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Factor lattices by tolerances

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1. Introduction

Given a lattice L, a binary, reflexive, symmetric *and compatible* relation $\varrho \subseteqq L \times L$ is said to be a tolerance relation (or shortly tolerance) of L. Tolerances of lattices were firstly investigated by CHAJDA and ZELINKA [2]. Recently the importance of this concept has grown: a finite lattice is monotone functionally complete iff it has the trivial tolerances only (cf. KINDERMANN [4]). Moreover, KINDERMANN [4] has shown that the algebraic functions on a finite lattice are just the monotone functions preserving its tolerances.

Our aims in the present paper are to introduce the concept of L/ϱ (i.e., factor lattice by a tolerance ϱ), to give a more handlable description of L/ϱ, and to give a structure-like theorem for lattices with the following consequence: every finite lattice is isomorphic to D/ϱ for a suitable finite distributive lattice D. A characterization for tolerances of lattices will be presented in Theorem 2.

Given a reflexive and symmetric relation ϱ over a non-empty set A, a subset H of A is called a *block* of ϱ if $H^2 \subseteqq \varrho$ but $G^2 \subseteqq \varrho$ for no $H \subset G \subseteqq A$. I.e., H is a block of ϱ if it is maximal with respect to the property: for any $a, b \in H$ $a\varrho b$. Let the set of all blocks be denoted by $\mathscr{C}_\varrho$. On the other hand, certain subsets of $P^+(A)$, the set of non-empty subsets of A, can be called *quasi-partitions* on A (cf. CHAJDA, NIEDERLE, and ZELINKA [1]). The connection of these two concepts (see [1] again) is the following. If ϱ is a reflexive and symmetric relation then $\mathscr{C}_\varrho$ is a quasi-partition. For a quasi-partition $\mathscr{C}$ the relation $\varrho_{\mathscr{C}} = \{(a, b) : \{a, b\} \subseteqq H$ for some $H \in \mathscr{C}\}$ is reflexive and symmetric. The map $\varrho \mapsto \mathscr{C}_\varrho$, from the set of reflexive and symmetric relations on A into the set of quasi-partitions on A, is bijective and its inverse map is $\mathscr{C} \mapsto \varrho_{\mathscr{C}}$. Moreover, a reflexive and symmetric relation ϱ is an equivalence iff $\mathscr{C}_\varrho$ is a partition. Therefore the following notion of factor latti-

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ces by tolerances seems to be a natural generalization of that of factor lattices by congruences.

For definition, let ϱ be a tolerance of a lattice L. For blocks G and H of and $\circ\in\{\wedge, \vee\}$ we define $G\circ H$ to be the unique block of ϱ for which $\{g\circ h: g\in G, h\in H\}\subseteq G\circ H$. (The correctness of this definition will be shown!). Now L/ϱ, the factor lattice by ϱ, is the set of all blocks of ϱ equipped with the above defined $\wedge$ and $\vee$ operations. I.e., the notation L/ϱ is used instead of $\mathscr{C}_\varrho$ and $L/\varrho=(L/\varrho; \wedge, \vee)$. It is worth mentioning that L/ϱ is the factor lattice in the usual sence whenever the tolerance ϱ happens to be a congruence relation.

2. L/ϱ is an algebra

In this section the correctness of the definition of L/ϱ will be shown. Suppose $G, H\in L/\varrho$. If $g_i\in G$, $h_i\in H$ $(i=1,2)$ then the compatibility of ϱ yields $(g_1\circ h_1, g_2\circ h_2)\in\varrho$. I.e., $\{g\circ h: g\in G, h\in H\}^2\subseteq\varrho$. Now Zorn Lemma applies and $\{g\circ h: g\in G, h\in H\}\subseteq E$ for some $E\in L/\varrho$.

To show the uniqueness of E some preliminaries are needed. In what follows in this section let ϱ be a fixed tolerance of a lattice L.

Lemma 1 (Chajda and Zelinka [2]). *For $a, b\in L$, $(a, b)\in\varrho$ if and only if $[a\wedge b, a\vee b]^2\subseteq\varrho$.*

Lemma 2. *The blocks of ϱ are convex sublattices of L.*

Proof. Let C be a block of ϱ, and suppose $a, b\in C$. For an arbitrary $x\in C$ $a\varrho x$ and $b\varrho x$, whence $a\vee b\varrho x\vee x=x$. I.e., $(C\cup\{a\vee b\})^2\subseteq\varrho$ and the maximality of C yields $a\vee b\in C$. Therefore C is a sublattice. If $a, b\in C$, $u\in L$, and $a\leqq u\leqq b$, then, for any $x\in C$, $a\wedge x\in C$ and $b\vee x\in C$. Thus $a\wedge x\varrho b\vee x$, and Lemma 1 yields $x\varrho c$. Finally, $u\in C$ follows from the maximality of C again. Q. e. d.

For a subset X of L let $[X)$ and $(X]$ denote the dual ideal and ideal generated by X, respectively. We write $[a)$ instead of $[\{a\})$, and dually.

Lemma 3 (Grätzer [3]). *For any convex sublattice C of L the equality $C=[C)\cap(C]$ holds. Moreover, if C is the intersection of a dual ideal D and an ideal I, then $D=[C)$ and $I=(C]$.*

Definition 1. For ideals I_1 and I_2 let $I_1\wedge I_2=I_1\cap I_2$, $I_1\vee I_2=\{x: x\leqq c\vee d$ for some $c\in I_1, d\in I_2\}=(I_1\cup I_2]$, and let $I_1\leqq I_2$ mean $I_1\subseteq I_2$. On the other hand for dual ideals D_1 and D_2 let $D_1\wedge D_2=\{x: x\geqq c\wedge d$ for some $c\in D_1, d\in D_2\}=[D_1\cup D_2)$, $D_1\vee D_2=D_1\cap D_2$, and let $D_1\leqq D_2$ mean $D_1\supseteq D_2$.

The motivation of this definition will be given in the remark to Lemma 4.

Proposition 1. *If* $(C]=(D]$ *for* $C, D\in L/\varrho$ *then* $C=D$.

Proof. First we show that $U=([C)\wedge[D))\cap(C]\in L/\varrho$. Suppose $x_1, x_2\in U$. Then $x_i\geqq c_i'\wedge d_i'$ for $c_i'\in[C)$ and $d_i'\in[D)$, $i=1,2$. Let $c\in C$ and $d\in D$, and set $c_i=c_i'\wedge c$, $d_i=d_i'\wedge d$ $(i=1, 2)$. Then, by Lemma 3, we have $x_i\geqq c\wedge d_i$, $c_i\in C$, and $d_i\in D$ for $i=1,2$. Set $a=x_1\vee c_1\vee x_2\vee c_2$ and $b=x_1\vee d_1\vee x_2\vee d_2$. By $(C]=(D]$ and Lemma 3 we obtain $a\in C$, $b\in D$, and $a\vee b\in C\cap D$. Since $c_1\wedge c_2\in C$ and $d_1\wedge d_2\in D$, $(c_1\wedge c_2, a\vee b)\in\varrho$ and $(d_1\wedge d_2, a\vee b)\in\varrho$ follow. The compatibility of ϱ yields $(c_1\wedge c_2\wedge d_1\wedge d_2, a\vee b)\in\varrho$. But $x_1, x_2\in[c_1\wedge d_1\wedge c_2\wedge d_2, a\vee b]$, whence Lemma 1 implies $(x_1, x_2)\in\varrho$. We have shown that $U^2\subseteqq\varrho$. $U\supseteqq[C)\cap(C]=C$ and the maximality of C yields $U=C\in L/\varrho$. By making use of $(C]=(D]$ we obtain $U\supseteqq[D)\cap(D]=D$ similarly. Therefore $U=D$ as well. Q. e. d.

Proposition 2. *Suppose* $C, D, E\in L/\varrho$ *and* $\{c\vee d\colon c\in C,\ d\in D\}\subseteqq E$. *Then* $[C)\vee[D)=[E)$.

Proof. Let $\{c\vee d\colon c\in C, d\in D\}$ be denoted by U. Since $[U)=[C)\cap[D)=[C)\vee[D)$, $[C)\vee[D)\subseteqq[E)$ follows easily. To show the required equality let $[C)\cap[D)=[C)\vee[D)\subset E$ be assumed. Then $[E)\setminus([C)\cap[D))\neq\emptyset$, and one can easily see that $E\setminus([C)\cap[D))\neq\emptyset$ as well. Therefore an element a can be chosen so that $a\in E$ and, e.g., $a\notin[C)$. Choosing elements $c\in C$ and $d\in D$ we can assume that $a\leqq c\vee d$. (Otherwise a could be replaced by $(c\vee d)\wedge a$, because $c\vee d, (c\vee d)\wedge a\in E$ and $(c\vee d)\wedge a\notin[C)$.) Evidently we have $a\wedge c\notin C$. For an arbitrary $x\in C$ we can proceed as follows. From $(x\vee c)\vee d\in U\subseteqq E$ and $a\in E$ we obtain $(x\vee c\vee d, a)\in\varrho$. From $x, c\in C$ and Lemma 2 $(x\vee c, x\wedge c)\in\varrho$ follows. By meeting we obtain $(x\vee c, a\wedge x\wedge c)\in\varrho$. From Lemma 1 $(x, a\wedge c)\in\varrho$ can be concluded. Consequently $(C\cup\{a\vee c\})^2\subseteqq\varrho$, a contradiction. Q. e. d.

Now Propositions 1 and 2 and their dual statements imply the correctness of the definition of L/ϱ.

3. L/ϱ is a lattice

Before proving what is stated in the title of this section, a more handlable description of L/ϱ is necessary.

Lemma 4. *Suppose* $E=C\vee D$ *and* $F=C\wedge D$ *for* $C, D, E, F\in L/\varrho$. *Then we have* $[C)\vee[D)=[E)$ *and* $(C]\vee(D]\leqq(E]$. *The dual statement,* $[C)\wedge[D)\geqq[F)$ *and* $(C]\wedge(D]=(F]$, *also holds.*

Remark. If for $X\in\{C, D, E, F\}\subseteqq L/\varrho$ X is an interval $[x_1, x_2]$, and $E=C\vee D$, $F=C\wedge D$, then Lemma 4 yields $c_1\vee d_1=e_1$, $c_2\vee d_2\leqq e_2$, $c_1\wedge d_1\geqq f_1$, and

$c_2 \wedge d_2 = f_2$. (This is always the case when L is a finite lattice.) This remark can supply a motivation of Definition 1.

Proof. Since $\{c \vee d: c \in C, d \in D\} \subseteq E$, we have $(C] \vee (D] = (\{c \vee d: c \in C, d \in D\}] \subseteq \subseteq (E]$, implying $(C] \vee (D] \leqq (E]$. The rest follows from Proposition 2 and the Duality Principle.

This lemma enables us to strengthen Proposition 1:

Corollary 1. *For* $C, D \in L/\varrho$ *we have* $[C) \leqq [D)$ *if and only if* $(C] \leqq (D]$.

Really, Proposition 1 follows from this corollary and Lemma 3.

Proof. Suppose $[C) \leqq [D)$, then $[C) \vee [D) = [D)$. Proposition 2 and the dual of Proposition 1 imply $C \vee D = D$. By making use of Lemma 4 we obtain $(C] \leqq \leqq (C] \vee (D] \leqq (D]$. The Duality Principle yields the converse implication. Q. e. d.

Theorem 1. *For any tolerance ϱ of an arbitrary lattice L, L/ϱ is a lattice again.*

Proof. By the Duality Principle it is enough to show that the $\vee$ operation is commutative and associative, and one of the absorption laws holds. Since the join for dual ideals in Definition 1 is commutative and associative, the commutativity and associativity are straightforward consequences of Proposition 2 and the dual of Proposition 1. To show $C \vee (C \wedge D) = C$, for $C, D \in L/\varrho$, by the dual of Proposition 1 it is enough to check $[C \vee (C \wedge D)) = [C)$. But, by Lemma 4, $[C) \geqq [C) \wedge [D) \geqq \geqq [C \wedge D)$, and so $[C \vee (C \wedge D)) = [C) \wedge [C \wedge D) = [C)$. Q. e. d.

The following theorem deals with the connection between tolerances and corresponding quasi-partitions on lattices. For a tolerance ϱ on a lattice L, $\mathscr{C}_\varrho = L/\varrho$ and $P^+(L)$ were defined in the Introduction.

Theorem 2. *Given a lattice L, for any $\mathscr{C} \subseteq P^+(L)$ the following two conditions are equivalent.*

(a) $\mathscr{C} = \mathscr{C}_\varrho (= L/\varrho)$ *for some tolerance ϱ on L.*

(b) *$\mathscr{C}$ has the following six properties:*

(C1) *The elements of $\mathscr{C}$ are convex sublattices of L;*

(C2) $\bigcup_{C \in \mathscr{C}} C = L$;

(C3) *For any $C, D \in \mathscr{C}$, $[C) = [D)$ is equivalent to $(C] = (D]$;*

(C4) *For any $C, D \in \mathscr{C}$ there exist $E, F \in \mathscr{C}$ such that $[C) \vee [D) = [E)$, $(C] \vee (D] \leqq (E]$, and $[C) \wedge [D) \geqq [F)$, $(C] \wedge (D] = (F]$;*

(C5) *Let $x \in L$, $d \in C \in \mathscr{C}$ be arbitrary. If for any $e \in C \cap (d]$ there exists C_e such that $\{e, x\} \subseteq C_e \in \mathscr{C}$ then $x \in (C]$, and, dually, if for any $f \in C \cap [d)$ there exists C_f such that $\{f, x\} \subseteq C_f \in \mathscr{C}$ then $x \in [C)$;*

(C6) *If U is a convex sublattice of L and for any $a, b \in U$ there exists $D \in \mathscr{C}$ containing both a and b, then $U \subseteq C$ for some $C \in \mathscr{C}$.*

Moreover, if L is a finite lattice then (C5) *and* (C6) *follow already from* (C1), (C2), (C3), *and* (C4).

Proof. (a) *implies* (b). (C1), (C3) and (C4) is involved in Lemma 2, Corrollary 1, and Lemma 4, respectively. Zorn Lemma yields (C2) and (C6). Suppose $x\in L$, $d\in C\in\mathscr{C}_\varrho=$ $=L/\varrho$, and for any $e\in C\cap(d]$ there exists $C_e\in L/\varrho$ such that $\{e,x\}\subseteq C_e$. Considering the set $X=\{x\}\cup(C\cap(d])$ we have $X^2\subseteq\varrho$. Extending X to an element of L/ϱ, say E, we obtain $[C)=[C\cap(d])\subseteq[X)\subseteq[E)$, i.e. $[C)\geqq[E)$. Corollary 1 yields $(C]\geqq(E]$. Hence $x\in X\subseteq E\subseteq(E]\subseteq(C]$. The proof of (C5) is completed by the Duality Principle.

(b) *implies* (a). Suppose $\mathscr{C}$ satisfies the requirements of (b) and let ϱ denote $\varrho_{\mathscr{C}}=\{(a,b)\in L^2\colon \{a,b\}\subseteq C$ for some $C\in\mathscr{C}\}$. The relation ϱ is evidently symmetric; and it is reflexive by (C2). If $C,D,E\in\mathscr{C}$, U denotes the set $\{c\vee d\colon c\in C, d\in D\}$, $[C)\vee[D)=[E)$, and $(C]\vee(D]\leqq(E]$ then $U\subseteq E$. Indeed, $U\subseteq[C)\cap[D)=[E)$, $U\subseteq$ $\subseteq(C]\vee(D]\subseteq(E]$, and, by Lemma 3, $E=[E)\cap(E]$. Now (C4) and the Duality Principle yield the compatibility of ϱ. Therefore ϱ is a tolerance on L, and $\mathscr{C}_\varrho=\mathscr{C}$ has to be shown. Suppose $C\in\mathscr{C}$. Then $C^2\subseteq\varrho$. If $(x,c)\in\varrho$ for any $c\in C$ then $x\in[C)\cap(C]=C$ by (C5) and Lemma 3. Thus $C\in\mathscr{C}_\varrho$ and $\mathscr{C}\subseteq\mathscr{C}_\varrho$. Conversely, if $U\in\mathscr{C}_\varrho$ then $U\subseteq C$ for some $C\in\mathscr{C}$ by (C6). But then both U and C belong to $\mathscr{C}_\varrho$, whence $U=C$. $\mathscr{C}=\mathscr{C}_\varrho$ has been shown.

Finally, suppose L is a finite lattice, $\mathscr{C}\subseteq P^+(L)$ and $\mathscr{C}$ satisfies (C1), (C2), (C3), and (C4). Since any convex sublattice of L is an interval, (C6) evidently holds. Suppose $x\in L$, $d\in C=[a,b]\in\mathscr{C}$ and for any $e\in C\cap(d]$ there exists C_e such that $\{e,x\}\subseteq$ $\subseteq C_e\in\mathscr{C}$. Then $\{a,x\}\subseteq C_a=[u,v]$. Since $u\leqq a$, we obtain $[C)\vee[C_a)=[C)$. Now (C4) together with (C3) yield $(C]\vee(C_a]=(C]$, i.e., $b\vee v=b$. Hence $x\leqq v\leqq b$, which implies $x\in(C]$. (C5) is satisfied by the Duality Principle. Q. e. d.

Note that usually it is convenient to give $\mathscr{C}_\varrho$ instead of ϱ. For example, let D be a five-element chain, say $D=\{0<1<2<3<4\}$, let $L=D^2\setminus\{(0,4)\}$, a sublattice of D^2, and let $\mathscr{C}_\varrho=\{[(0,0),(2,1)],\ [(3,0),(4,1)],\ [(3,2),(4,4)],\ [(0,2),(2,3)],$ $[(1,2),(2,4)]\}$. Then Theorem 2 makes it easy to check that ϱ is a tolerance and L/ϱ is isomorphic to N_5, the five-element non-modular lattice.

Proposition 1 yields that for any tolerance ϱ on a finite lattice L, L/ϱ cannot have more element than L. That is why the following example can be of some interest. Define ϱ over Q, the set of rational numbers, by $\varrho=\{(x,y)\colon |x-y|\leqq 1\}$. Armed with the usual ordering Q turns into a lattice and ϱ is a tolerance on it. By making use of the results of this section it is easy to check that the factor lattice Q/ϱ is isomorphic to R, the set of real numbers with the usual ordering. (Indeed, the map $Q/\varrho\to R$, $C\mapsto\inf C$ is an isomorphism.)

4. Lattices as tolerance-factors of distributive lattices

The first example in the previous section indicates that forming factor lattices by tolerances preserves neither distributivity nor modularity. It is a naturally arising question which lattice identities are preserved. No non-trivial ones, as it will appear from the forthcoming theorem. Let **T**, **I**, **H**, **S**, **P**, and $\mathbf{P}_f$ denote the operators of taking factor lattices by tolerances, isomorphic lattices, homomorphic images, sublattices, direct products, and direct products of finite families, respectively. Note, that $\mathbf{H}\,V \subseteq \mathbf{IT}\,V$ for any class V of lattices. Moreover, as it can be deduced from Theorem 2, $\mathbf{IT}\,V = \mathbf{ITT}\,V$ for any class V of lattices. (To keep the size of the paper limited, the proof, which is similar to that of Homomorphism Theorem, will be omitted.) Let **2** denote the two-element lattice.

Theorem 3. $\mathbf{ISTSP}\{\mathbf{2}\}$ *is the class of all lattices, while* $\mathbf{ITSP}_f\{\mathbf{2}\}$ *is the class of all finite lattices.*

Proof. Only one argument is needed to prove this theorem consisting of two statements, just we have to show that our embeddings are surjective for the case of finite lattices. We have to show that an arbitrary (finite, respectively) lattice L belongs to $\mathbf{ISTSP}\{\mathbf{2}\}$ (to $\mathbf{ITSP}_f\{\mathbf{2}\}$, resp.). First of all we can assume that L is complete, since the map $L \to I(L)$, $x \mapsto (x]$ is an (surjective for finite L) embedding of L into its ideal lattice, i.e., into a complete lattice.

Claim 1. There are complete distributive lattices D_0 and D_1 in $\mathbf{P}\{\mathbf{2}\}$ and injective 0-and 1-preserving maps $\varphi_0: L \to D_0$, $\varphi_1: L \to D_1$ such that φ_0 preserves arbitrary joins and φ_1 preserves arbitrary meets. If L is finite then $D_0, D_1 \in \mathbf{P}_f\{\mathbf{2}\}$.

Proof. Let D_1 be $P(L \setminus \{0\})$, the Boolean lattice of all subsets of $L \setminus \{0\}$, and define $\varphi_1: L \to D_1$ as $x \mapsto (x] \setminus \{0\}$. The completeness of L yields $(\bigwedge (x_\gamma : \gamma \in \Gamma)] = \bigcap ((x_\gamma] : \gamma \in \Gamma)$, whence the required properties of φ_1 are trivial. Moreover, D_1 is isomorphic to $\mathbf{2}^{|L|-1}$. Q. e. d.

Now let D be $D_0 + D_1$, the ordinal sum of D_0 and D_1. I.e., D is the disjoint union of D_0 and D_1 equipped with the following ordering: $x \leqq y$ iff $x \in D_0$ and $y \in D_1$, or $x, y \in D_i$ and $x \leqq y$ for some $i \in \{0, 1\}$. Note that D is complete and it can be embedded into the direct square of $\mathbf{2}^{|L|-1}$, thus it is in $\mathbf{ISP}\{\mathbf{2}\}$ (in $\mathbf{ISP}_f\{\mathbf{2}\}$ for finite L). With the help of functions in Claim 1 define $\mathscr{C} \subseteq P^+(D)$ by

$$\mathscr{C} = \{C : \varnothing \neq C \subseteq D, \text{ for any } c, d \in C \text{ there exists } a \in L \text{ such that}$$

$$\{c, d\} \subseteq [a\varphi_0, a\varphi_1], \text{ and } C \text{ is maximal with respect to this property}\}.$$

Now, by making use of Theorem 2, we show that $\mathscr{C} = \mathscr{C}_\varrho (= D/\varrho)$ for some tolerance ϱ on D.

To check (C1) suppose $x, y\in C\in\mathscr{C}$. For an arbitrary $z\in C$ there exist $a, b\in L$ such that $x, z\in[a\varphi_0, a\varphi_1]$ and $y, z\in[b\varphi_0, b\varphi_1]$. Since φ_0 preserves joins and φ_1 is monotone, we obtain $x\vee y, z\in[a\varphi_0\vee b\varphi_0, a\varphi_1\vee b\varphi_1]\subseteq[(a\vee b)\varphi_0, (a\vee b)\varphi_1]$. From the maximality of C we obtain $x\vee y\in C$, showing that C is a sublattice. Let $c, d\in C, x\in D$ and $c<x<d$. Suppose that, e.g., $x\in D_0$, and let z be an arbitrary element of C. Then $c, z\in[a\varphi_0, a\varphi_1]$ for some $a\in L$. But $a\varphi_1\in D_1$ implies $x<a\varphi_1$, whence $x, z\in[a\varphi_0, a\varphi_1]$. The maximality of C yields $x\in C$, i.e. C is a convex sublattice. By the maximality of C, $1\varphi_0\in C$, so C is not empty.

From $[0\varphi_0, 0\varphi_1]\cup[1\varphi_0, 1\varphi_1]=L$ and Zorn Lemma (C2) follows.

Now suppose that, in contrary to (C3), $[C)=[E)$ and $(C]\neq(E]$ for $C, E\in\mathscr{C}$. Then one of $(C]\setminus(E]$ and $(E]\setminus(C]$, say $(C]\setminus(E]$ is not empty. Fix an element d from $C\setminus(E]$ and let x be an arbitrary element of E. Since $d\wedge x\in(C]\wedge[E)=[C)=[E)$, Lemma 3 yields $d, d\wedge x\in C$ and $x, d\wedge x\in E$. Hence $a\varphi_0\leqq d\wedge x\leqq d\leqq a\varphi_1$ and $b\varphi_0\leqq d\wedge x\leqq x\leqq b\varphi_1$ for some $a, b\in L$. By forming join we obtain $(a\vee b)\varphi_0=a\varphi_0\vee b\varphi_0\leqq d\wedge x\leqq d\vee x\leqq a\varphi_1\vee b\varphi_1\leqq(a\vee b)\varphi_1$. Thus $x, d\in[(a\vee b)\varphi_0, (a\vee b)\varphi_1]$, contradicting the maximality of E. The rest of (C3) follows from the Duality Principle.

To show (C4), let $C, E\in\mathscr{C}$ and define $X=\{c\vee e: c\in C, e\in E\}$. For any two elements in X, say $c_1\vee e_1$ and $c_2\vee e_2$ $(c_i\in C, e_i\in E)$, there exists an $u\in L$ such that $c_i\vee e_i\in[u\varphi_0, u\varphi_1]$ for $i=1,2$. Indeed, $c_i\in[a\varphi_0, a\varphi_1]$ and $e_i\in[b\varphi_0, b\varphi_1]$ $(i=1,2$ and $a, b\in L)$, and u can be defined as $a\vee b$. From Zorn Lemma we obtain the existence of an $F\in\mathscr{C}$ such that $X\subseteq F$. Since $(C]\vee(E]=(X]\leqq(F]$ is evident, $[C)\vee[E)=[F)$ has to be shown. If $x\in[C)\vee[E)=[C)\cap[E)$ then $x\geqq c$ and $x\geqq e$ for $c\in C, e\in E$. Hence $x\geqq c\vee e\in F$ implies $x\in[F)$, showing that $[C)\vee[E)\subseteq[F)$. Suppose that $[C)\vee[E)\subset[F)$. Then $F\setminus([C)\cap[E))$ and so, e.g., $F\setminus[C)$ are not empty. Fix elements d, c, and e in $F\setminus[C), C$, and E, respectively. For an arbitrary $x\in C$ we have $x\wedge c, x\vee c\in C$ and $d, ((x\vee c)\vee e)\vee d\in F$. Therefore $a\varphi_0\leqq x\wedge c\leqq x\vee c\leqq a\varphi_1$ and $b\varphi_0\leqq d\leqq x\vee c\vee e\vee d\leqq b\varphi_1$ for some $a, b\in L$. By meeting we obtain $(a\wedge b)\varphi_0\leqq a\varphi_0\wedge b\varphi_0\leqq x\wedge c\wedge d\leqq x\vee c\leqq a\varphi_1\wedge b\varphi_1=(a\wedge b)\varphi_1$. Now $c\wedge d\notin C$ and $x, c\wedge d\in[(a\wedge b)\varphi_0, (a\wedge b)\varphi_1]$ contradicts the maximality of C. The rest of (C4) is settled by the Duality Principle.

Before going on we show that

$$(*)\qquad [u\varphi_0, u\varphi_1]\in\mathscr{C}\quad\text{for any}\quad u\in L.$$

Only the maximality of $[u\varphi_0, u\varphi_1]$ has to be shown. Suppose $[u\varphi_0, u\varphi_1]$ is not maximal, then $[u\varphi_0, u\varphi_1]\subset C$ for some $C\in\mathscr{C}$. Fix an element c in $C\setminus[u\varphi_0, u\varphi_1]$. Since C is a sublattice, $c_0=c\wedge u\varphi_0$ and $c_1=c\vee u\varphi_1$ are in C, and either $c_0<u\varphi_0$ or $c_1>u\varphi_1$. If, e.g., $c_0<u\varphi_0$, then $c_0, u\varphi_1\in C$ implies $a\varphi_0\leqq c_0<u\varphi_0<u\varphi_1\leqq a\varphi_1$ for some $a\in L$. Hence $a\varphi_0\neq u\varphi_0, (a\vee u)\varphi_0=a\varphi_0\vee u\varphi_0=u\varphi_0$, and $u\varphi_1=a\varphi_1\wedge u\varphi_1=(a\wedge u)\varphi_1$. The injectivity of φ_0 and φ_1 yields $a\neq u, a\vee u=u$, and $a\wedge u=u$, a contradiction.

Now suppose $x\in L$, $d\in C\in\mathscr{C}$ and for any $e\in C\cap(d]$ there exists $C_e\in\mathscr{C}$ such that $\{e,x\}\subseteq C_e$. Then for any $e\in C\cap(d]$ there exists $a_e\in L$ such that $e,x\in[a_e\varphi_0, a_e\varphi_1]$. Set $u=\bigwedge(a_e: e\in C\cap(d])$ and $h=\bigwedge(e: e\in C\cap(d])$. Since φ_1 preserves arbitrary meets and φ_0 is monotone, we obtain $u\varphi_0\leqq\bigwedge(a_e\varphi_0: e\in C\cap(d])\leqq h$ and $x\leqq\bigwedge(a_e\varphi_1: e\in C\cap(d])=u\varphi_1$, i.e., $h,x\in[u\varphi_0,u\varphi_1]=E$. From $(*)$ we conclude that $E\in\mathscr{C}$. Since $u\varphi_0\leqq h\leqq y$ holds for any $y\in C$ (indeed, $h\leqq y\wedge d\in C\cap(d]$), $[E)\leqq[C)$. Now (C3) and (C4) imply $(E]\leqq(C]$ (cf. the proof of Corollary 1). Therefore $x\in(C]$ follows from $x\in E\subseteq(E]\subseteq(C]$. The rest of (C5) follows from the Duality Principle.

Now let U be a convex sublattice of D and suppose that for any $a,b\in U$ there exists $E\in\mathscr{C}$ containing both a and b. Then $a,b\in[u\varphi_0,u\varphi_1]$ for some $u\in L$, and Zorn Lemma implies (C6).

We have shown that $\mathscr{C}$ is associated with a tolerance ϱ on D. Let $D/\varrho=\mathscr{C}$ denote the corresponding factor lattice. For $u\in L$ let $u\psi$ denote $[u\varphi_0,u\varphi_1]$. Then, by $(*)$, ψ is a map from L into D/ϱ. If $u,v\in L$ then $[(u\vee v)\psi)=[(u\vee v)\varphi_0)=$ $=[u\varphi_0\vee v\varphi_0)=[u\varphi_0)\vee[v\varphi_0)=[u\psi)\vee[v\varphi)$. Lemma 4 and the dual of Proposition 1 imply $(u\vee v)\psi=u\psi\vee v\psi$, showing that ψ is a homomorphism. Since φ_0 is injective, so is ψ. Therefore $L\in\mathbf{IST}\{D\}$.

In case L is finite, so is D. Then any convex sublattice and, in particular, any element of $\mathscr{C}$ is an interval. Hence ψ is surjective, and $L\in\mathbf{IT}\{D\}$. Q. e. d.

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