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On the duality of interpolation spaces of several Banach spaces

DICESAR LASS FERNANDEZ

Introduction

Since the work by ARONSZAJN—GAGLIARDO ([1]) appeared, the problem of the duality of interpolation spaces of two Banach spaces has attracted the interest of many authors. See for instance LIONS [10] for the trace method, LIONS—PEETRE [12] for the mean methods, SCHERER [14], LACROIX—SONRIER [9], PEETRE [13] for the J- and K-methods, and CALDERÓN [4] for the complex method.

Although the study of interpolation spaces has been mainly restricted to couples of Banach spaces, many papers concerning interpolation spaces of several Banach spaces have appeared. See for instance LIONS [11], YOSHIKAWA [16], FAVINI [5], SPARR [15], FERNANDEZ [6], [7] and [8]. Thus, it is natural to pose the question of duality for the theories of interpolation of several Banach spaces. The purpose of this paper is to study the duality between the J- and K- interpolation methods for several Banach spaces introduced in FERNANDEZ [6]. The distinguishing feature of the J- and K- methods studied in [6] is that they deal with 2^n spaces and n-parameters. This permits us to show that the two methods are equivalent, in the sense that they generate the same interpolation spaces. The equivalence of the two methods is fundamental to the study of the duality problem. Also, the idea used in the proof of the equivalence is the same one used to prove a density theorem, which is another crucial point in the duality theory. In this way we have the tools to show that the J- and K- methods for 2^n spaces "are in duality" as is the case for $n=1$.

For the duality of the complex method for 2^n Banach spaces see BERTOLO [3].

Through this paper we shall use the following notations: (A) if $a=(a_1, \dots, a_d)$, $b=(b_1, \dots, b_d)\in\mathbf{R}^d$ then we set (i) $a\leqq b$ iff $a_j\leqq b_j$, $j=1, 2, \dots, d$; (ii) $a\cdot b=a_1b_1+\dots a_db_d$; (iii) $a\circ b=(a_1b_1, \dots, a_db_d)$; (iv) $|a|=a_1+\dots+a_d$; (v) $a^b=a_1^{b_1}\dots a_d^{b_d}$; (vi) $2^b=2^{b_1}\dots 2^{b_d}$; (B) $\mathbf{1}=(1, \dots, 1)$, (C) $L_*^Q=L_*^Q(\mathbf{R}^d)$ stands for the L^Q spaces with mixed norms of BENEDEK—PANZONE [2] with respect to the measure $d_*t=d_*t_1 \dots d_*t_d=dt_1/t_1 \dots dt_d/t_d$.

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1. Interpolation of 2^d Banach spaces

We shall first give a summary of facts on the theory of interpolation of 2^d Banach spaces. Also, we give the discretization of the methods here considered and a density theorem which has not appeared before.

1.1. Generalities. 1.1.1. The set of $k=(k_1, \dots, k_d)\in\mathbf{R}^d$ such that $k_j=0$ or 1 will be denoted by $\Box$. We have $\Box=\{0, 1\}$ when $d=1$ and $\Box=\{(0, 0), (1, 0), (0, 1), (1, 1)\}$ when $d=2$. The families of objects we shall consider will depend on indices in $\Box$.

1.1.2. We shall consider families of 2^d Banach spaces $\mathbf{E}=(E_k|k\in\Box)$ embedded, algebraically and continuously, in one and the same linear Hausdorff space V. Such a family will be called an admissible family of Banach spaces (in V).

1.1.3. If $\mathbf{E}=(E_k|k\in\Box)$ is an admissible family of Banach spaces, the linear hull $\Sigma\mathbf{E}$ and the intersection $\cap\mathbf{E}$ are defined in the usual way. They are Banach spaces under the norms

(1) $$\|x\|_{\Sigma\mathbf{E}} = \inf\{\Sigma_k\|x_k\|_{E_k} \mid x = \Sigma_k x_k;\ x_k\in E_k,\ k\in\Box\}$$

and

(2) $$\|x\|_{\cap\mathbf{E}} = \max\{\|x\|_{E_k} \mid k\in\Box\}.$$

The spaces $\cap\mathbf{E}$ and $\Sigma\mathbf{E}$ are continuously embedded in V.

1.1.4. A Banach space E which satisfies

(1) $$\cap\mathbf{E}\subset E\subset\Sigma\mathbf{E}$$

will be called an intermediate space (with respect to $\mathbf{E}$). (Hereafter $\subset$ will denote a continuous embedding.)

1.2. Intermediate spaces. 1.2.1. Let $\mathbf{E}=(E_k|k\in\Box)$ be an admissible family of Banach spaces. Suppose $t=(t_1, \dots, t_d)>0$ and $t^k=t_1^{k_1}\dots t_d^{k_d}$. For $x\in\Sigma\mathbf{E}$, we set

(1) $$K(t;\ x) = K(t;\ x;\ \mathbf{E}) = \inf\{\Sigma_k t^k\|x_k\|_{E_k} \mid x = \Sigma_k x_k,\ x_k\in E_k,\ k\in\Box\}$$

and for $x\in\cap\mathbf{E}$

(2) $$J(t;\ x) = J(t;\ x;\ \mathbf{E}) = \max\{t^k\|x\|_{E_k} \mid k\in\Box\}.$$

Now, assume $0<\Theta=(\theta_1, \dots, \theta_d)<1$ and $1\leqq Q=(q_1, \dots, q_d)\leqq\infty$.

1.2.2. Definition. We define $\mathbf{E}_{\Theta;Q;K}=(E_k|k\in\Box)_{\Theta;Q;K}$ to be the space of all elements $x\in\Sigma\mathbf{E}$ for which

(1) $$t^{-\Theta}K(t;\ x)\in L_*^Q,$$

and $\mathbf{E}_{\Theta;Q;J}=(E_k|k\in\square)_{\Theta;Q;J}$ to be the space of all elements $x\in\Sigma\mathbf{E}$ for which there exists a strongly measurable function $u\colon \mathbf{R}_+^d\to\cap\mathbf{E}$ such that

(2) $$x=\int_{\mathbf{R}_+^d} u(t)\,d_*t \quad (\text{in } \Sigma\,\mathbf{E}),$$

and

(3) $$t^{-\Theta}J(t;u(t))\in L_*^Q.$$

1.2.3. Proposition. *The spaces* $\mathbf{E}_{\Theta;Q;K}$ *and* $\mathbf{E}_{\Theta;Q;J}$ *are Banach spaces under the norms*

(1) $$\|x\|_{\Theta;Q;K}=\|t^{-\Theta}K(t;x)\|_{L^Q},$$

and

(2) $$\|x\|_{\Theta;Q;J}=\inf\left\{\left\|t^{-\Theta}J(t;u(t))\right\|_{L_*^Q}\mid x=\int u(t)\,d_*t\right\},$$

respectively. Furthermore, the spaces $\mathbf{E}_{\Theta;Q;K}$ *and* $\mathbf{E}_{\Theta;Q;J}$ *are intermediate spaces with respect to* $\mathbf{E}$.

1.2.4. We shall say the spaces $\mathbf{E}_{\Theta;Q;K}$ and $\mathbf{E}_{\Theta;Q;J}$ are generated by the K- and J-methods, respectively.

The following result gives a connection between the spaces generated by the K-and J- method and says that those are actually equivalent.

1.2.5. Proposition. *If* $0<\Theta=(\theta_1,\dots,\theta_d)<1$ *and* $1\leqq Q=(q_1,\dots,q_d)<\infty$ *we have*

(1) $$\mathbf{E}_{\Theta;Q;K}=\mathbf{E}_{\Theta;Q;J}.$$

1.2.6. When we have no need to specify which interpolation method has generated the intermediate space we shall write simply $\mathbf{E}_{\Theta,Q}$ for the spaces $\mathbf{E}_{\Theta;Q;K}$ and $\mathbf{E}_{\Theta;Q;J}$.

For the proofs of the above results see FERNANDEZ [6].

1.3. The discretization on the K- and J-method. Let $\mathbf{E}=(E_k|k\in\square)$ be an admissible family of Banach spaces.

1.3.1. Proposition. *Let* $x\in\Sigma\mathbf{E}$. *Then* $x\in(E_k|k\in\square)_{\Theta;Q;K}$ *iff*

(1) $$\left(e^{-N\cdot\Theta}K(e^N;x)\right)_{N\in\mathbf{Z}^d}\in l^Q(\mathbf{Z}^d).$$

Moreover

(2) $$\|x\|_{\Theta;Q;K}\cong\left\|\left(e^{-N\cdot\Theta}K(e^N;x)\right)_{N\in\mathbf{Z}^d}\right\|_{l^Q(\mathbf{Z}^d)}.$$

Proof. If $t^{-\Theta}K(t;x)=t_1^{-\theta_1}\dots t_d^{-\theta_d}K(t_1,\dots,t_d;x)$, we have

$$\|x\|_{\Theta;Q;K}=\left(\sum_{m_d=-\infty}^{\infty}\int_{e^{m_d}}^{e^{m_d+1}}\dots\left(\sum_{m_1=-\infty}^{\infty}\int_{e^{m_1}}^{e^{m_1+1}}\left(t^{-\Theta}K(t;x)\right)^{q_1}d_*t_1\right)^{q_2/q_1}\dots d_*t_d\right)^{1/q_d}.$$

On the other hand, if $e^{m_j} \leqq t_j \leqq e^{m_j+1}$, $j=1, 2, \dots, d$ we have

$$K(e^{m_1}, \dots, e^{m_d}; x) \leqq K(t_m, \dots, t_d; x) \leqq e^N K(e^{m_1}, \dots, e^{m_d}; x)$$

and

(3) $$e^{-\Theta \cdot M} K(e^M; x) \leqq t^{-\Theta} K(t; x) \leqq e^d e^{-\Theta} K(e^M; x).$$

These inequalities imply (2) at once and prove the assertion.

1.3.2. Proposition. *Let* $x \in \Sigma \mathbf{E}$. *Then,* $x \in (E_k \mid k \in \square)_{\Theta; Q; J}$ *iff there is* $u_M \in \cap \mathbf{E}$, $M \in \mathbf{Z}^d$, *such that*

(1) $$x = \sum_{M \in \mathbf{Z}^d} u_M \quad (in\ \Sigma \mathbf{E})$$

and

(2) $$e^{-M \cdot \Theta} J(e^M; u_M)_{M \in \mathbf{Z}^d} \in l^Q (\mathbf{Z}^d).$$

Moreover

(3) $$\|x\|_{\Theta; Q; J} \cong \inf \left\{ \|(e^{-M \cdot \Theta} J(e^M; u_M))_{M \in \mathbf{Z}^d}\|_{l^Q(\mathbf{Z}^d)} \mid x = \Sigma_M u_M \right\}.$$

Proof. Let $x \in (E_k \mid k \in \square)_{\Theta; Q; J}$ and $u = u(t)$ be as in 1.2.2. If $M = (m_1, \dots, m_d)$, let us set

$$u_m = u_{m_1 \cdots m_d} = \int_{e^{m_d}}^{e^{m_d+1}} \cdots \int_{e^{m_1}}^{e^{m_1+1}} u(t_1, \dots, t_d)\, d_* t_1 \dots d_* t_d.$$

Then we have

$$x = \int_{\mathbf{R}_+^d} u(t)\, d_* t = \sum_{M \in \mathbf{Z}^d} u_M$$

and

(4) $$\left\|(e^{-M \cdot \Theta} J(e^M; u_M))_{M \in \mathbf{Z}^d}\right\|_{l^Q(\mathbf{Z}^d)} \leqq C \left\|t^{-\Theta} J(t; u)\right\|_{L_*^Q}.$$

Taking the infimum in the above inequality we get one half of (3).

We proceed similarly to obtain the converse inequality in (4), which will imply the other half of (3). The proof is complete.

1.4. Density theorems. Let $\mathbf{E} = (E_k \mid k \in \square)$ be an admissible family and let us denote by $\overline{\cap \mathbf{E}}^K$ and $\overline{\cap \mathbf{E}}^J$ the closure of $\cap \mathbf{E}$ in $\mathbf{E}_{\Theta; Q; K}$ and $\mathbf{E}_{\Theta; Q; J}$ respectively. Of course, we have $\overline{\cap \mathbf{E}}^K = \overline{\cap \mathbf{E}}^J = \overline{\cap \mathbf{E}}$.

1.4.1. Proposition. *If* $0 < \Theta < 1$ *and* $1 \leqq Q < \infty$ *we have*

(1) $\overline{\cap \mathbf{E}}^K \subset \mathbf{E}_{\Theta; Q; K}$; (2) $\mathbf{E}_{\Theta; Q; J} \subset \overline{\cap \mathbf{E}}^J$.

Proof. The inclusion (1) is obvious. To prove (2), let $x \in \mathbf{E}_{\Theta; Q; J}$ and let $u = u(t)$ be as in 1.2.2(2)—(3). Let us set

$$x_M = x_{m_1 \dots m_d} = \int_{1/m_d}^{m_d} \cdots \int_{1/m_1}^{m_1} u(t_1, \dots, t_d)\, d_* t_1 \dots d_* t_d.$$

Then

$$x - x_M = \int_{\mathbf{R}_+^d} Y_M(t) u(t)\, d_* t,$$

where $Y_M(t) = Y_{m_1 \ldots m_d}(t) = 0$, if $1/m_j < t_j < m_j$ $(j = 1, 2, \ldots, d)$ and $=1$ otherwise. Consequently

$$\|x - x_M\|_{\theta; Q; J} \leqq \left\|t^{-\theta} J\big(t; Y_M(t) u(t)\big)\right\|_{L_*^Q} = \left\|t^{-\theta} Y_M(t)\, J\big(t; u(t)\big)\right\|_{L_*^Q}.$$

Finally, since $Y_M(t) \to 0$ as $M \to \infty$, the result follows.

1.4.2. Corollary. *We have* $\overline{\bigcap \mathbf{E}} = \mathbf{E}_{\theta; Q}$.

Proof. It follows at once from 1.4.1(1)—(2) and the equivalence theorem.

2. Duality

2.1. Dual families. For a given admissible family $\mathbf{E} = (E_k \mid k \in \square)$ of Banach spaces there is a natural duality between $\bigcap \mathbf{E}$ and $\Sigma \mathbf{E}$, and $\mathbf{E}_{\theta; Q; K}$ and $\mathbf{E}_{\theta; Q; J}$. In order to examine this duality let us set the following hypothesis (H) on the admissible family $\mathbf{E}$:

(H) $\qquad \bigcap \mathbf{E}$ is dense in each E_k, $k \in \square$.

Let $\mathbf{E}' = (E_k' \mid k \in \square)$ be the family given by the duals of the elements of the family $\mathbf{E} = (E_k \mid k \in \square)$.

Since $\bigcap \mathbf{E} \subset E_k$, the spaces E_k', $k \in \square$, can be canonically embedded in $(\bigcap \mathbf{E})'$. The density hypothesis assures that this embedding does not identify distinct elements in E_k' with the same element in $(\bigcap \mathbf{E})'$. In this way, the family $\mathbf{E}' = (E_k' \mid k \in \square)$ of dual spaces is an admissible family of Banach spaces.

2.1.1. Proposition. *Let* $\mathbf{E} = (E_k \mid k \in \square)$ *be an admissible family which satisfies the hypothesis* (H) *and let* $\mathbf{E}' = (E_k' \mid k \in \square)$ *be its dual family. Then*

(1) $$(\bigcap \mathbf{E})' = \Sigma \mathbf{E}'$$

and

(2) $$\|x'\|_{\Sigma \mathbf{E}'} = \sup \{|\langle x, x' \rangle_{\cap}| / \|x\|_{\cap \mathbf{E}} \mid x \in \bigcap \mathbf{E}\};$$

(3) $$(\Sigma \mathbf{E})' = \bigcap \mathbf{E}'$$

and

(4) $$\|x'\|_{\cap \mathbf{E}'} = \sup \{|\langle x, x' \rangle_{\Sigma}| / \|x\|_{\cap \mathbf{E}} \mid x \in \Sigma \mathbf{E}\};$$

where $\langle\, , \rangle_{\cap}$ *denotes the duality between* $\bigcap \mathbf{E}$ *and* $(\bigcap \mathbf{E})'$ *and* $\langle\, , \rangle_{\Sigma}$ *between* $\Sigma \mathbf{E}$ *and* $(\Sigma \mathbf{E})'$.

Proof. Since $E'_k \subset (\bigcap \mathbf{E})'$, for each $k \in \square$, it follows that $\Sigma \mathbf{E}' \subset (\bigcap \mathbf{E})'$.

Conversely, if $\Phi \in (\bigcap \mathbf{E})'$, the linear form

$$\psi\colon (x_k \mid k \in \square) \to \Phi(2^{-d} \Sigma_k x_k)$$

is bounded in the diagonal subspace of $\oplus_k E_k$, with the norm $\max_k \|x_k\|_{E_k}$. By the Hahn—Banach theorem there is an $(x'_k \mid k \in \square) \in \oplus_k E'_k$ such that

$$\Sigma_k \langle x, x'_k \rangle_{E_k} = \psi(x)$$

for all $x \in \bigcap \mathbf{E}$, and

$$\Sigma_k \|x'_k\|_{E'_k} \leqq \|\Phi\|_{(\cap \mathbf{E})'}.$$

Now, if we take $x_k = x$, $k \in \square$, it follows that

$$\Phi(x) = \Sigma_k \langle x, x'_k \rangle_{E_k}, \quad x \in \bigcap \mathbf{E}.$$

Finally, by the density hypothesis (H), the linear forms x'_k, $k \in \square$, are determined by their values in $\bigcap \mathbf{E}$ and

$$\|\Phi\|_{(\cap \mathbf{E})'} \leqq \Sigma_k \|x'_k\|_{E'_k}.$$

Similarly we prove (3) and (4).

As a corollary of proposition 2.1.1 we get the following result on the K- and J-functional norms.

2.1.2. Proposition. *Let* $\mathbf{E} = (E_k \mid k \in \square)$ *be an admissible family of Banach spaces which satisfies the density hypothesis* (H) *and let* $\mathbf{E}' = (E'_k \mid k \in \square)$ *be its dual family. Then*

(1) $$K(t; x'; \mathbf{E}') = \sup \{ |\langle x, x' \rangle| / J(t^{-1}; x; \mathbf{E}) \mid x \in \bigcap \mathbf{E} \}$$

and

(2) $$J(t; x'; \mathbf{E}') = \sup \{ |\langle x, x' \rangle / K(t^{-1}; x; \mathbf{E}) \mid x \in \Sigma \mathbf{E} \}.$$

Proof. Let E be a normed space and $t > 0$. Let us denote space E with the norm $t\|\cdot\|_E$ by tE. Then we have $(tE)' = t^{-1}E'$.

Now, if we consider the family $(t^k E_k \mid k \in \square)$ we see that (1) and (2) follow at once from 2.1.1(2) and 2.1.1(4), respectively.

2.2. The duality of spaces $\mathbf{E}_{Q,\theta}$. Let $\mathbf{E}$ be an admissible family of Banach spaces which satisfies the density hypothesis (H). Then we can consider intermediate spaces with respect to the dual family $\mathbf{E}'$, and in particular the interpolation spaces $\mathbf{E}_{\theta,Q}$.

Let E be an intermediate space with respect to the admissible family $\mathbf{E}$. Then, a necessary and sufficient condition for E' to be an intermediate space with respect to the dual family $\mathbf{E}'$ is that $\bigcap \mathbf{E}$ be dense in E. Thus, if $E = \mathbf{E}_{\theta,Q}$ the density result of proposition 1.4.1 assures that $E' = (\mathbf{E}_{\theta,Q})'$ is an intermediate space with respect to the dual family $\mathbf{E}'$.

We shall now study the relationship between the spaces $\mathbf{E}_{\Theta,Q'}$ and $(\mathbf{E}_{\Theta,Q})'$. To this end we shall use again the notation $\mathbf{E}_{\Theta;Q;K}$ and $\mathbf{E}_{\Theta;Q;J}$ for the spaces generated by the K- and J-method, respectively.

2.2.1. Proposition. *Let* $\mathbf{E}=(E_k \mid k\in\square)$ *be an admissible family which satisfies the density hypothesis* (H) *and let* $\mathbf{E}'=(E_k' \mid k\in\square)$ *be its dual family. Suppose* $1\leqq Q=(q_1, \dots, q_d)<\infty$ *and* $0<\Theta=(\theta_1, \dots, \theta_d)<1$. *Then*

(1) $$\mathbf{E}'_{\Theta;Q'} = (\mathbf{E}_{\Theta;Q})'$$

where $1/Q+1/Q'=1$ (i.e., $1/q_j+1/q_j'=1$, $j=1, 2, \dots, d$).

Proof. We shall prove that

(2) $$\mathbf{E}'_{\Theta;Q';K} = (\mathbf{E}_{\Theta;Q;J})'.$$

By Prop. 2.1.1 it follows that

$$\mathbf{E}'_{\Theta;Q';K}\subset\Sigma\mathbf{E}' = (\cap\mathbf{E})'.$$

Now, if $x'\in\mathbf{E}'_{\Theta;Q;K}$ and $\langle\,,\rangle_\cap$ is the duality between $\cap\mathbf{E}$ and $(\cap\mathbf{E})'$, the relation $\langle x, x'\rangle_\cap$ makes sense for $x\in\mathbf{E}_{\Theta;Q;J}\cap(\cap\mathbf{E})$. Thus, by definition, there is a strongly measurable function $u\colon \mathbf{R}^d\to\cap\mathbf{E}$ such that $u\in L^1_*(\mathbf{R}^a_+;\cap\mathbf{E})$ and satisfies 1.2.2(2). From 2.1.2(1) it follows that

(3) $$\int_{\mathbf{R}^d} |\langle u(t), x'\rangle|\, d_*t \leqq \int_{\mathbf{R}^d_+} J(t;\, u(t))K(t^{-1};\, x')\, d_*t =$$
$$= \int_{\mathbf{R}^d_+} t^{-\Theta}J(t;\, u(t))\, t^{\Theta}K(t^{-1}, x')\, d_*t \leqq \left\|t^{-\Theta}J(t;\, u(t))\right\|_{L^Q_*} \|t^{\Theta}K(t^{-1}, x')\|_{L^{Q'}_*}.$$

This shows that $x'ou\in L^1_*(\mathbf{R}^d_+)$ and thus

(4) $$\int_{\mathbf{R}^d_+} \langle u(t), x'\rangle\, d_*t = \left\langle \int_{\mathbf{R}^d_+} u(t)\, d_*t;\, x' \right\rangle = \langle x, x'\rangle.$$

From (2) and (3) we get the following inequality of Hölder type

(5) $$|\langle x, x'\rangle| \leqq \|x\|_{\Theta;Q;J}\|x'\|_{\Theta;Q';K}.$$

This Hölder inequality implies at once that $\langle\,,\rangle$ is a bounded linear form on a dense subspace of $\mathbf{E}_{\Theta;Q;J}$. Thus $\langle\,,\rangle$ can be extended boundedly to all $\mathbf{E}_{\Theta;Q;J}$. Hence, $x'\in(\mathbf{E}_{\Theta;Q;J})'$ and we have, for the dual norm

$$\|x'\|_{(\mathbf{E}_{\Theta;Q;J})'} = \sup\{|\langle x, x'\rangle|/\|x\|_{\Theta;Q;J} \mid x\in\mathbf{E}_{\Theta;Q;J}\} \leqq \|x'\|_{\Theta;Q';K}.$$

From this inequality we obtain one half of (1).

Conversely, let $x' \in (\mathbf{E}_{\theta;Q;J})'$. By 2.1.2(1), given $\varepsilon > 0$ there is $Y_N = Y_{n_1 \dots n_d} \in \in \bigcap \mathbf{E}$ with $Y_N \neq 0$ and such that

$$\varepsilon K(e^N;\ x';\ \mathbf{E}') \leqq \langle y_N / J(e^N;\ y_N),\ x' \rangle.$$

Next, we denote by $l^{\theta,Q}(\mathbf{Z}^d)$ the space of all multiple sequences of real number $(x_N)_{N \in \mathbf{Z}^d}$ such that

$$\|(x_N)_{N \in \mathbf{Z}^d}\|_{l^{\theta,Q}} = \|(e^{N \cdot \theta} x_N)_{N \in \mathbf{Z}^d}\|_{l^Q} < \infty.$$

Now, if $\alpha = (\alpha_N)_N \in l^{\theta,Q}(\mathbf{Z}^d)$ and

$$x_\alpha = \sum_{N \in \mathbf{Z}^d} \alpha_N y_N / J(e^N;\ y_N),$$

it follows that

$$\|x_\alpha\|_{\theta;Q;J} \leqq \left\| \left(e^{-N \cdot \theta} J\left(e^N;\ \alpha_N y_N / J(e^N;\ y_N)\right) \right)_{N \in \mathbf{Z}^d} \right\|_{l^Q(\mathbf{Z}^d)} =$$

$$= \left\| \left(e^{-N \cdot \theta} |\alpha_N| \right)_{N \in \mathbf{Z}^d} \right\|_{l^Q(\mathbf{Z}^d)} = \|\alpha\|_{l^{\theta,Q}} < \infty.$$

Thus $x_\alpha \in \mathbf{E}_{\theta;Q;J}$. Also

$$\langle x_\alpha,\ x' \rangle = \langle \Sigma_N \alpha_N y_N / J(e^N,\ y_N),\ x' \rangle \geqq \varepsilon \Sigma_N \alpha_N K(e^{-N};\ x')$$

therefore

$$\varepsilon \Sigma_N e^{-N} \alpha_N e^N K(e^{-N};\ x') \leqq \|\alpha\|_{\theta,Q} \|x'\|_{\theta;Q;J}. \tag{6}$$

Since $l^{\theta,Q}(\mathbf{Z}^d)$ and $l^{1-\theta,Q}(\mathbf{Z}^d)$ are in duality via the duality

$$\langle \alpha,\ \delta \rangle = \sum_{N \in \mathbf{Z}^d} e^{-|N|} \alpha_N \delta_N,$$

by taking the supremum over all $\alpha \in l^{\theta,Q}(\mathbf{Z}^d)$ with $\|\alpha\|_{l^{\theta,Q}} \leqq 1$ in (6) we obtain

$$\varepsilon \|e^N K(e^{-N};\ x')\|_{l^{\theta,Q'}} \leqq \|x'\|_{Q;\theta;J},$$

that is

$$\varepsilon \|x'\|_{\theta;Q';K} \leqq \|x'\|_{\theta;Q;J}.$$

Since ε is arbitrary we obtain the second half of (2).

From (2) and the equivalence theorem we obtain (1) and the proof is complete.

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INSTITUTO DE MATEMÁTICA
UNIVERSIDADE ESTADUAL DE CAMPINAS
CX. POSTAL 6155
13100 — CAMPINAS — S. P. — BRASIL