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Functionally complete algebras in congruence distributive varieties

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We say that $\varrho\supset A^h$ is central if $(a_1, ..., a_h)\in\varrho$ whenever $a_i=a_j$ for some $1\leqq i<j\leqq h$, ϱ is invariant under permutations of coordinates and $\varrho\supseteqq C\times A^{h-1}\neq\emptyset$. We prove: A finite at least three-element algebra $A=\langle A; F\rangle$ in a congruence distributive variety is functionally complete if and only if A is simple, monotonic with respect to no bounded partial order on A and A^h admits no central subalgebra for $h=2, ..., |A|-1$. For two-element algebras the condition simplifies to nonmonotonicity. If the variety **K** satisfies $\Delta_2(\mathbf{K})$, the absence of central subalgebras of A^h $(h=2, ..., |A|-1)$ may be restricted to $h=2$.

Recall that a finite algebra A is functionally complete (other names: complete or Sheffer with constants) if each finitary operation on the same universe is algebraic over A (i.e. obtainable from the operations of A, the projections and the constants via iterated composition). It is known (see e.g. [7; § 79] that all finite algebras in an arithmetical variety (i.e. in a congruence distributive and permutable equational class) are functionally complete. Recently McKenzie [5] has shown that with the exception of affine algebras a finite algebra in a congruence permutable variety is functionally complete if and only if it is simple (see also [2]; a short proof is in [15]). R. W. QUACKENBUSH in [7] asks for an analog of McKenzie's result for congruence distributive varieties. Combining JÓNSSON's Mal'cev-type conditions [4] with the results of [8, 9] we answer this question.

For a set A and h positive integer we say that $\varrho\subset A^h$ is *central* if ϱ is totally reflexive ($(a_1, ..., a_n)\in\varrho$ whenever $a_i=a_j$ for some $1\leqq i<j\leqq h$), invariant under all transpositions of coordinates and contains $C\times A^{h-1}\neq\emptyset$. The main result is:

Theorem. *A finite at least three-element algebra $A=\langle A; F\rangle$ in a congruence distributive variety is functionally complete if and only if A is simple, monotonic with respect to no bounded partial order on A and A^h admits no central subalgebra for $h=2, ..., |A|-1$.*

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Proof. The *necessity* is obvious. For example, if a nontrivial equivalence θ is a congruence of A then even the set of all operations on A admitting θ as a congruence is not a functionally complete set of operations.

Sufficiency. Let e_i $(i=1,2,3)$ denote the ternary projections (defined by $e_i(x_1, x_2, x_3)=x_i$ for all $x_1, x_2, x_3 \in A$). It is well known [4] (quoted also in [1, Ch. 5, Ex. 70]) there exist $n \geqq 2$ and ternary operations $e_1=t_0, t_1, \ldots, t_{n-1}, t_n=e_3$ derived from A such that for $i=0, 1, \ldots, n-1$ the identities

(1) $t_i(x, y, x) = x,$

(2) $t_i(x, x, y) = t_{i+1}(x, x, y)$ (i even),

(3) $t_i(x, y, y) = t_{i+1}(x, y, y)$ (i odd),

hold for all $x, y \in A$. To prove the functional completeness of A it suffices to verify that the operations of A augmented by the constants satisfy the following six types of conditions [8, 9, 11]. The first type of condition is just the nonmonotonicity. The conditions of the second type (the absence of the automorphisms of certain types) are taken care of by the constants. The third type of condition applies only if $|A|=p^m$, p prime and requires A to be non affine. Here A is *affine* if all $f \in F$ are of the type

(4) $$f(x_1, \ldots, x_n) = B_1x_1 + \ldots + B_nx_n + C$$

where $+$ denotes the addition of an m-dimensional (column) vector space of characteristic p on A and B_i and C are $m \times m$ and $m \times 1$ matrices over $\mathbf{p}=\{0, 1, \ldots, p-1\}$.

The following simple statement will be needed once more later.

Claim 1. *For at least one* $t \in T=\{t_1, \ldots, t_n\}$ *the following conditions* (i)—(iii) *are not equivalent:*

(i) $t(x, x, y)=t(x, y, x)=x,$

(ii) $t(x, y, y)=t(x, y, x)=x,$

(iii) $t=e_1.$

Proof. Suppose (i)—(iii) are equivalent for all t_i $(i=1, \ldots, n)$. From $t_0=e_1$, (2) and (1) we obtain that t_1 satisfies (i), hence $t_1=e_1$ by (iii). From this, (3) and (1) it follows that t_2 satisfies (ii) and again $t_2=e_1$. Continuing in this way we finally arrive at the contradiction $e_3=t_n=e_1$. □

Using (4) it is easily proved that the conditions (i)—(iii) are equivalent for t affine. Consequently, not all t_i are affine and therefore A is not affine.

The fourth condition is the simplicity of A. The fifth condition is that no h-ary central relation is a subalgebra of A^h for $h=1, \ldots, |A|-1$. Our assumptions do not cover the unary central relations (proper subsets of A) but these are taken care of by the constants.

For the sixth type of condition, the first point to notice is that $t_0, \ldots, t_n$ are all surjective (as maps $A^3 \to A$). We claim that at least one of the t_i is essentially at least binary. If not, then by (1) each t_i is either e_1 or e_3. Let j be the least index such that $t_j = e_3$. Then clearly $0 < j \leqq n$ and using (2) or (3) (for $z=x$ or $z=y$) we obtain the contradiction

$$x = e_1(x, z, y) = t_{j-1}(x, z, y) = t_j(x, z, y) = y.$$

Thus $T = \{t_1, \ldots, t_n\}$ is not included in the set of all essentially unary or nonsurjective operations which is a particular instance of the sixth type (the Słupecki condition [14]). We show that T satisfies the remaining conditions as well. To this end we must define *wreath algebras.* Let $h \geqq 3$, $m > 1$ and $n > 1$ be integers and let $\mathbf{h} = \{0, \ldots, h-1\}$, $M = \{1, \ldots, m\}$, $N = \{1, \ldots, n\}$ and $B = \mathbf{h}^m$. A *wreath operation* on B is an n-ary operation w on B associated to permutations p_i of $\mathbf{h}$ $(i=1, \ldots, m)$ and maps $r\colon M \to N$ and $s\colon M \to M$ as follows. For $x_i = (x_{i1}, \ldots, x_{im}) \in B$ $(i \in N)$ set

$$w(x_1, \ldots, x_n) = \big(p_1(x_{r(1)s(1)}), \ldots, p_m(x_{r(m)s(m)})\big). \tag{5}$$

Now an algebra A is said to be a *wreath algebra* if it is isomorphic to an algebra on B having wreath operations only.

Next we prove the following:

Claim 2. *For a ternary wreath operation t on B the conditions* (i)—(iii) *above are equivalent.*

Proof. Let p_1, p_2, p_3 be the permutations of $\mathbf{h}$ and let $r\colon M \to \{1, 2, 3\}$ and $s\colon M \to M$ be the maps in the representation (5) of t. Set $R_i = r^{-1}\{i\}$ $(i=1, 2, 3)$. To prove (i)⇒(iii) observe that in (5) we need $R_2 = R_3 = \varnothing$; moreover s and p_i $(i=1, 2, 3)$ must be identities on M and $\mathbf{h}$. The same argument proves (ii)⇒(iii). Finally (iii)⇒(i) and (iii)⇒(ii) are obvious. □

Using an idea from [13] (applied also in [10]) it was shown in [12] that a surjective algebra does not satisfy the remaining conditions of the sixth type if and only if it is a wreath algebra. Combining both claims we obtain that $\langle A, T\rangle$ cannot be a wreath algebra; consequently A is not a wreath algebra and the proof is complete. □

Remark 1. For algebras on a two-element set the situation is much simpler. There are only two bounded partial orders (dual to each other) on such a set and therefore a single type of monotonicity. Similarly there is but one type of linearity. It is well known and follows from Post's criterion [6, 3] that a two-element algebra is functionally complete if and only if it is neither monotonic nor linear. The same argument as above may be used to remove the nonlinearity stipulation yielding: *A two-element algebra in a congruence distributive variety is functionally complete if and only if it is not monotonic.*

Remark 2. Observe that there are more exceptional cases than in congruence permutable varieties; moreover they are of a different nature. Lattices — a typical instance of congruence distributive varieties — provide examples of functionally noncomplete algebras that are monotonic and possibly not simple.

For a congruence distributive variety $\mathbf{K}$ let $\Delta_n(\mathbf{K})$ denote the existence of $e_1 = t_0, t_1, \ldots, t_n = e_3$ satisfying (1)—(3) for the least n [4]. For example, $\Delta_2(\mathbf{K})$ means that each algebra in $\mathbf{K}$ has a majority ternary operation M among its derived operations (i.e. M satisfies the identity $M(x, x, y) = M(x, y, x) = M(y, x, x) = x$). We have:

Corollary. *Let $\mathbf{K}$ be an equational class of algebras satisfying $\Delta_2(\mathbf{K})$. A finite algebra A in $\mathbf{K}$ is functionally complete if and only if it is simple, monotonic with respect to no bounded partial order on A and admits no central subalgebra of A^2.*

Proof. Let $2 < h < |A|$ and let ϱ be central. Then there exists $c \in A$ such that $c \times A^{h-1} \subseteq \varrho$. Let M be the corresponding majority operation. We maintain that ϱ is not a subalgebra of $\langle A; M \rangle^h$. Assume it is and choose $a_1, \ldots, a_h \in A$. From $M(a_1, a_1, c) = a_1$, $M(a_2, c, a_2) = a_2$, $M(c, a_i, a_i(i = 3, \ldots, h)$ and $(a_1, a_2, c, \ldots, c) \in \varrho$, $(a_1, c, a_3, \ldots, a_h) \in \varrho$, $(c, a_2, \ldots, a_h) \in \varrho$ it follows that $(a_1, \ldots, a_h) \in \varrho$ leading to the contradiction $\varrho = A^h$. □

Remark 3. Applying an argument from [12] it can be shown that for a surjective algebra (i.e. the operations are all onto maps) in a congruence distributive variety the absence of central subalgebras can be restricted to the nonexistence of binary central subalgebras. Now for $n > 2$ and $2 < l < |A| < \aleph_0$ we construct a functionally noncomplete algebra A_l in a variety K satisfying $\Delta_n(\mathbf{K})$ having central h-ary subalgebras exactly for h in the range from l to $|A| - 1$ and satisfying all the other conditions of the theorem. For this end we first exhibit a ternary algebra $\mathbf{T}_c = \langle A; t_0, \ldots, t_n \rangle$ satisfying (1)—(3) and admitting every central h-ary relation ϱ containing $\{c\} \times A^{h-1}$ $(h = 2, \ldots, |A| - 1)$ where c is a fixed element of A. Set

$$t_1(x, x, y) = x, \quad t_i(x, y, x) = x \quad (i = 1, \ldots, n-1), \tag{6}$$

$$t_{n-1}(x, x, y) = y \quad (n \text{ odd}), \qquad t_{n-1}(x, y, y) = y \quad (n \text{ even}) \tag{7}$$

and $t_i(x, y, z) = c$ in all remaining cases. To establish that a central relation ϱ containing $\{c\} \times A^{h-1}$ is a subalgebra of T_c^h it suffices to prove the following claim. If $1 \leqq i \leqq n$, $a = (a_1, \ldots, a_h) \in A^h \setminus \varrho$ and $t_i(x_{1j}, x_{2j}, x_{3j}) = a_j$ $(j = 1, \ldots, h)$, then $a = (x_{m1}, \ldots, x_{mh})$ for at least one $1 \leqq m \leqq 3$. Note that all a_j are distinct from c

because $a \notin \varrho$. For $i=1$ by virtue of (6) then either $x_{1j}=x_{2j}=a_j$ or $x_{1j}=x_{3j}=a_j$ and therefore $a=(x_{11}, \ldots, x_{1h})$. Similarly by (7) we obtain $a=(x_{31}, \ldots, x_{3h})$ for $i=n-1$ while $a=(x_{11}, \ldots, x_{1h})$ follows directly from (6) for $1<i<n-1$.

Let U_l denote the set of all finitary operations on A whose range has less than l elements (i.e. $|f(A^n)|<l$). Finally $A_l=\langle A; U_l \cup \{t_0, \ldots, t_n\}\rangle$ provides the required example. Indeed for $l \leqq h<|A|$ due to total reflexivity each central h-ary relation ϱ is a subalgebra of $\langle A; f\rangle^h$ for every $f \in U_l$ whereas for $1<h<l$ there is always a range h operation f not admitting ϱ as a subalgebra of $\langle A; f\rangle^h$.

To conclude we mention the following problem arising in this connection. Suppose A is a finite algebra in a congruence distributive variety which is not functionally complete (e.g. a lattice). Functional completeness is achieved by adjoing new operations. The problem is to describe conditions for these added operations. These will depend on the conditions of the theorem failed by A and *a fortiori* by $t_0, \ldots, t_n$ which may impose certain structure on $t_0, \ldots, t_n$ (for example the monotonicity of all $t_0, \ldots, t_n$ with respect to a bounded partial order) and allow us to restrict the conditions for new operations.

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