{dist}\left(b_k - X_0 \ldots X_{k-1} c, \bigvee_{j=0}^{k-1} W_j \mathfrak{H}(t_{j+1})\right) < (2k)^{-1}. \tag{1}$$

Further it is

$$c = P_{\mathfrak{H}(A_k)}(\varphi_0 \ldots \varphi_{k-1} c') = P_{\mathfrak{H}(A_k)} g_k d \tag{2}$$

for some $c' \in \mathfrak{H}(A_k)$, $d \in H^2(E_\infty)$. In the given orthonormal basis in the space E_∞ d is represented by a sequence $d = \{d_j\}_{j=1}^{\infty}$, $d_j \in H^2$. Further there exists a sequence $h = \{h_j\}_{j=1}$ of H^∞ functions such that $\sum_{j=1}^{\infty} |h_j(\lambda)| \leqq K$ for some constant K independent on $\lambda \in D$ and

$$|d - h|_{H^2(E_\infty)} < (4\|X_0\| \ldots \|X_{k-1}\|)^{-1} \tag{3}$$

(we suppose $\mathfrak{H}(A_j) \neq \{0\}$ so by Lemma 10 $X_j \neq 0$ for every j). By Lemma 8 we can choose matrices Δ_k, Λ_k and B_k such that

$$\sum_{j=1}^{\infty} |f_j(\lambda) - h_j(\lambda)| < (4\|X_0\| \ldots \|X_{k-1}\|)^{-1} \tag{4}$$

where $f = \{f_j\}_{j=1}^{\infty}$ is the first column of the matrix Δ_k.

Denote $e = P_{\mathfrak{H}(t_{k+1})} 1$, $e \in \mathfrak{H}(t_{k+1})$. Then (for some $w \in H^2$)

$$Z_k e = P_{\mathfrak{H}(s_k)} g_k P_{\mathfrak{H}(t_{k+1})} 1 = P_{\mathfrak{H}(s_k)} g_k + P_{\mathfrak{H}(s_k)} g_k t_{k+1} w = P_{\mathfrak{H}(s_k)} g_k$$

where $g_k = s_k / t_{k+1}$ (see the Remark after Theorem 9). Further

$$X_k Z_k e = P_{\mathfrak{H}(A_k)} \Delta_k P_{\mathfrak{H}(s_k)} (g_k, 0, 0, \ldots)^T = P_{\mathfrak{H}(A_k)} \Delta_k P_{\mathfrak{H}(B_k)} (g_k, 0, \ldots)^T =$$

$$= P_{\mathfrak{H}(A_k)} \Delta_k (g_k, 0, \ldots)^T + P_{\mathfrak{H}(A_k)} \Delta_k B_k w' =$$

$$= P_{\mathfrak{H}(A_k)} g_k \Delta_k (1, 0, \ldots)^T + P_{\mathfrak{H}(A_k)} A_k \Lambda_k w' = P_{\mathfrak{H}(A_k)} g_k (f_1, f_2, \ldots)^T$$

(for some $w' \in H^2(E_\infty)$). Finally,

$$|X_k Z_k e - c|_{\mathfrak{H}(A_k)} = |P_{\mathfrak{H}(A_k)} g_k (f_1, f_2, \ldots)^T - P_{\mathfrak{H}(A_k)} g_k (d_1, d_2, \ldots)^T|_{\mathfrak{H}(A_k)} \leqq$$

$$\leqq |g_k f - g_k d|_{H^2(E_\infty)} = |f - d|_{H^2(E_\infty)} \leqq$$

$$\leqq |f - h|_{H^2(E_\infty)} + |h - d|_{H^2(E_\infty)} < (2\|X_0\| \ldots \|X_{k-1}\|)^{-1}$$

(we used the fact that g_k is inner). Hence

$$|W_k e - X_0 \ldots X_{k-1} c|_{\mathfrak{H}(A)} \leqq \|X_0\| \ldots \|X_{k-1}\| \, |X_k Z_k e - c|_{H^2(E_\infty)} < (2k)^{-1}$$

and (1) implies $\operatorname{dist}\left(b_k, \bigvee_{j=0}^{k} W_j \mathfrak{H}(t_{j+1})\right) < k^{-1}$.

This completes the proof.

Remark. It is well-known (see [8]) that Theorem 13 implies that the operators T and $\bigoplus_{j=1}^{n} S(\psi/\mathcal{E}_j(\Omega))$ are even quasisimilar. Relation $T \prec \bigoplus_{j=1}^{n} S(\psi/\mathcal{E}_j(\Omega))$ follows by considering the adjoint operator T^*.

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