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Amalgamated free product of lattices. I. The common refinement property

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1. Introduction. The common refinement property has been investigated for many algebraic constructions. Intuitively, we say that the common refinement property holds for the construction $*$ (e.g., direct product or free product) if, whenever A_0, A_1, B_0, B_1 are algebras for which $*$ is defined, $L=A_0 * A_1=B_0 * B_1$, and $A_0, A_1, B_0, B_1 \subseteq L$, then

(1) $A_i = (A_i \cap B_0) * (A_i \cap B_1), \quad i = 0, 1,$

(2) $B_j = (A_0 \cap B_j) * (A_1 \cap B_j), \quad j = 0, 1,$

(3) $L = (A_0 \cap B_0) * (A_0 \cap B_1) * (A_1 \cap B_0) * (A_1 \cap B_1).$

This is, of course, not a definition; we did not even specify what is meant by the right side of (3). In most concrete cases, however, the meaning of (1), (2), and (3) is clear: direct product of groups and rings, direct product of lattices with 0, free product of lattices (G. GRÄTZER and J. SICHLER [4]), and free product of algebras in a regular variety (B. JÓNSSON and E. NELSON [6]) are examples of algebraic constructions satisfying the common refinement property.

The present investigation was prompted by Problem VI. 2 in G. GRÄTZER [1], asking whether or not free $\{0, 1\}$-product of bounded lattices satisfies the common refinement property. We answer this question in the affirmative; the method of the proof, however, leads much farther. It will be shown that two free products amalgamated over the same finite lattice Q always have a common refinement. The Theorem gives, for an arbitrary lattice Q and any two representations of a lattice L as free Q-products, a necessary and sufficient condition for the existence of a common refinement.

2. Results. To define the concept of an amalgamated free product, let Q, A_0, A_1 be lattices ($Q=\varnothing$ is allowed), let Q be a sublattice of both A_0 and A_1, and let

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$A_0 \cap A_1 = Q$. Then $A_0 \cup A_1$ is a partial lattice in a natural way (see Section 3 for a detailed definition). The free lattice generated by this partial lattice will be called the free product of A_0 and A_1 amalgamated over Q, or the Q-free product of A_0 and A_1; it will be denoted by $A_0 *_Q A_1$. In this paper, the formula $L = A_0 *_Q A_1$ always assumes that L is a lattice, A_0 and A_1 are sublattices of L, $Q = \varnothing$ or Q is a sublattice of both A_0 and A_1.

Our main theorem is as follows (for a more complete version see Section 4):

Theorem. *Let $L = A_0 *_Q A_1 = B_0 *_Q B_1$. These two decompositions of L have a common refinement, that is, conditions* (1)—(3) *of Section* 1 *hold for $*_Q$ if and only if for any $i, j \in \{0, 1\}$, $x \in A_i$, $y \in B_j$, the inequality $x \leqq y$ in L implies the existence of a $z \in A_i \cap B_j$ such that $x \leqq z$ in A_i and $z \leqq y$ in B_j.*

This theorem has several consequences.

Corollary 1. *If Q satisfies the Ascending Chain Condition or the Descending Chain Condition, then any two Q-free decompositions of a lattice have a common refinement.*

Clearly, the special case $Q = \{0, 1\}$ of Corollary 1 answers Problem VI. 2 of [1] in the affirmative.

Corollary 2. *Let $L = A_0 *_Q A_1 = B_0 *_Q B_1$. If, for any $i, j \in \{0, 1\}$, either A_i or B_j is convex in $A_i \cup B_j$, then the two decompositions have a common refinement.*

The most important open problem in this investigation is whether the condition given in the Theorem is a tautology or not; that is, whether Q-free products always have common refinements.

It follows easily from the main result of G. Grätzer and J. Sichler [4] that the free factors of a lattice L form a distributive lattice. This statement remains valid for Q-free factors ($Q \subseteq L$) if Q-free products always have common refinements (see Section 8). The next two corollaries establish distributivity like properties of the set of all Q-free factors for an arbitrary Q.

Corollary 3. *If $A_0 *_Q A_1 = A_0 *_Q A_2$ and $A_1 \subseteq A_2$, then $A_1 = A_2$.*

Corollary 4. *If $A_0 *_Q A_1 = A_0 *_Q A_2 = A_1 *_Q A_2$, then $Q = A_0 = A_1 = A_2$.*

3. Amalgamated free products. We need a lemma before we give the definition of an amalgamated free product.

Lemma 1. *Let A_0 and A_1 be lattices, let Q be a sublattice of both A_0 and A_1 or $Q = \varnothing$, and let $A_0 \cap A_1 = Q$. Then there exists a smallest partial lattice on the set $A_0 \cup A_1$ extending the operations of A_0 and A_1.*

Proof. Since the Amalgamation Property holds for lattices, there is an embedding of $A_0 \cup A_1$ into a lattice preserving the operations of A_0 and A_1. Restricting the operations of this lattice to $A_0 \cup A_1$, we get a partial lattice on the set $A_0 \cup A_1$. Therefore, the set of all partial lattices on the set $A_0 \cup A_1$ whose operations are extensions of the operations of A_0 and A_1 is nonempty. Now let $\langle A_0 \cup A_1; \wedge_\gamma, \vee_\gamma \rangle$, $\gamma \in \Gamma$, be partial lattices on the set $A_0 \cup A_1$. Let $\wedge$ and $\vee$ be the intersection of the $\wedge_\gamma$'s and $\vee_\gamma$'s, respectively ($\wedge_\gamma$ and $\vee_\gamma$ are sets, in fact, they are subsets of $(A_0 \cup A_1)^2 \times (A_0 \cup A_1)$). We shall prove that $\langle A_0 \cup A_1; \wedge, \vee \rangle$ is a partial lattice. This, will prove Lemma 1.

Here we need N. Funayama's characterization of partial lattices (see, e.g., G. GRÄTZER [1]): A partial algebra $\langle H; \wedge, \vee \rangle$ is a partial lattice if and only if, for arbitrary $a, b, c \in H$, the following five conditions and their duals hold.

(i) $a \wedge a$ exits and $a \wedge a = a$.
(ii) If $a \wedge b$ exists, then $b \wedge a$ exists and $a \wedge b = b \wedge a$.
(iii) If $a \wedge b$, $(a \wedge b) \wedge c$, $b \wedge c$ exist, then $a \wedge (b \wedge c)$ exists, and $(a \wedge b) \wedge c = a \wedge (b \wedge c)$. If $b \wedge c, a \wedge (b \wedge c), a \wedge b$ exist, then $(a \wedge b) \wedge c$ exists and $(a \wedge b) \wedge c = a \wedge (b \wedge c)$.
(iv) If $a \wedge b$ exists, then $a \vee (a \wedge b)$ exists, and $a = a \vee (a \wedge b)$.
(v) If $[a) \vee [b) = [c)$ in $D_0(H)$, then $a \wedge b$ exists in H and equals c. (Here $D_0(H)$ denotes the lattice consisting of $\varnothing$ and all dual ideals of H. $D_0(H)$ is ordered by inclusion.)

Now we prove (v) for $\langle A_0 \cup A_1; \wedge, \vee \rangle$, the proof of the other four conditions is similar. Every $\langle A_0 \cup A_1; \wedge_\gamma, \vee_\gamma \rangle$, $\gamma \in \Gamma$, is a partial lattice, therefore, (v) holds for $\langle A_0 \cup A_1; \wedge_\gamma, \vee_\gamma \rangle$. Assume that $[a) \vee [b) = [c)$ in $D_0(\langle A_0 \cup A_1; \wedge, \vee \rangle)$. Then $[a) \vee [b) = [c)$ in $D_0(\langle A_0 \cup A_1; \wedge_\gamma, \vee_\gamma \rangle)$ for all $\gamma \in \Gamma$. In fact, $\wedge_\gamma$ is an extension of $\wedge$; therefore, the dual ideals generated by a and b relative to $\wedge_\gamma$ contain the dual ideals generated by a and b relative to $\wedge$, respectively. Thus $[a) \vee [b) \supseteq [c)$ in $D_0(\langle A_0 \cup A_1; \wedge_\gamma, \vee_\gamma \rangle)$. The reverse inclusion is trivial. Now, by (v), $a \wedge_\gamma b = c$ for all $\gamma \in \Gamma$. Hence $a \wedge b = c$. This completes the proof.

Definition 1. Let $Q, A_0; A_1$ be as in Lemma 1. Let $P(A_0, A_1, Q)$ denote the smallest partial lattice of Lemma 1. If $Q = A_0 \cap A_1$ is understood, we write $P(A_0, A_1)$. for $P(A_0, A_1, Q)$. Then the free lattice generated by $P(A_0, A_1, Q)$ will be called the *free product of A_0 and A_1 amalgamated over Q*, and it will be denoted by $A_0 *_Q A_1$.

A warning is in order here. We can partially order $A_0 \cup A_1$ by the smallest partial order containing the ordering of A_0 and the ordering of A_1. If we take $A_0 \cup A_1$ together with all the existing g.l.b.'s and l.u.b.'s relative to this ordering, then the resulting partial lattice is generally different from the one defined above.

Definition 1 can easily be extended to a definition of the Q-free product of an arbitrary finite number of lattices containing Q. In particular, if $L=A_0*_Q A_1=$ $=B_0*_Q B_1$, then $(A_0\cap B_0)*_Q(A_0\cap B_1)*_Q(A_1\cap B_0)*_Q(A_1\cap B_1)$ is the free lattice generated by the smallest partial lattice on the set $(A_0\cap B_0)\cup(A_0\cap B_1)\cup(A_1\cap B_0)\cup$ $\cup(A_1\cap B_1)$ whose operations extend the operations of all $A_i\cap B_j$, $i,j=0,1$.

We shall need a description of the ordering and of the ideals of $P(A_0, A_1)$.

Lemma 2. *Let $x\in A_0$ and $y\in A_1$. Then $x\leqq y$ in $P(A_0, A_1)$ if and only if there is a $z\in Q$ with $x\leqq z$ in A_0 and $z\leqq y$ in A_1.*

Proof. Define $\leqq$ on $A_0\cup A_1$ as follows: $\leqq$ retains its meaning on A_0 and A_1; for $x\in A_0$ and $y\in A_1$ (or $x\in A_1$ and $y\in A_0$) define $\leqq$ as in the lemma. It is obvious that $\leqq$ is a partial ordering on $A_0\cup A_1$. (This is used in the proof of the Amalgamation Property for lattices.) Consider the partial lattice $\langle A_0\cup A_1; \wedge, \vee\rangle$, where $a\wedge b=c$ iff c is the greatest lower bound of a and b with respect to $\leqq$; $a\vee b=c$ is defined dually.

Let $\leqq_1$ denote the ordering of $P(A_0, A_1)$. Since $P(A_0, A_1)$ is the smallest partial lattice on $A_0\cup A_1$, $\leqq_1$ must be contained in $\leqq$. To prove the converse, let $a\leqq b$, $a, b\in A_0\cup A_1$. If $a, b\in A_i$ for some i in $\{0, 1\}$, then $a\leqq b$ in A_i. Hence, by the definition of $P(A_0, A_1)$, $a\leqq_1 b$. Therefore, and by symmetry, we may assume that $a\in A_0$ and $b\in A_1$. Thus there is an element c in $A_0\cap A_1$ such that $a\leqq c$ in A_0 and $c\leqq b$ in A_1. The same inequalities hold in $P(A_0, A_1)$, that is, $a\leqq_1 c\leqq_1 b$, as claimed.

Lemma 3. *Every ideal of $P(A_0, A_1)$ is the union of an ideal I_0 of A_0 and an ideal I_1 of A_1 satisfying $I_0\cap Q=I_1\cap Q$. Conversely, if I_0 is an ideal of A_0 and I_1 is an ideal of A_1 with $I_0\cap Q=I_1\cap Q$, then $I_0\cup I_1$ is an ideal of $P(A_0, A_1)$.*

Proof. Let I be an ideal of $P(A_0, A_1)$. Then $I_i=I\cap A_i$ is an ideal of A_i, $i=0, 1$, and $I_0\cap Q=I\cap A_0\cap Q=I\cap A_0\cap A_1=I\cap A_1\cap Q=I_1\cap Q$, which proves the first statement.

To prove the converse, consider the partial algebra $\langle A_0\cup A_1; \vee, \wedge\rangle$, where $x\wedge y$ (resp., $x\vee y$) is defined if and only if x and y are in the same A_i and $x\wedge y$ (resp., $x\vee y$) is the meet (resp., join) of x and y in A_i. Call a set I an ideal of the partial algebra $\langle A_0\cup A_1; \wedge, \vee\rangle$ if, whenever $x, y\in I$ and $x\vee y$ is defined, then $x\vee y\in I$ and whenever $x\in I, y\in A_0\cup A_1$, and $y\leqq x$, then $y\in I$. (The partial order $\leqq$ was defined in Lemma 2.) Now let I_0 be an ideal of A_0 and let I_1 be an ideal of A_1 with $I_0\cap Q=I_1\cap Q$. The latter condition ensures that $I_0\cup I_1$ is an ideal of $\langle A_0\cup A_1;$ $\wedge, \vee\rangle$. Now we prove that $I_0\cup I_1$ is an ideal of $P(A_0, A_1)$. In fact, $P(A_0, A_1)$ is the smallest partial lattice in which, besides the partial operations of $\langle A_0\cup A_1; \wedge, \vee\rangle$, all the meets and joins are defined that follow by iterated application of conditions

(i) to (v) and their duals. Therefore, it is sufficient to check that a single application of any one of (i) to (v) and their duals does not change the ideals; this is evident.

4. Smooth representations of ideals. The proofs in G. GRÄTZER and J. SICHLER [4] rely on two facts:

1. In a free product $L=A_0*A_1$ every element has a lower A_0-cover, which is an element of $(A_0)^b$ (that is, A_0 with a new 0 and 1 adjoined);

2. Forming lower A_0-covers is a homomorphism of L into $(A_0)^b$.

In general, these statements do not hold for amalgamated free products. In this section we find some statements that hold for amalgamated free products; these statements can be viewed as substitutes for the two facts mentioned above.

Throughout this section, let Q, A_0, A_1, L be lattices, let $L=A_0*_Q A_1$, and let $A=P(A_0, A_1, Q)$ as defined in Section 3. Let $I(A)$ (respectively, $I(A_i)$) denote the ideal lattice of A (respectively, of A_i). For any ideal I of L or of A define

$$(I)_i = I\cap A_i, \quad i=0, 1$$

and for an ideal I of L define

$$I_A = I\cap A.$$

For a principal ideal I of L, the ideals $(I)_i$ and I_A correspond to the usual lower covers (see, e.g., [1]), however, $I\to(I)_i, I\in I(L)$, is not a homomorphism, that is,

$$(1) \qquad \big(p(I_0, \ldots, I_{n-1})\big)_i = p\big((I_0)_i, \ldots, (I_{n-1})_i\big)$$

does not hold for all polynomials p. For certain polynomials, however, (1) does hold (see Definition 2) and it will turn out (Lemma 8) that this happens often enough, making it possible to carry out some of the proofs of [4] under more general conditions.

Definition 2. Let $p=p(x_0, \ldots, x_{n-1})$ be an n-ary lattice polynomial, let $I, I_0, \ldots, I_{n-1}$ be ideals of L (of A, A_i, respectively), and let $I=p(I_0, \ldots, I_{n-1})$ in $I(L)$ (in $I(A), I(A_i)$, respectively). We say that $p(I_0, \ldots, I_{n-1})$ is a *smooth representation* of I (or that $p(I_0, \ldots, I_{n-1})$ *is smooth*) iff one of the following conditions holds:

a) $p=x_i$;

b) $p=p_0\wedge p_1$ and both $p_0(I_0, \ldots, I_{n-1})$ and $p_1(I_0, \ldots, I_{n-1})$ are smooth;

c) $p=p_0\vee p_1$, both $p_0(I_0, \ldots, I_{n-1})$ and $p_1(I_0, \ldots, I_{n-1})$ are smooth, and, for any $q\in Q$,

$$q\in p(I_0, \ldots, I_{n-1}) \text{ implies that } q\in p_0(I_0, \ldots, I_{n-1}) \text{ or}$$

$$q\in p_1(I_0, \ldots, I_{n-1}).$$

The following lemma shows that every representation of an element of L can be turned into a smooth representation.

Lemma 4. *Let* $a \in L, a_0, \ldots, a_{n-1} \in A_0 \cup A_1$, *and let* $a = p(a_0, \ldots, a_{n-1})$ *where* p *is a lattice polynomial. Then there exist an integer* $m \geqq 0$, *a polynomial* $\tilde{p}$ *in* $n+m$ *variables, and subsets* $Q_0, \ldots, Q_{m-1}$ *of* Q *such that*

$$(a] = \tilde{p}((a_0], \ldots, (a_{n-1}], (Q_0], \ldots, (Q_{m-1}])$$

is a smooth representation of $(a]$ *in* $I(L)$.

Proof. We prove this statement by induction on the rank of p.

If $p = x_i$, then we can choose $m = 0, \tilde{p} = p$.

If $p = p_0 \vee p_1$, then, by the induction hypothesis, there exist an $m \geqq 0$, polynomials $\tilde{p}_0$ and $\tilde{p}_1$ of $n+m-1$ variables, and subsets $Q_0, \ldots, Q_{m-2}$ of Q such that

$$\tilde{p}_i((a_0], \ldots, (a_{n-1}], (Q_0], \ldots, (Q_{m-2}])$$

is a smooth representation of $p_i((a_0], \ldots, (a_{n-1}])$ for $i=0$ and 1. Let $Q_{m-1} = (a] \cap Q$. We claim that

$$\tilde{p}_0((a_0], \ldots, (a_{n-1}], (Q_0], \ldots, (Q_{m-2}]) \vee$$

$$\vee (\tilde{p}_1((a_0] \ldots, (a_{n-1}], (Q_0], \ldots, (Q_{m-2}]) \vee (Q_{m-1}])$$

is a smooth representation of $(a]$. Indeed, by the definitions of $\tilde{p}_i$ and of Q_{m-1}, this ideal equals $(a]$. Moreover, $\tilde{p}_1((a_0], \ldots) \vee (Q_{m-1}]$ is smooth because its components are smooth and if, for $q \in Q, q \in \tilde{p}_1((a_0], \ldots) \vee (Q_{m-1}]$, then $q \in (a]$; thus, $q \in (Q_{m-1}]$ by the definition of Q_{m-1}. Similarly, $\tilde{p}_0((a], \ldots) \vee (\tilde{p}_1((a_0], \ldots) \vee (Q_{m-1}])$ is smooth.

Finally, if $p = p_0 \wedge p_1$, then let $\tilde{p}_i((a_0], \ldots, (a_{n-1}], (Q_0], \ldots, (Q_{m-1}])$ be a smooth representation of $p_i((a_0], \ldots, (a_{n-1}])$ for $i=0$ and 1. The meet of these two polynomials is obviously a smooth representation of $(a]$.

In the remainder of this section we have to compute polynomials in $L, I(L)$, $I(A)$, and $I(A_i)$, $i=0, 1$. We shall distinguish between the operations in $I(A)$ and $I(A_i)$ by superscripting them by A and i, respectively.

The following lemma is a consequence of the solution of the word problems for lattices freely generated by a partial lattice (see, e.g., G. Grätzer, A. Huhn, and H. Lakser [2]).

Lemma 5. *Let* $x, y \in L$. *Then*

$(x \vee y] \cap A = ((x] \cap A) \vee^A ((y] \cap A)$, *and* $(x \wedge y] \cap A = ((x] \cap A) \wedge^A ((y] \cap A)$.

Lemma 6. *Let* I *and* J *be ideals of* L. *Then*

$$(I \vee J)_A = (I)_A \vee^A (J)_A.$$

Furthermore, if $I \vee J$ *is smooth, then so is* $(I)_A \vee^A (J)_A$.

Proof. We prove that $(I\vee J)_A\subseteq(I)_A\vee^A(J)_A$ (the reverse inclusion is obvious). Let $a\in(I\vee J)_A$. Then $a\in A$ and there exist $i\in I$ and $j\in J$ such that $a\leqq i\vee j$. From Lemma 5, it follows that

$$a\in(i\vee j]\cap A\subseteq((i]\cap A)\vee^A((j]\cap A)\subseteq(I)_A\vee^A(J)_A.$$

This proves the first half of the lemma.

Assume now that $I\vee J$ is smooth. We have to prove that so is $(I)_A\vee^A(J)_A$. Let $q\in Q$ and let

$$q\in(I)_A\vee^A(J)_A.$$

Then $q\in I\vee J$; thus, $q\in I$ or $q\in J$, say $q\in I$. Since $q\in Q\subseteq A$, we have $q\in I\cap A=(I)_A$. This completes the proof.

Most of the results of this section are summarized in the following two lemmas that show that one can work with smooth representations as if forming lower covers were a homomorphism.

Lemma 7. *Let I and J be ideals of A and let us assume that $I\vee^A J$ is smooth. Then*

$$(I\vee^A J)_i=(I)_i\vee^i(J)_i \quad \textit{for} \quad i=0,1$$

and the right side of the equation is smooth.

Proof. We claim that

$$((I)_0\vee^0(J)_0)\cap Q=((I)_1\vee^1(J)_1)\cap Q.$$

Indeed, let $q\in Q$ and let $q\in(I)_0\vee^0(J)_0$. Then $q\in I\vee^A J$; therefore, q is in I or J, say, $q\in I$. Then $q\in(I)_1\subseteq(I)_1\vee^1(J)_1$, which verifies that the left side is contained in the right side. Repeating this argument starting with the right side, we verify the claim.

This claim, by Lemma 3, shows that

$$((I)_0\vee^0(J)_0)\cup((I)_1\vee^1(J)_1)$$

is an ideal of $P(A_0, A_1)$; obviously, it is the smallest ideal containing both I and J, that is,

$$I\vee^A J=((I)_0\vee^0(J)_0)\cup((I)_1\vee^1(J)_1).$$

Now we compute (using the above claim again):

$$\begin{aligned}(I\vee^A J)_0 &=\\ &=(((I)_0\vee^0(J)_0)\cup((I)_1\vee^1(J))_1)\cap A_0=\\ &=((I)_0\vee^0(J)_0)\cup(((I)_1\vee^1(J)_1)\cap Q)=\\ &=((I)_0\vee^0(J)_0)\cup(((I)_0\vee^0(J)_0)\cap Q)=\\ &=(I)_0\vee^0(J)_0.\end{aligned}$$

Finally, we can see that $(I)_0 \vee^0 (J)_0$ is smooth arguing as we did in Lemma 6.

Lemma 8. *Let $p=p(x_0, \ldots, x_{n-1})$ be a lattice polynomial and let $I_0, \ldots, I_{n-1}$ be ideals of L, such that $p(I_0, \ldots, I_{n-1})$ is smooth. Then*

$$\big(p(I_0, \ldots, I_{n-1})\big)_i = p\big((I_0)_i, \ldots, (I_{n-1})_i\big)$$

is a smooth representation of $\big(p(I_0, \ldots, I_{n-1})\big)_i$.

Proof. By induction: if $p=x_i$ or $p=p_0 \wedge p_1$, then Lemma 8 is trivial; if $p=p_0 \vee p_1$, then Lemma 8 is a combination of Lemmas 6 and 7.

5. Amalgamated free products of sublattices. It was proved in B. Jónsson [5], that, if a variety V has the Amalgamation Property, then the following statement holds: for arbitrary algebras A_0 and A_1 in V and subalgebras A_0' of A_0 and A_1' of A_1 the set $A_0' \cup A_1'$ generates a subalgebra in the free product $A_0 * A_1$ canonically isomorphic to $A_0' * A_1'$. "Canonically" means that the isomorphism is the identity map on A_0' and on A_1'. Jónsson's proof is valid not only for varieties but also for classes closed under the formation of subalgebras and of direct products. Thus the proof works for Q-lattices, that is, lattices containing Q as a sublattice such that the elements of Q are regarded as nullary operations. This yields the following lemma.

Lemma 9. *Let $L=A_0 *_Q A_1$, let A_0' and A_1' be sublattices of A_0 and A_1, respectively, and let $Q \subseteq A_0'$ and $Q \subseteq A_1'$. Then the sublattice of $A_0 *_Q A_1$ generated by $A_0' \cup A_1'$ is canonically isomorphic to $A_0' *_Q A_1'$.*

There is an alternative proof by using the solution to the word problem for lattices generated by a partial lattice. For the case $Q=\varnothing$, such a proof appears in G. Grätzer, H. Lakser, and C. R. Platt [3]. (See also G. Grätzer [1].)

6. Proof of the Theorem. We introduce some new notation. For an ideal I of L, let I_{A_0} denote the ideal of L generated by $I \cap A_0$; we call I_{A_0} the lower A_0-cover of I. Similarly for I_{A_1}, I_{B_0}, and I_{B_1}. Note that Lemma 8 holds also for lower A_i (resp., B_j)-covers.

For arbitrary fixed $i, j \in \{0, 1\}$, we define $I_{ij}(L)$ as the set of principal ideals of L and the lower A_i-covers and lower B_j-covers of principal ideals of L.

We prove the main theorem in a stronger form:

Theorem. *Let $L=A_0 *_Q A_1 = B_0 *_Q B_1$. Then the following conditions are equivalent.*

(i) $L=(A_0 \cap B_0) *_Q (A_0 \cap B_1) *_Q (A_1 \cap B_0) *_Q (A_1 \cap B_1)$.
(ii) $A_i=(A_i \cap B_0) *_Q (A_i \cap B_1)$, *for* $i=0, 1$.
(iii) $B_j=(A_0 \cap B_j) *_Q (A_1 \cap B_j)$, *for* $j=0, 1$.

(iv) *For any* $i, j \in \{0, 1\}$, $x \in A_i$, *and* $y \in B_j$, $x \leqq y$ *in* L *implies the existence of a* $z \in A_i \cap B_j$ *such that* $x \leqq z$ *in* A_i *and* $z \leqq y$ *in* B_j.

(v) *For any* $i, j \in \{0, 1\}$ *and for any ideal* I *of* L, $I = (I \cap A_i] = (I \cap B_j]$ *implies that* $I = (I \cap A_i \cap B_j]$.

(iv) *For any* $i, j \in \{0, 1\}$ *and for any ideal* $I \in I_{ij}(L)$, $I = (I \cap A_i] = (I \cap B_j]$ *implies that* $I = (I \cap A_i \cap B_j]$.

Proof. We prove the theorem by the following scheme: (i)↔(ii), (i)↔(iii); (i), (ii), and (iii) jointly imply (iv); (iv)→(v)→(vi)→(ii).
(ii)→(i) is clear from the definition of the right side of (i) (given after Definition 1).
(i)→(ii). Let $a \in A_0$. Then, by (i), a can be expressed in the form

$$a = p(x_{00}, x_{00}', \ldots, x_{01}, x_{01}', \ldots, x_{10}, x_{10}', \ldots, x_{11}, x_{11}', \ldots)$$

where $x_{ij}, x_{ij}', \ldots \in A_i \cap B_j$, $i, j \in \{0, 1\}$ and p is a lattice polynomial. By Lemma 4, $(a]$ has a smooth representation in $I(L)$:

$$(a] = \tilde{p}((x_{00}], \ldots, (x_{01}], \ldots, (x_{10}], \ldots, (x_{11}], \ldots, (Q_0], \ldots),$$

where $Q_0, \ldots \subseteq Q$. Then, by Lemma 8,

$$(a] = (a]_{A_0} = \tilde{p}((x_{00}]_{A_0}, \ldots, (x_{10}]_{A_0}, \ldots, (x_{01}]_{A_0}, \ldots, (x_{11}]_{A_0}, \ldots, (Q_0], \ldots).$$

We claim that, $(x_{10}]_{A_0}$, as well as $(x_{11}]_{A_0}$, is generated as an ideal by elements of Q. Indeed, let $(x_{10}]_{A_0}$ be generated by x_γ, $\gamma \in \Gamma$, in A_0. By Lemma 2, for every $\gamma \in \Gamma$, there is a $y_\gamma \in Q$ with $x_\gamma \leqq y_\gamma \leqq x_{10}$. Thus $\{y_\gamma | \gamma \in \Gamma\}$ generates $(x_{10}]_{A_0}$, and $\{y_\gamma | \gamma \in \Gamma\}$ is a subset of Q. Summarizing,

$$(a] = \tilde{p}((x_{00}], \ldots, (x_{10}], \ldots, (Q_{10}], \ldots, (Q_{11}], \ldots, (Q_0], \ldots),$$

where $Q_{10}, \ldots, Q_{11}, \ldots, Q_0, \ldots \subseteq Q$. Hence a can be expressed by $x_{00}, \ldots, x_{10}, \ldots$, and elements of Q. Thus, a is in the sublattice generated by $(A_0 \cap B_0) \cup (A_0 \cap B_1)$. Therefore, $A_0 \cap B_0$ and $A_0 \cap B_1$ generate A_0. It follows from Lemma 9 that they generate A_0 freely over Q.
(i)↔(iii) follows by symmetry.
(i), (ii), and (iii) jointly imply (iv). By (ii) and (iii), the sublattice $[A_0 \cup B_0]$ generated by $A_0 \cup B_0$ is also generated by $A_0 \cap B_0$, $A_0 \cap B_1$, and $A_1 \cap B_0$. By (i), $A_0 \cap B_0$, $A_0 \cap B_1$, $A_1 \cap B_0$ freely generate over Q. Thus $[A_0 \cup B_0]$ is freely generated by $(A_0 \cap B_0) \cup (A_0 \cap B_1) \cup (A_1 \cap B_0)$. Hence, it is also freely generated by $[(A_0 \cap B_0) \cup \cup (A_0 \cap B_1)] \cup [(A_0 \cap B_0) \cup (A_1 \cap B_0)]$. By (ii) and (iii), this set is $A_0 \cup B_0$, and the relative sublattice of L on this subset is the partial lattice $P(A_0, B_0, A_0 \cap B_0)$. Therefore, $[A_0 \cup B_0]$ is the free $(A_0 \cap B_0)$-product of A_0 and B_0. Thus, Lemma 2 gives us (iv) for $i = j = 0$. Since (i), (ii), and (iii) are symmetric in i and j, condition (iv) now follows.

(iv)→(v). Let the ideal I be generated by $\{x_\gamma|\gamma\in\Gamma\}\subseteq A_i$ and by $\{y_\delta|\delta\in\Delta\}\subseteq B_j$; we can assume that $\{y_\delta|\delta\in\Delta\}$ is closed under finite joins. By (iv), for any $\gamma\in\Gamma$ we can choose a $\gamma'\in\Delta$ and a $z_{\gamma'}\in A_i\cap B_j$ satisfying $x_\gamma\leqq z_{\gamma'}\leqq y_{\gamma'}$. Obviously, $\{z_{\gamma'}|\gamma'\in\Gamma\}\subseteq\subseteq A_i\cap B_j$ generates I.

(v)→(vi) is obvious since (vi) is a special case of (v).

(vi)→(ii). By Lemma 9 and by symmetry, it suffices to prove that A_0 is generated by $(A_0\cap B_0)\cup(A_0\cap B_1)$.

For $a\in A_0$, there exist a polynomial p and elements $b_0, b_0', \ldots\in B_0, b_1, b_1', \ldots\in\in B_1$, such that $a=p(b_0, b_0', \ldots, b_1, b_1', \ldots)$. By Lemma 4, there exist a polynomial $\tilde{p}$ and $Q_0, Q_1, \ldots\subseteq Q$ such that

$$(a] = \tilde{p}((b_0], (b_0'], \ldots, (b_1], (b_1'], \ldots, (Q_0], (Q_1], \ldots)$$

is a smooth representation of $(a]$. Then, by Lemma 8,

$$(a] = (a]_{A_0} = \tilde{p}((b_0]_{A_0}, \ldots, (b_1]_{A_0}, \ldots, (Q_0]_{A_0}, \ldots).$$

In this expression, $(Q_0]_{A_0}=(Q_0], \ldots$. Furthermore, we shall prove the claim that $(b_0]_{A_0}, (b_0']_{A_0}, \ldots,$ and $(b_1]_{A_0}, (b_1']_{A_0}, \ldots$ are generated by elements of $A_0\cap B_0$ and $A_0\cap B_1$, respectively. Thus, each ideal occurring in the representation is generated by elements of $(A_0\cap B_0)\cup(A_0\cap B_1)$. Therefore, so is $(a]$. We conclude that $a\in\in[(A_0\cap B_0)\cup(A_0\cap B_1)]$, which was to be proved.

To verify the claim, it is sufficient to prove by symmetry that $(b_0]_{A_0}$ is generated by its elements in $A_0\cap B_0$.

First, we verify that $(b_0]_{A_0}$ is generated by its elements in B_0.

We start with a smooth representation

$$(b_0] = q((a_0], (a_0'], \ldots, (a_1], (a_1'], \ldots, (R_0], (R_1], \ldots),$$

where $a_0, a_0', \ldots\in A_0, a_1, a_1', \ldots\in A_1$ and $R_0, R_1, \ldots\subseteq Q$. Then

$$(b_0]_{A_0} = q((a_0]_{A_0}, \ldots, (a_1]_{A_0}, \ldots, (R_0]_{A_0}, \ldots) = q((a_0], \ldots, (a_1]_{A_0}, \ldots, (R_0], \ldots)$$

and, applying Lemma 8 twice, we obtain

$$(b_0]_{A_0} = ((b_0]_{B_0})_{A_0} = q(((a_0]_{B_0})_{A_0}, \ldots, ((a_1]_{B_0})_{A_0}, \ldots, ((R_0]_{B_0})_{A_0}, \ldots).$$

Hence,

$$(b_0]_{A_0} = q((a_0], \ldots, (a_1]_{A_0}, \ldots, (R_0], \ldots) \supseteqq q((a_0]_{B_0}, \ldots, (a_1]_{A_0}, \ldots, (R_0], \ldots) \supseteqq$$
$$\supseteqq q(((a_0]_{B_0})_{A_0}, \ldots, ((a_1]_{B_0})_{A_0}, \ldots, ((R_0]_{B_0})_{A_0}, \ldots) = (b_0]_{A_0}$$

therefore,

$$(b_0]_{A_0} = q((a_0]_{B_0}, \ldots, (a_1]_{A_0}, \ldots, (R_0], \ldots).$$

The ideals $(a_0]_{B_0}, \ldots$ are, by definition, generated by elements of B_0; the ideals $(a_1]_{A_0}, \ldots$ are generated by elements of Q by Lemma 2. Since $Q_0, \ldots\subseteq Q$, we conclude that $(b_0]_{A_0}$ is generated by elements of B_0.

Finally, since $(b_0]_{A_0}$ has been proved to be generated by its elements in B_0, and $(b_0]_{A_0}$ is by definition generated by its elements in A_0, and $(b_0]_{A_0} \in I_{00}(L)$, all the hypotheses of (vi) are satisfied. Condition (vi) yields that $(b_0]_{A_0}$ is generated by elements of $A_0 \cap B_0$, which completes the proof of the claim.

This finishes the proof of the implication (vi)→(ii) and of the Theorem.

7. Proof of Corollaries 1—4. Proof of Corollary 1. Let Q satisfy, for example, the Ascending Chain Condition, and let $L = A_0 *_Q A_1 = B_0 *_Q B_1$. We claim that, for any $i, j \in \{0, 1\}$, $I_{ij}(L)$ consists of all principal ideals of L. Indeed, let us take a smooth representation of the principal ideal $(x]$:

$$(x] = p((a_0], (a_0'], \ldots, (a_1], (a_1'], \ldots, (Q_0], (Q_1], \ldots),$$

$a_0, a_0', \ldots \in A_0$, $a_1, a_1', \ldots \in A_1$, and $Q_0, Q_1, \ldots \subseteqq Q$. Then

$$(x]_{A_0} = p((a_0], (a_0'], \ldots, (a_1]_{A_0}, (a_1']_{A_0}, \ldots, (Q_0], (Q_1], \ldots).$$

It follows from Lemma 2 that the ideals $(a_1]_{A_0}, \ldots,$ are generated by elements of Q; thus, by the Ascending Chain Condition, these ideals and also $(Q]_0, \ldots$ are principal. Therefore, $(x]_{A_0}$ is a principal ideal. This proves the claim for $i=j=0$. By symmetry, the claim is proved.

Using this claim, it is easy to establish condition (vi) of the Theorem: if the single generating element of an ideal in $I_{ij}(L)$ is both in A_i and in B_j, then it is in $A_i \cap B_j$. Thus the Theorem shows the existence of a common refinement.

Proof of Corollary 2. Let $L = A_0 *_Q A_1 = B_0 *_Q B_1$, and let us assume that the hypotheses of Corollary 2 hold, that is, for any i, j, A_i or B_j is convex in $A_i \cup B_j$. We are going to establish condition (v) of the Theorem. Let $i, j \in \{0, 1\}$, let, for instance, A_i be convex in $A_i \cup B_j$. Let $I \in I(L)$, such that $I = (I \cap A_i] = (I \cap B_j]$. Let G be a generating set of I in A_i and let H be a generating set of I in B_j. We can assume that both G and H are closed under finite joins. Then

$$I = \{x \mid x \leqq g \text{ for some } g \in G\} = \{x \mid x \leqq h \text{ for some } h \in H\}.$$

Thus, for any $g \in G$ there exists an $h_g \in H$ satisfying $g \leqq h_g$ and for $h_g \in H$ there exists an $g' \in G$ with $h_g \leqq g'$. Therefore, $g \leqq h_g \leqq g'$, so by the convexity of A_i in $A_i \cup B_j$, we conclude that $h_g \in A_i$; since $h_g \in G \subseteqq B_j$, $h_g \in A_i \cap B_j$. Now it is clear that $K = \{h_g \mid g \in G\}$ generates I and $K \subseteqq A_i \cap B_j$, verifying condition (v) of the Theorem.

Proof of Corollaries 3 and 4. Under the conditions of the Corollaries, $[A_1 \cup A_2]$ is the free product of A_1 and A_2 amalgamated over $A_1 \cap A_2$. Thus we may apply Lemma 2 to $A_1 \cup A_2$. Therefore, both corollaries follow from the following observation (due to E. Fried):

Let $L=A_0 *_Q A_1 = A_0 *_Q A_2$. If the conclusion of Lemma 2 holds for $A_1 \cup A_2$ (that is, for $x \in A_1$ and $y \in A_2$, $x \leqq y$ iff $x \leqq z \leqq y$ for some $z \in A_1 \cap A_2$ and symmetrically for $x \in A_2$ and $y \in A_1$), then $A_1 = A_2$.

Indeed, under these conditions (iv) of the Theorem holds, hence there is a common refinement. Applying condition (ii) of the Theorem we obtain

$$A_1 = (A_0 \cap A_1) *_Q (A_2 \cap A_1) = Q *_Q (A_2 \cap A_1).$$

Similarly, $A_2 = Q *_Q (A_1 \cap A_2)$, hence $A_1 = A_2$.

8. Open problems. We repeat the question already mentioned in Section 2.

Problem 1. Is there a lattice Q such that Q-free products do not always have common refinements?

An equally important question arises in connection with Corollaries 3 and 4. In fact, they suggest, that some sort of a distributive law must be valid for Q-free factors.

Problem 2. Do Q-free factors of a lattice L form a distributive sublattice of the lattice of all sublattices of L? Does there exist some "natural" generalization of distributivity that holds for Q-free factors and implies Corollaries 3 and 4?

A negative answer to Problem 1 would answer both questions of Problem 2 in the affirmative; this can be seen from the following observations.

Let us assume that for a lattice Q, any two Q-free products of a lattice L have a common refinement. Let L be a lattice and let Q be a sublattice of L. Then $L = A *_Q A' = B *_Q B'$ implies that

$$L = (A \cap B) *_Q [A' \cup B'];$$

thus the Q-free factors form a sublattice of the lattice of all sublattices of L. Now let A, B, C be Q-free factors of L, that is, let

$$L = A *_Q A' = B *_Q B' = C *_Q C'.$$

Then

$$A \cap [B \cup C] = [(A \cap B) \cup (A \cap C)],$$

since both sides are the Q-free products of $A \cap B \cap C$, $A \cap B \cap C'$, and $A \cap B' \cap C$.

9. Appendix: On the definition of amalgamated free products. In Section 3 we defined $A_0 *_Q A_1$ as the free lattice generated by the smallest partial lattice on the set $A_0 \cup A_1$ $(A_0 \cap A_1 = Q)$ extending the operations of A_0 and A_1. We denoted this partial lattice by $P(A_0, A_1, Q)$. Here we prove the following characterization:

$P(A_0, A_1, Q)$ is the smallest *weak partial lattice* on the set $A_0 \cup A_1$ extending the operations of A_0 and A_1.

By a weak partial lattice (see [1]) we mean a partial algebra $\langle H; \wedge, \vee\rangle$ satisfying conditions (i)—(iv) of Section 3 and their duals.

This result means the following: by definition, $P(A_0, A_1, Q)$ is formed by taking $A_0 \cup A_1$, and extending the $\wedge$ and $\vee$ of A_0 and A_1 by iterating (i)—(v) and their duals; according to the result of this appendix, condition (v) and its dual are not needed in this process.

Let $WP(A_0, A_1, Q) = WP$ be the smallest weak partial lattice on $A_0 \cup A_1$ extending the operations of A_0 and A_1. The existence of WP can be proved along the lines of the proof of Lemma 1. The proof of Lemma 2 shows that the partial ordering on WP is the same as the partial ordering on $P(A_0, A_1, Q)$. We are going to prove that WP is a partial lattice, that is, (v) and its dual hold. Then obviously $WP= =P(A_0, A_1, Q)$.

By duality, it is sufficient to verify (v). To do that, let $a, b, c \in A_0 \cup A_1$ such that $(a] \vee (b] = (c]$ in the ideal lattice of WP. We have to show that $a \vee b$ exists and $a \vee b = c$ in WP.

If $a \vee b$ exists, then $(a \vee b]$ is obviously $(a] \vee (b]$, hence $(a \vee b] = (c]$. We conclude that $a \vee b = c$. Therefore, it is sufficient to show that if $(a] \vee (b] = (c]$, then $a \vee b$ exists.

If $a, b \in A_0$ or $a, b \in A_1$, then $a \vee b$ exists. Hence we can assume that $a \in A_0$ and $b \in A_1$. By symmetry, we can also assume that $c \in A_1$.

By the general description of join of ideals in a weak partial lattice (see Exercise 5.22 of [1]), $(a] \vee (b] = (c]$ implies the existence of a natural number n and elements

(1) $a = a_0 \leqq a_1 \leqq \ldots \leqq a_n$ in A_0,

(2) $b = b_0 \leqq b_1 \leqq \ldots \leqq b_n = c$ in A_1,

(3) $r_0 \leqq q_1 \leqq r_1 \leqq q_2 \leqq \ldots \leqq q_n \leqq q$ in Q

such that

(4) $r_i \leqq b_i, \quad 0 \leqq i \leqq n$,

(5) $q_i \leqq a_i, \quad 1 \leqq i \leqq n$,

(6) $b_{i+1} = b_i \vee q_{i+1}, \quad 0 \leqq i < n$,

(7) $a_{i+1} = a_i \vee r_{i+1}, \quad 0 \leqq i < n$,

(8) $a_n \leqq q \leqq b_n$.

(The symmetric case with $q_0 \leqq r_1 \leqq q_1 \leqq \ldots \leqq q_{n-1}, q_i \leqq a_i, 0 \leqq i \leqq n, r_i \leqq b_i, 1 \leqq i \leqq n, a_{i+1} = a_i \vee r_{i+1}, 0 \leqq i < n, b_{i+1} = b_i \vee q_i, 0 \leqq i < n$ is handled similarly.)

In the proof we shall utilize the following two properties of weak partial lattices:

(P1) If $x \vee y = z$ and $x \leqq u \leqq z$, then $u \vee y$ exists and $u \vee y = z$.

Indeed, by the associative identity,

$$u \vee (x \vee y) = (u \vee x) \vee y,$$

the left side exists and equals z; $u \vee x$ exist and equals u, hence $u \vee y$ exists and equals z, as claimed.

(P2) If $x \vee y = z$, $x = x_1 \vee x_2$, and $x_2 \leqq y$, then $x_1 \vee y$ exists and $x_1 \vee y = z$.

Indeed, by the associative identity,

$$(x_1 \vee x_2) \vee y = x_1 \vee (x_2 \vee y),$$

the left side exists and equals z; in the right side $x_2 \vee y$ exists and equals y, hence by (iii), $x_1 \vee y$ exists and $x_1 \vee y = z$, as claimed.

Now we prove $a \vee b = c$ by induction on n. Let $n = 1$. Then we have the elements $a_0 = a$, $b_0 = b$, $b_1 = c$, r_0, q_1, q, and $r_0 \leqq q_1 \leqq q$, $r_0 \leqq b_0$, $r_0 \leqq q_1 \leqq a_1$, $a_1 = a \vee r_0$, $b_1 = b_0 \vee q_1$, $a_1 \leqq q \leqq b_1 = c$.

By (P1), $q_1 \vee b = c$ and $q_1 \leqq a \leqq c$ implies that $a_1 \vee b$ exist and $a_1 \vee b = c$. Since $a_1 = a \vee r_0$ and $r_0 \leqq b$, by (P2), $a \vee b$ exists and $a \vee b = c$, as claimed.

Now let $n > 1$. It is clear, that the elements $a_1 \leqq \ldots \leqq a_n, b_1 \leqq \ldots \leqq b_n, r_1 \leqq q_2 \leqq \ldots \leqq q_n \leqq q$ satisfy (1)—(8) with $n-1$. Therefore, $a_1 \vee b_1$ exists and $a_1 \vee b_1 = c$. By (P2), $c = a_1 \vee b_1 = a_1 \vee (q_1 \vee b) = a_1 \vee b$, since $q_1 \leqq a_1$. Again by (P2), $c = a_1 \vee b = (a \vee r_0) \vee b = a \vee b_1$ since $r_0 \leqq b$. This proves the theorem.

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