Acta Sci. Math., 44 (1982), 13—16

Tolerance Hamiltonian varieties of algebras

IVAN CHAJDA

The concept of Hamiltonian algebras was first introduced for groups. *A group $\mathfrak{G}$ is Hamiltonian* if every subgroup of $\mathfrak{G}$ is normal, i.e., a class of some congruence on $\mathfrak{G}$. EVANS [6] introduced the Hamiltonian property for loops and KLUKOVITS [7] generalized this concept for universal algebras and varieties: *an algebra $\mathfrak{A}$ is Hamiltonian* if every subalgebra of $\mathfrak{A}$ is a class of some congruence on $\mathfrak{A}$; *a variety $\mathcal{V}$ is Hamiltonian* if each $\mathfrak{A}\in\mathcal{V}$ has this property.

In [7], Hamiltonian varieties are characterized by a nice $\forall\exists$-condition. Such conditions are also used for characterizations of varieties fulfilling given tolerance identities [3]. It is natural to ask whether the Hamiltonian property can be extended also for tolerances (see e.g. [5]) and which $\forall\exists$-condition characterizes such varieties.

By a *tolerance* on an algebra $\mathfrak{A}=(A, F)$ is meant a reflexive and symmetric binary relation T on A having the *Substitution Property* with respect to F (i.e. T is a subalgebra of the direct product $\mathfrak{A}\times\mathfrak{A}$). Thus each congruence is a tolerance but not vice versa.

Definition 1. Let T be a tolerance on an algebra $\mathfrak{A}=(A, F)$. Call $\emptyset\neq B\subseteqq A$ a *block of T* provided

(i) $B\times B\subseteqq T$,

(ii) B is a maximal subset of A with respect to (i), i.e. if $B\subseteqq C$ and $C\times C\subseteqq T$, then $B=C$.

Clearly, if a tolerance T on $\mathfrak{A}$ is a congruence on $\mathfrak{A}$, every block of T is a congruence class of T and vice versa. Thus blocks of tolerances are generalizations of congruence classes.

The paper [2] contains a characterization of the property that every block of each tolerance on $\mathfrak{A}$ is a subalgebra of $\mathfrak{A}$. The objective of this paper is to describe the converse situation, namely:

Received November 20, 1979.

Definition 2. *An algebra $\mathfrak{A}$ is tolerance Hamiltonian* if every subalgebra of $\mathfrak{A}$ is a block of some tolerance on $\mathfrak{A}$. *A variety $\mathscr{V}$ is tolerance Hamiltonian* if each $\mathfrak{A}\in\mathscr{V}$ has this property.

Although [1], [2] contain necessary and sufficient conditions under which a subset of an algebra is a block of some tolerance on it, these conditions cannot be used in the way as Mal'cev's Lemma in [7]. The proof of our Theorem 1 is based on a characterization given by Lemma 3 below.

For the sake of brevity, write $\vec{x}_j$ instead of $x_1, \ldots, x_n$ and $\vec{y}_i$ instead of $y_1, \ldots, y_m$ if the integers n, m are given.

Theorem 1. *Let $\mathscr{V}$ be a variety of algebras. The following conditions are equivalent:*

(1) *$\mathscr{V}$ is tolerance Hamiltonian.*

(2) *For every $(m+n+k)$-ary polynomial p and for every $(m+n+1)$-ary polynomial t there exists an $(m+n+1)$-ary polynomial q over $\mathscr{V}$ such that*

$$p\big(t(\vec{y}_i, \vec{x}_j, z) \ldots, t(\vec{y}_i, \vec{x}_j, z), x_1, \ldots, x_n, v_1, \ldots, v_k\big) = t(\vec{y}_i, \vec{x}_j, z)$$

implies

$$p\big(y_1, \ldots, y_m, t(\vec{y}_i, \vec{x}_j, z), \ldots, t(\vec{y}_i, \vec{x}_j, z), v_1, \ldots, v_k\big) = q(\vec{y}_i, \vec{x}_j, z).$$

Let us begin the proof of Theorem 1 with some lemmas. If T is a binary relation on a set $\mathfrak{A}$, we denote $[z]_T = \{a\in A; \langle a, z\rangle\in T\}$.

Lemma 1. *Let $\mathfrak{A}=(A, F)$ be an algebra and $z\in B\subseteqq A$. The following conditions are equivalent:*

(a) *$B=[z]_T$ for some tolerance T on $\mathfrak{A}$.*

(b) *For every $(m+n)$-ary algebraic function φ over $\mathfrak{A}$,*

$$\varphi(z, \ldots, z, b_1, \ldots, b_n) = z \quad \textit{for some} \quad b_i\in B$$

implies

$$\varphi(a_1, \ldots, a_m, z, \ldots, z)\in B \quad \textit{for each} \quad a_j\in B.$$

Proof. $(a)\Rightarrow(b)$: Routine.

$(b)\Rightarrow(a)$: Let $R=\{\langle x, x\rangle;\ x\in A\}\cup\{\langle x, z\rangle;\ x\in B\}\cup\{\langle z, x\rangle;\ x\in B\}$. Let T be the set of all $\langle a, b\rangle$ such that $a=\varphi(a_1, \ldots, a_k)$, $b=\varphi(b_1, \ldots, b_k)$ for some $\langle a_i, b_i\rangle\in R$ and for some algebraic function φ over $\mathfrak{A}$. Clearly T is a tolerance on $\mathfrak{A}$. It only remains to prove $B=[z]_T$. Evidenty, $B\subseteqq[z]_T$. Suppose $c\in[z]_T$. Then $\langle c, z\rangle\in T$, i.e. $c=\psi(a_1, \ldots, a_k)$, $z=\psi(b_1, \ldots, b_k)$ for some $\langle a_i, b_i\rangle\in R$ and some k–ary algebraic function ψ. We can suppose, that $k=m+n+k'$ $(m\geqq 0, n\geqq 0, k'\geqq 0)$, moreover, $b_i=z$ for $i=1, \ldots, m$ and $a_i=z$ for $i=m+1, \ldots, m+n$ and $a_i=b_i$ for $i=m+n+1, \ldots, k$. Put

$$\varphi(\xi_1, \ldots, \xi_{m+n}) = \psi(\xi_1, \ldots, \xi_{m+n}, a_{m+n+1}, \ldots, a_k).$$

Since $z=\varphi(z, \ldots, z, b_1, \ldots, b_n)$, by (b) we obtain $c=\varphi(a_1, \ldots, a_m, z, \ldots, z)\in B$ proving the reverse inclusion $[z]_T\subseteqq B$.

Lemma 2. *Let $\mathfrak{A}=(A, F)$ and let T be a tolerance on $\mathfrak{A}$. For $\emptyset\neq B\subseteqq A$ the following conditions are equivalent:*

(a) *B is a block of T.*

(b) $B=\bigcap\{[z]_T;\ z\in B\}$.

Proof. Routine.

Lemma 3. *Let $\mathfrak{A}=(A, F)$ and $\emptyset\neq B\subseteqq A$. The following conditions are equivalent:*

(a) *B is a block of some tolerance on $\mathfrak{A}$.*

(b) *For every $(m+n)$-ary algebraic function φ over $\mathfrak{A}$ and for each $z\in B$,*

$$\varphi(z, \ldots, z, b_1, \ldots, b_n)=z \quad \textit{for some} \quad b_i\in B$$

implies

$$\varphi(a_1, \ldots, a_m, z, \ldots, z)\in B \quad \textit{for each} \quad a_j\in B.$$

This follows directly from Lemmas 1 and 2.

Proof of Theorem 1. (1)$\Rightarrow$(2): Let p and t be $(m+n+k)$-ary and $(m+n+1)$-ary polynomials over $\mathscr{V}$, respectively, such that

$$(*) \qquad p\big(t(\vec{y}_i, \vec{x}_j, z), \ldots, t(\vec{y}_i, \vec{x}_j, z), x_1, \ldots, x_n, v_1, \ldots, v_k\big)=t(\vec{y}_i, \vec{x}_j, z).$$

Let $\mathfrak{A}=(A, F)=\mathfrak{F}_{m+n+k+1}$ be the $\mathscr{V}$-free algebra with the set of free generators $\{x_1, \ldots, x_n, y_1, \ldots, y_m, v_1, \ldots, v_k, z\}$ and $\mathfrak{B}=(B, F)=\mathfrak{F}_{m+n+1}$ the $\mathscr{V}$-free algebra with generators $\{x_1, \ldots, x_n, y_1, \ldots, y_m, z\}$. Hence $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$. Since $\mathscr{V}$ is tolerance Hamiltonian, B is a block of some tolerance on $\mathfrak{A}$. By Lemma 3, $(*)$ yields $p(y_1, \ldots, y_m, t(\vec{y}_i, \vec{x}_j, z), \ldots, t(\vec{y}_i, \vec{x}_j, z), v_1, \ldots, v_k)\in B$. Since $\mathfrak{B}=\mathfrak{F}_{m+n+1}$ there exists an $(m+n+1)$-ary polynomial q over $\mathscr{V}$ such that (2) of Theorem 1 is valid.

(2)$\Rightarrow$(1): Let $\mathscr{V}$ be a variety fulfilling (2), $\mathfrak{A}=(A, F)\in\mathscr{V}$, $\mathfrak{B}=(B, F)$ a subalgebra of $\mathfrak{A}$ and $z\in B$. Let φ be an arbitrary $(m+n)$-ary algebraic function over $\mathfrak{A}$ and p its generating polynomial, i.e. $\varphi(\xi_1, \ldots, \xi_{m+n})=p(\xi_1, \ldots, \xi_{m+n}, c_1, \ldots, c_k)$ for some $c_1, \ldots, c_k\in A$. If $\varphi(z, \ldots, z, b_1, \ldots, b_n)=z$ for some $b_i\in B$, then, by (2),

$$\varphi(a_1, \ldots, a_m, z, \ldots, z)=q(a_1, \ldots, a_m, b_1, \ldots, b_n, z)\in B$$

for each $a_1, \ldots, a_m\in B$. By Lemma 3, $\mathfrak{A}$ and also $\mathscr{V}$ are tolerance Hamiltonian.

Theorem 2. *The tolerance Hamiltonian property is local, i.e. an algebra $\mathfrak{A}$ is tolerance Hamiltonian if and only if every finitely generated subalgebra of $\mathfrak{A}$ is a block of some tolerance on $\mathfrak{A}$.*

Proof. It is a direct consequence of Lemma 3: if $\mathfrak{B}=(B, F)$ is a subalgebra of $\mathfrak{A}$ which is not a block of any tolerance on $\mathfrak{A}$ and every finitely generated subalgebra is, then there exist $z \in B$ and an $(m+n)$-ary algebraic function φ over $\mathfrak{A}$ such that $\varphi(z, \ldots, z, b_1, \ldots, b_n)=z$ and $\varphi(a_1, \ldots, a_m, z, \ldots, z) \notin B$ for some $a_1, \ldots, a_m, b_1, \ldots, b_n \in B$. Hence the subalgebra $\mathfrak{C}$ of $\mathfrak{A}$ generated by $\{a_1, \ldots, a_m, b_1, \ldots, b_n, z\}$ is not a block of any tolerance on A which contradicts the assumptions. The converse implication is trivial.

Theorem 3. *The variety of all semilattices is tolerance Hamiltonian (but not Hamiltonian).*

Proof. If p, t are semilattice polynomials fulfilling the assumptions of the condition (2) of Theorem 1, then clearly p does not depend on $v_1, \ldots, v_k$ and the statement of (2) is evident. Thus Theorem 3 is a direct consequence of Theorem 1. By the theorem of Klukovits [7], this variety is evidently not Hamiltonian.

Remark. As it was proved by Zelinka [8], on every at least three element semilattice there exists a tolerance which is not a congruence.

Theorem 4. *No non-trivial variety of lattices is tolerance Hamiltonian.*

Proof. Let p and t be $(2+0+1)$-ary (i.e. ternary) lattice polynomials given as follows:

$$p(x, y, z) = x \vee (y \wedge z), \quad t(x, y, z) = z.$$

Then we have $p(t(y_1, y_2, z), t(y_1, y_2, z), v_1)=p(z, z, v_1)=z=t(y_1, y_2, z)$, thus the assumptions of (2) of Theorem 1 are valid, but $p(y_1, y_2, v_1)$ is essentially dependent on v_1. Hence, no polynomial q of the required type exists.

References

[1] I. Chajda, Partitions, coverings and blocks of compatible relations, *Glasnik Matem. Zagreb,* **14** (1979), 21—26.

[2] I. Chajda, Characterizations of relational blocks, *Algebra Universalis,* **10** (1980), 65—69.

[3] I. Chajda, Distributivity and modularity of lattices of tolerances, *Algebra Universalis,* **12** (1981), 247—255.

[4] I. Chajda, J. Duda, Blocks of binary relations, *Ann. Univ. Sci. Budapest,* Sectio Math., **22—23** (1979—1980), 3—9.

[5] I. Chajda—B. Zelinka, Lattices of tolerances, *Časop. pěst. matem.,* **102** (1977), 10—24.

[6] T. Evans, Properties of algebras almost equivalent to identities, *J. London Math. Soc.,* **35** (1962), 53—59.

[7] L. Klukovits, Hamiltonian varieties of universal algebras, *Acta Sci. Math.,* **37** (1975), 11—15.

[8] B. Zelinka, Tolerance relations on semilattices, *CMUC* **16, 2** (1975), 333—338.

TŘÍDA LIDOVÝCH MILICÍ 22
750 00 PŘEROV
CZECHOSLOVAKIA