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Compact approximants

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§ 1. **Introduction.** Interest in approximating a given (bounded linear) operator T on a fixed Hilbert space $\mathfrak{H}$ goes back to [3] and [4], among other references. Each of the preceding sources constructed a compact operator C such that $\|T-C\|$ equaled the distance from T to the compact operators; such an operator C is said to be a compact approximant. Although much attention has been focused on the Calkin algebra and discovering compact approximants with various algebraic properties, only [5] seems to have studied the structure of the set of compact approximants. The main results of [5] show that the set of compact approximants has no extreme points except in the case that a multiple of T is a compact perturbation of a maximal partial isometry and the existence of a finite rank compact approximant is characterized.

This paper attempts to clarify where the investigation of compact approximants stands and to extend it in several directions. The next section compares the methods of [3] and [4] and shows that the resulting compact approximants are essentially the same. The new derivation of the Gohberg—Krein compact approximant will play a key role in several subsequent proofs. Section § 3 gives a simplified criterion for when T has a finite rank compact approximant. A similar criterion is given for T to have a compact approximant which belongs to the Schatten p-class. Section § 4 gives a condition which is necessary and sufficient for T to have a compact approximant with maximal norm.

Throughout this work $U|T|$ will be the polar factorization of T where U is a maximal partial isometry and $|T|$ is $(T^*T)^{1/2}$. For T compact let $s_1(T), s_2(T), \ldots$ be the eigenvalues of $|T|$ in nonincreasing order repeated according to multiplicity. If for some $p \geqq 1$ one has

$$\sum_j s_j(T)^p < \infty$$

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then one says that T belongs to the Schatten p-class C_p which is normed with

$$\|T\|_p = (\sum_j s_j(T)^p)^{1/p}.$$

The quantities $\|T\|_e$ and $r_e(T)$ are defined to be the norm and spectral radius, respectively, of the coset of T in the Calkin algebra.

§ 2. Constructing compact approximants. Since the existence of a compact approximant is proved in [3] as a by-product of the extension of s-numbers from compact operators to bounded operators and the latter is only outlined, a brief development of the Gohberg—Krein compact approximant is offered. Through the use of the characterization of the essential spectrum for a self-adjoint operator, a much quicker derivation is achieved. For any normal operator the essential spectrum coincides with the Weyl spectrum which is all the points in the spectrum except isolated eigenvalues with finite multiplicity. See [p. 376, 6], [2], [1]. First, a fundamental lemma is required.

Lemma 2.1. $\|T\|_e = \||T|\|_e = r_e(|T|)$.

Proof. Let π denote the canonical map of the operators on $\mathfrak{H}$ into the Calkin algebra $\mathscr{C}$. Since $\mathscr{C}$ is a C^*-algebra and π is a *-homomorphism, one knows that $\|\pi(T)\| = \||\pi(T)|\|$ and

$$|\pi(T)| = (\pi(T)^*\pi(T))^{1/2} = (\pi(T^*T))^{1/2} = \pi((T^*T)^{1/2}) = \pi(|T|).$$

Thus, $\|T\|_e = \|\pi(T)\| = \|\pi(|T|)\| = \||T|\|_e$. Since $\pi(|T|)$ is normal in $\mathscr{C}$, its norm equals its spectral radius and the lemma is proved.

It is now clear that the spectrum of $|T|$ in the open interval $(\|T\|_e, \infty)$ consists entirely of isolated eigenvalues with finite multiplicity; let $\{\lambda_1, \lambda_2, \ldots\}$ be a nonincreasing enumeration of that possibly finite set with each eigenvalue repeated according to its multiplicity. Let $E(\cdot)$ be the spectral measure for $|T|$ and denote $E([0, \|T\|_e])\mathfrak{H}$ and $E((\|T\|_e, \infty))\mathfrak{H}$ by $\mathfrak{H}_0$ and $\mathfrak{H}_1$, respectively. Let $\{e_1, e_2, \ldots\}$ be an orthonormal sequence of eigenvectors of $|T|$ such that e_j corresponds to λ_j for $j=1, 2, \ldots$ and note that the spectral representation of $|T|$ restricted to $\mathfrak{H}_1$, denoted $|T|\,|\mathfrak{H}_1$, is

$$\sum_j \langle \cdot, e_j\rangle \lambda_j e_j.$$

If $U|T|$ is the polar factorization of T then the Gohberg—Krein compact approximant of T, denoted by K henceforth, is

$$K = \sum_j \langle \cdot, e_j\rangle(\lambda_j - \|T\|_e)Ue_j.$$

Since $\{\lambda_1, \lambda_2, \ldots\}$ cannot have an accumulation point in $(\|T\|_e, \infty)$, either the above sum is finite or $\{\lambda_1, \lambda_2, \ldots\}$ converges to $\|T\|_e$. In either case, it is apparent that

K is the limit of finite rank operators and consequently K is compact. The following calculation shows that K is a compact approximant for T.

$$\|T-K\| = \|U|T| - U(0|\mathfrak{H}_0 \oplus (|T| - \|T\|_e)|\mathfrak{H}_1)\| \leqq \||T| - 0|\mathfrak{H}_0 \oplus (|T| - \|T\|_e)|\mathfrak{H}_1\| =$$
$$= \max\left\{\||T||\mathfrak{H}_0\|, \|T\|_e I|\mathfrak{H}_1\|\right\} = \|T\|_e.$$

In sharp contrast to the above contruction Holmes and Kripke obtain a compact approximant for T without using the polar factorization of T. They note that if there is an orthogonal projection P with finite codimension such that TP does not assume its norm — i.e. $\|TPx\| = \|TP\|\|x\|$ implies $x=0$ — then $T(I-P)$ is a finite rank compact approximant. In the case that T does not have a finite rank compact approximant, the compact approximant constructed by Holmes and Kripke, denoted by L henceforth, is

$$L = \sum_j \langle \cdot, f_j \rangle (\|Tf_j\| - \|T\|_e) Tf_j / \|Tf_j\|$$

where $\{f_1, f_2, \ldots\}$ is an orthonormal sequence such that $\|Tf_1\| = \|T\|$ and $\|Tf_{j+1}\| = \|TP_j\|$ where P_j is the orthogonal projection onto the orthogonal complement of $\{f_1, \ldots, f_j\}$ for $j = 1, 2, \ldots$.

Since $\|Tx\| = \|U|T|x\| = \||T|x\|$, one has $\||T|f_1\| = \||T|\|$ and $\||T|f_{j+1}\| = \||T|P_j\|$ for $j = 1, 2, \ldots$. This implies that

$$|T|f_1 = \||T|\|f_1 \quad \text{and} \quad |T|f_{j+1} = \||T|P_j\|f_{j+1} \quad \text{for} \quad j = 1, 2, \ldots.$$

Clearly one can choose $f_j = e_j$ for $j = 1, 2, \ldots$ with $\{e_1, e_2, \ldots\}$ given as in the construction of the Gohberg—Krein compact approximant. The formula for L becomes

$$L = \sum_j \langle \cdot, e_j \rangle (\lambda_j - \|T\|_e) Te_j / \|Te_j\| \quad \text{or} \quad L = \sum_j \langle \cdot, e_j \rangle (\lambda_j - \|T\|_e) Ue_j$$

where $\lambda_j = \||T|P_j\|$ for $j = 0, 1, \ldots$ and $P_0 = I$. Here it is used that

$$Te_j / \|Te_j\| = U|T|e_j / \|U|T|e_j\| = U\lambda_j e_j / \|U\lambda_j e_j\| = Ue_j.$$

It is straightforward to see that the formulas for K and L can be restated in forms which are independent of the choices of bases for the eigenspaces of $|T|$. Thus the following theorem has been proved.

Theorem 2.2. *For any operator T which does not have a finite rank compact approximant the Holmes—Kripke compact approximant L coincides with the Gohberg—Krein compact approximant K.*

A slight refinement of the Holmes—Kripke construction produces a unique compact approximant even in the case that T has a finite rank compact approximant. If n is the infimum of the codimension of orthogonal projections P such that TP does not assume its norm then the Holmes—Kripke construction produces a unique rank n compact approximant which coincides with the Gohberg—Krein compact approximant.

§ 3. Compact approximants in C_p. In [5] it is shown that T has a finite rank compact approximant if and only if there is no infinite dimensional closed subspace $\mathscr{E}\subset\mathfrak{H}$ with $\|Tx\|>\|T\|_e\|x\|$ for all nonzero $x\in\mathscr{E}$. Following [5] the set of compact approximants of T is denoted $\mathfrak{K}_T$.

Theorem 3.1. *The following conditions are equivalent.*

(i) $\mathfrak{K}_T$ *contains a finite rank operator.*

(ii) $|T|$ *has only finitely many eigenvalues in* $(\|T\|_e, \infty)$.

(iii) *The Gohberg—Krein compact approximant K has finite rank.*

Proof. The alternative derivation of the Gohberg—Krein compact approximant makes it clear that (ii) implies (iii) which implies (i). Thus, it suffices to show that (i) implies (ii).

Let A be a finite rank operator in $\mathfrak{K}_T$ and, for the sake of a contradiction, assume $|T|$ has infinitely many eigenvalues $\{\lambda_1, \lambda_2, \ldots\}$ in the open interval $(|T|_e, \infty)$. Let $\mathscr{E}$ be the closed span of the eigenspaces of $|T|$ corresponding to $\{\lambda_1, \lambda_2, \ldots\}$. It is easy to see that

$$\||T|x\| > \|T\|_e\|x\| \quad \text{for every} \quad x\in\mathscr{E}, \quad x\neq 0$$

and so

$$\|Tx\| = \|U|T|x\| = \||T|x\| > \|T\|_e\|x\| \quad \text{for such } x.$$

The argument is finished as in [5]. Since the restriction of A to $\mathscr{E}$ must have nontrivial kernel, there is some nonzero $y\in\mathscr{E}\cap\ker A$ and $\|(T-A)y\|=\|Ty\|>\|T\|_e\|y\|$ which contradicts that $A\in\mathfrak{K}_T$.

For a given operator T it is much easier to construct T^*T and check the number of eigenvalues in $(\|T\|_e^2, \infty)$ than it is to examine all possible subspaces $\mathscr{E}$. It is not difficult to see that if T has infinitely many eigenvalues in $\{z\colon |z|>\|T\|_e\}$ then there is an infinite dimensional subspace $\mathscr{E}$. But the converse of the preceding statement is false. Thus, it appears that the criterion for a finite rank compact approximant cannot be simplified any further.

The results in the preceding theorem can be refined to provide a condition which is necessary and sufficient for $\mathfrak{K}_T$ to contain an operator from the Schatten p-class C_p.

Theorem 3.2. *The following conditions are equivalent.*

(i) $\mathfrak{K}_T$ *contains an operator in* C_p.

(ii) *If* $\{\lambda_1, \lambda_2, \ldots\}$ *is a nonincreasing enumeration of the eigenvalues of* $|T|$ *in* $(\|T\|_e, \infty)$, *repeated according to multiplicity, then*

$$\sum_j (\lambda_j - \|T\|_e)^p < \infty.$$

(iii) *The Gohberg—Krein compact approximant K for T belongs to* C_p.

Proof. The alternative derivation of the Gohberg—Krein compact approximant given in section § 2 makes it reasonably clear that (ii) implies (iii). That (iii) implies (i) is trivial and so it suffices to show that (i) implies (ii).

Of course, the spectrum of $|T|$ in $(\|T\|_e, \infty)$ belongs to the complement of the essential spectrum of $|T|$ and, thus, it consists of isolated eigenvalues $\{\lambda_1, \lambda_2, \dots\}$ each with finite multiplicity. Furthermore, the only possible accumulation point of $\{\lambda_1, \lambda_2, \dots\}$ is $\|T\|_e$. Let $\{e_1, e_2, \dots\}$ be an orthonormal sequence such that e_j is an eigenvector for $|T|$ corresponding to λ_j for $j=1, 2, \dots$. Each λ_j is repeated according to its multiplicity.

Let A be a C_p-operator in $\mathfrak{K}_T$ and let $U|T|$ be the polar factorization of T. Note that U^*A is a C_p-operator and

$$\||T| - U^*A\| = \|U^*U|T| - U^*A\| \leqq \|T-A\| = \|T\|_e.$$

Furthermore, one has

$$\|T\|_e^2 \geqq \||T| - U^*A\|^2 \geqq \|(|T| - \operatorname{re} U^*A)^2 + (\operatorname{im} U^*A)^2\| \geqq \||T| - \operatorname{re} U^*A\|^2.$$

Thus $\operatorname{re}(U^*A)$ belongs to $\mathfrak{K}_{|T|}$ and it is routine to see that it is a C_p-operator.

Let C denote $\operatorname{re}(U^*A)$ henceforth, let $\alpha_j = \langle Ce_j, e_j\rangle$ for $j=1, 2, \dots$ and $Ce_j = \alpha_j e_j + x_j$ with $x_j \perp e_j$. Note that

$$\|T\|_e^2 \geqq \||T|e_j - Ce_j\|^2 = \|\lambda_j e_j - Ce_j\|^2 = \|\lambda_j e_j - \alpha_j e_j - x_j\|^2 = (\lambda_j - \alpha_j)^2 + \|x_j\|^2.$$

Thus, $\|T\|_e \geqq |\lambda_j - \alpha_j|$ or $\lambda_j - \|T\|_e \leqq \alpha_j \leqq \lambda_j + \|T\|_e$. This makes it apparent that $\alpha_j \geqq 0$ and $\sum_j (\lambda_j - \|T\|_e)^p \leqq \sum_j \alpha_j^p$. According to [item 5, p. 94, 3]

$$\|C\|_p^p \geqq \sum_j \langle |C| e_j, e_j\rangle^p$$

and since $|C| \geqq C$ it is apparent that

$$\langle |C| e_j, e_j\rangle \geqq \langle Ce_j, e_j\rangle = \alpha_j \quad \text{for} \quad j = 1, 2, \dots .$$

Thus, it is proved that $\sum_j (\lambda_j - \|T\|_e)^p < \infty$ as desired.

§ 4. A compact approximant with maximal norm. Recall that an operator T is said to "assume its norm" provided there is a nonzero vector f such that $\|Tf\| = \|T\|\|f\|$. Such f is said to be a maximal vector for T. It is easy to see that T assumes its norm if and only if $\|T\|^2$ is an eigenvalue of T^*T. Note that $\|T\|^2\|f\|^2 = \|Tf\|^2 = \langle T^*Tf, f\rangle$ and $\|(\|T\|^2 - T^*T)^{1/2} f\|^2 = 0$ are equivalent. This makes it clear, for example, that any compact operator assumes its norm.

The condition that T assume its norm played a key role in [4] and now it plays a key role in determining when $\mathfrak{K}_T$ contains an operator with maximal norm — i.e. $A \in \mathfrak{K}_T$ such that $B \in \mathfrak{K}_T$ implies $\|B\| \leqq \|A\|$.

Theorem 4.1. *There is a compact approximant A of T, i.e. $A\in\mathfrak{K}_T$, with maximal norm if and only if T assumes its norm.*

Proof. First it is shown that if T does not assume its norm then $\mathfrak{K}_T$ does not contain an operator with maximal norm. For any $B\in\mathfrak{K}_T$ and f a maximal unit vector for B one has

$$\|T\|_e \geqq \|(B-T)f\| \geqq \|Bf\|-\|Tf\| = \|B\|-\|Tf\|$$

or

$$\|T\|_e+\|T\| > \|T\|_e+\|Tf\| \geqq \|B\|.$$

Thus, it would suffice to show that $\|T\|_e+\|T\|$ is the supremum of the norms of operators in $\mathfrak{K}_T$.

Since T does not assume its norm, $|T|$ does not assume its norm. Since $\|T\|$ is not an eigenvalue for $|T|$, it must be an accumulation point for the spectrum. Consequently $\||T|\|_e$ equals $\||T|\|$ and equivalently $\|T\|_e$ equals $\|T\|$. Let $E(\cdot)$ be the spectral measure for $|T|$ and choose a unit vector f_n from $E([\|T\|-1/n, \|T\|])\mathfrak{H}$. Define C_n by

$$C_n = \langle\cdot, f_n\rangle(2\|T\|-1/n)f_n.$$

Note that C_n is rank one and $\|C_n\|$ converges to $2\|T\|=\|T\|+\|T\|_e$.

It now suffices for this half of the proof to show that C_n is a compact approximant for $|T|$. Denote $E([0, \|T\|-1/n))\mathfrak{H}$ and $E([\|T\|-1/n, \|T\|])\mathfrak{H}$ by $\mathfrak{H}_0$ and $\mathfrak{H}_1$, respectively. Since $\mathfrak{H}_0$ reduces $|T|-C_n$ to $|T|\,|\mathfrak{H}_0$ it suffices to sow that

$$\|(|T|-C_n)|\mathfrak{H}_1\| \leqq \|T\|_e = \||T|\|_e$$

where $A|\mathfrak{H}_1$ denotes the restriction of A to $\mathfrak{H}_1$. Since the above restriction is self-adjoint it clearly suffices to show that

$$\langle(|T|-C_n)g, g\rangle\in[-\|T\|, \|T\|]$$

for every unit vector g in $\mathfrak{H}_1$. Since the numerical range of C_n is $[0, 2\|T\|-1/n]$, one has

$$-\|T\| = \|T\|-1/n-(2\|T\|-1/n) \leqq \|T\|-1/n-\langle C_n g, g\rangle \leqq$$
$$\leqq \langle(|T|-C_n)g, g\rangle \leqq \|T\|-\langle C_n g, g\rangle \leqq \|T\|.$$

This shows that $\|T\|_e+\|T\|=2\|T\|$ is the supremum of the norms of the operators UC_n which belong to $\mathfrak{K}_T$, where $U|T|$ is the polar factorization of T. Thus, half of the theorem is proved.

Now it is assumed that T has a maximal vector and it is to be shown that $\mathfrak{K}_T$ contains an operator with norm $\|T\|_e+\|T\|$. Since T assumes its norm, $\|T\|^2$ is an eigenvalue for T^*T and this implies $\|T\|$ is an eigenvalue for $|T|$. First, consider the case that $\|T\|$ has finite multiplicity for $|T|$ and let P be the orthogonal projection onto the corresponding eigenspace. For brevity sake let β denote

$\|T\|_e + \|T\|$. In order to show that $\beta P \in \mathfrak{K}_{|T|}$ it is noted that the restriction of $|T| - \beta P$ to $(I-P)\mathfrak{H}$ is just $|T|\,|(I-P)\mathfrak{H}$. Thus, it suffices to show that

$$\|(|T| - \beta P)|P\mathfrak{H}\| \leqq \|T\|_e.$$

Since $(|T| - \beta P)|P\mathfrak{H}$ is just $-\|T\|_e P|P\mathfrak{H}$, the above inequality is clear and $\beta P \in \mathfrak{K}_{|T|}$. It follows that βUP belongs to $\mathfrak{K}_T$ where $U|T|$ is the polar factorization of T.

It only remains to deal with the case that $\|T\|$ is an infinite dimensional eigenvalue of $|T|$. In this case it is clear that $\|T\|_e = \||T|\|_e = \|T\|$. Let P be the orthogonal projection onto some nontrivial finite dimensional subspace of the eigenspace for $|T|$ correspoding to $\|T\|$. Since $(I-P)\mathfrak{H}$ reduces $|T| - 2\|T\|P$ to $|T|\,|(I-P)\mathfrak{H}$ and $P\mathfrak{H}$ reduces it to $-\|T\|P|P\mathfrak{H}$, it is apparent that $2\|T\|P$ belongs to $\mathfrak{K}_{|T|}$. Thus $2\|T\|UP$ belongs to $\mathfrak{K}_T$ and the proof of the theorem is complete.

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