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Bands of power joined semigroups

STOJAN BOGDANOVIĆ

A *band* is a semigroup in which every element is an idempotent. A semigroup S is called *power joined* if for each pair of elements $a, b\in S$ there exist positive integers m, n with $a^m=b^n$. We say that a semigroup S is a *band of power joined semigroups* if there exists a congruence ϱ such that S/ϱ is a band and each class modϱ is a power joined semigroup. In this case ϱ is called a *band congruence*. One defines analogously semilattices, rectangular bands and left zero bands of power joined semigroups. Bands of power joined semigroups are studied by T. NORDAHL [1] in the *medial* case ($xaby= =xaby$). In the present paper we consider the general case.

For non-defined notions we refer to [2] and [3].

Theorem 1. *A semigroup S is a band of power joined semigroups if and only if*

(A) $$(\forall a, b\in S)(\forall m, n\in N)(\exists r, s\in N)\big((ab)^r=(a^m b^n)^s\big).$$

Proof. Let S be a band Y of power joined semigroups $S_\alpha, \alpha\in Y$. For $a\in S_\alpha$, $\alpha\in Y$ and $b\in S_\beta, \beta\in Y$ we have $a^m b^n\in S_{\alpha\beta}$ for every $m, n\in N$, and thus

$$(ab)^r=(a^m b^n)^s \quad \text{for some} \quad r, s\in N.$$

Conversely, let S satisfy condition (A). We define a relation ϱ on a semigroup S as follows:

(1) $$a\varrho b \Leftrightarrow (\exists m, n\in N)(a^m=b^n).$$

It is clear that ϱ is an equivalence on S. Let $a\varrho b$, then

$$(ab)^t=(a^m b^n)^p=(a^{m+m})^p=a^{2pm}.$$

Hence, each ϱ-class is a power joined subsemigroup of S. We shall show that ϱ is a congruence on S. Suppose $a\varrho b$ and $c\in S$. Then $a^m=b^n$ for some $m, n\in N$, and by (A) we have

(2) $$(ac)^k=(a^m c^t)^r \quad \text{for some} \quad k, r\in N,$$

(3) $$(bc)^{k_1}=(b^n c^t)^{r_1} \quad \text{for some} \quad k_1, r_1\in N.$$

It follows from (2) and (3) that

$$(ac)^{kr_1}=(a^m c^t)^{rr_1}=[(a^m c^t)^{r_1}]^r=[(b^n c^t)^{r_1}]^r=[(bc)^{k_1}]^r=(bc)^{k_1 r}.$$

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Hence, $ac\varrho bc$. Similarly, we obtain $ca\varrho cb$. Consequently, ϱ is a congruence and since $a\varrho a^2$ for every $a\in S$, we have that S is a band of power joined semigroups.

Theorem 2. *A semigroup S is a semilattice of power joined semigroups if and only if*

(B) $$(\forall a, b\in S)(\forall m, n\in N)(\exists r, s\in N)((ba)^r = (a^m b^n)^s).$$

Proof. Let S be a semilattice Y of power joined semigroups $S_\alpha, \alpha\in Y$. For $a\in S_\alpha, \alpha\in Y$ and $b\in S_\beta, \beta\in Y$ we have $a^m b^n, ba\in S_{\alpha\beta}$ for every $m, n\in N$. Hence,

$$(ba)^r = (a^m b^n)^s \quad \text{for some} \quad r, s\in N.$$

Conversely, let S satisfy condition (B). Then

(4) $$(ba)^{r_1} = (ab)^{s_1} \quad \text{for some} \quad r_1, s_1\in N.$$

From (B) and (4) we have

(5) $$(ab)^{s_1 r} = (ba)^{rr_1} = (a^m b^n)^{sr_1}$$

for every $m, n\in N$ and for some $r, s\in N$. It follows from (5) and Theorem 1 that the relation ϱ on S (from (1)) is a band congruence and every equivalence class of S mod ϱ is a power joined semigroup. It follows from (4) that $ab\varrho ba$, so ϱ is a semilattice congruence.

Theorem 3. *A semigroup S is a rectangular band of power joined semigroups if and only if*

(C) $$(\forall a, b, c\in S)(\exists r, s\in N)((abc)^r = (ac)^s).$$

Proof. Let S satisfy condition (C). Then

$$(a^m b^n)^r = (a(a^{m-1} b^{n-1}) b)^r = (ab)^s$$

for every $m, n\in N$ and for some $r, s\in N$. Hence, the condition (A) holds and from this ϱ (from (1)) is a band congruence on S (Theorem 1) and every equivalence class of S mod ϱ is a power joined semigroup. It follows from (C) that ϱ is a rectangular band congruence.

The converse follows immediately.

Corollary. *A semigroup S is a left zero band of power joined semigroups if and only if*

(D) $$(\forall a, b\in S)(\exists r, s\in N)((ab)^r = a^s).$$

References

[1] T. Nordahl, Bands of power joined semigroups, *Semigroup Forum*, **12** (1976), 299—311.

[2] J. M. Howie, *An Introduction to Semigroup Theory*, Academic Press (1976).

[3] M. Petrich, *Introduction to Semigroups*, Merill Publishing Company (1973).

INSTITUTE OF MATHEMATICS
UNIVERSITY OF NOVI SAD
21001 NOVI SAD, P. O. B. 224
YUGOSLAVIA