Acta Sci. Math., 44 (1982), 67—75

A general moment inequality for the maximum of partial sums of single series

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1. The main result

Let $(X, \mathcal{A}, \mu)$ be a (not necessarily finite or σ-finite) positive measure space. Let $\{\xi_k=\xi_k(x)\colon k=1, 2, \ldots\}$ be a given sequence of functions, defined on X, measurable with respect to $\mathcal{A}$, and such that $|\xi_k|^\gamma$ are integrable over X with respect to μ, where γ is a fixed real number, $\gamma\geqq 1$; i.e., our permanent assumption is that $\xi_k\in L^\gamma(X, \mathcal{A}, \mu)$ for each k. Set

$$S(b, l) = \sum_{k=b+1}^{b+l} \xi_k \quad \text{and} \quad M(b, m) = \max_{1\leqq l\leqq m} |S(b, l)|,$$

where b is a nonnegative integer, l and m are positive integers.

In the following, $f(b, m)$ denotes a nonnegative function defined for integral $b\geqq 0$ and $m\geqq 1$, which possesses the 'superadditivity' property:

(1.1) $\quad f(b, k)+f(b+k, l) \leqq f(b, k+l) \quad$ for $\quad b \geqq 0, \quad k \geqq 1, \quad$ and $\quad l \geqq 1$.

We shortly explain the origin of the term 'superadditivity' in connection with the property expressed by (1.1). The fact is that $f(b, k)$ is actually a function of the interval $(b, b+k]=I_1$ with nonnegative integer endpoints. Considering the intervals $I_2=(b+k, b+k+l]$ and $I=(b, b+k+l]$ too, we can see that the union $I_1\cup I_2$ is a disjoint representation of I. Now (1.1) can be rewritten as follows

$$f(I_1)+f(I_2) \leqq f(I) \quad \text{where} \quad f(I_1) = f(b, k), \quad \text{etc.}$$

In the additive or subadditive case the relation '$\leqq$' should be replaced by '$=$' or '$\geqq$', respectively.

Received September 30, 1981.

This research was conducted while the author was on leave from Szeged University and a Visiting Professor at Ulm University, West Germany (supported by DAAD).

Further, by $\varphi(t, m)$ we denote a nonnegative function defined for real $t \geqq 0$ and integral $m \geqq 1$. We assume that $\varphi(t, m)$ is nondecreasing in both variables, i.e.,

$$\varphi(t_1, m_1) \leqq \varphi(t_2, m_2) \quad \text{whenever} \quad 0 \leqq t_1 \leqq t_2 \quad \text{and} \quad 1 \leqq m_1 \leqq m_2.$$

Our main result can be formulated as follows.

Theorem. *Let $\gamma \geqq 1$ be given. Suppose that there exist a nonnegative and superadditive function $f(b, m)$, and a nonnegative function $\varphi(t, m)$, nondecreasing in both variables, such that for every $b \geqq 0$ and $m \geqq 1$ we have*

$$\int |S(b, m)|^\gamma \, d\mu \leqq f(b, m) \, \varphi^\gamma(f(b, m), m). \tag{1.2}$$

Then for every $b \geqq 0$ and $m \geqq 2$ we have both the inequality

$$\int M^\gamma(b, m) \, d\mu \leqq 3^{\gamma-1} f(b, m) \left\{ \sum_{k=0}^{[\log m]-1} \varphi\left(\frac{f(b, m)}{2}, \left[\frac{m}{2^{k+1}}\right]\right) \right\}^\gamma \tag{1.3}$$

and the inequality

$$\int M^\gamma(b, m) \, d\mu \leqq \frac{5}{2} f(b, m) \left\{ \sum_{k=0}^{[\log m]} \varphi\left(\frac{f(b, m)}{2^k}, \left[\frac{m}{2^k}\right]\right) \right\}^\gamma. \tag{1.4}$$

Here and in the sequel the integrals are taken over the whole space X, $[t]$ denotes the integral part of t, and all logarithms are with base 2.

Remark 1. It is striking that the factor 5/2 in (1.4) does not depend on γ, in contrast to the factor $3^{\gamma-1}$ in (1.3). On the other hand, we have to take $[m/2^k]$ in the argument of φ on the right-hand side of (1.4), instead of $[m/2^{k+1}]$, which is the case in (1.3).

2. Special cases

We are going to present the riches of applicability of our Theorem, without aiming at completeness.

Let us take $\varphi(t, m) = t^{(\alpha-1)/\gamma}$ with an $\alpha > 1$. Then

$$\tilde{\Phi}(t, m) = \sum_{k=0}^{[\log m]} \varphi\left(\frac{t}{2^k}, \left[\frac{m}{2^k}\right]\right) \leqq (1 - 2^{(1-\alpha)/\gamma})^{-1} t^{(\alpha-1)/\gamma},$$

independently of m.

Corollary 1. *Let $\alpha > 1$ and $\gamma \geqq 1$ be given. Suppose that there exists a nonnegative and superadditive function $f(b, m)$ such that for every $b \geqq 0$ and $m \geqq 1$ we have*

$$\int |S(b, m)|^\gamma \, d\mu \leqq f^\alpha(b, m).$$

Then for every $b \geqq 0$ and $m \geqq 1$ we have

$$\int M^\gamma(b, m) \, d\mu \leqq \frac{5}{2} (1 - 2^{(1-\alpha)/\gamma})^{-\gamma} f^\alpha(b, m).$$

This result, apart from the factor 5/2 on the right-hand side, was proved by the present author in [3, Theorem 1], and somewhat later (with another constant) by LONGNECKER and SERFLING [2, Theorem 1].

Now take $\varphi(t, m)=t^{(\alpha-1)/\gamma} w(t)$, where again $\alpha>1$ and $w(t)$ is a (not necessarily nondecreasing, but positive) slowly varying function, i.e., $w(t)$ is defined and positive for real $t>0$, and for every fixed real $C>0$ we have

$$\frac{w(Ct)}{w(t)} \to 1 \quad \text{as} \quad t \to \infty.$$

For example, $w(t)=\{\log(1+t)\}^{\beta}\{\log\log(2+t)\}^{\delta}$ is such a function, where β and δ are arbitrary real numbers. It is not hard to check that we again have

$$\tilde{\Phi}(t, m) \leqq C(\alpha, \gamma, w)\, t^{(\alpha-1)/\gamma} w(t),$$

where $C(\alpha, \gamma, w)$ is a positive constant depending only on α, γ, and $w(t)$.

Corollary 2. *Let $\alpha>1$ and $\gamma\geqq 1$ be given. Suppose that there exist a nonnegative and superadditive function $f(b, m)$, and a slowly varying positive function $w(t)$, such that $t^{(\alpha-1)/\gamma} w(t)$ is nondecreasing and that for every $b\geqq 0$ and $m\geqq 1$ we have*

$$\int |S(b, m)|^{\gamma}\, d\mu \leqq f^{\alpha}(b, m)\, w^{\gamma}(f(b, m)).$$

Then for every $b\geqq 0$ and $m\geqq 1$ we have

$$\int M^{\gamma}(b, m)\, d\mu \leqq \frac{5}{2} C(\alpha, \gamma, w) f^{\alpha}(b, m)\, w^{\gamma}(f(b, m)).$$

Next take $\varphi(t, m)=\lambda(m)$, where $\{\lambda(m)\colon m=1, 2, \ldots\}$ is a nondecreasing sequence of positive numbers.

Corollary 3. *Let $\gamma\geqq 1$ be given. Suppose that there exist a nonnegative and superadditive function $f(b, m)$, and a positive and nondecreasing sequence $\lambda(m)$ such that for every $b\geqq 0$ and $m\geqq 1$ we have*

$$\int |S(b, m)|^{\gamma}\, d\mu \leqq f(b, m)\, \lambda^{\gamma}(m).$$

Then for every $b\geqq 0$ and $m\geqq 1$ we have

$$\int M^{\gamma}(b, m)\, d\mu \leqq 3^{\gamma-1} f(b, m) \left\{\sum_{k=1}^{[\log m]} \lambda\left(\left[\frac{m}{2^k}\right]\right)\right\}^{\gamma}. \tag{2.1}$$

This moment inequality, apart from the factor $3^{\gamma-1}$ on the right-hand side, was already proved by the present author in a slightly different form in [3, Theorem 4].

Finally, it is quite obvious that in any case we can state the following

Corollary 4. *Under the conditions of the Theorem, for every* $b \geqq 0$ *and* $m \geqq 2$ *we have*

$$\int M^{\gamma}(b, m)\, d\mu \leqq 3^{\gamma-1} f(b, m)\, \varphi^{\gamma}\left(f(b, m), \left[\frac{m}{2}\right]\right)(\log m)^{\gamma}. \tag{2.2}$$

In the special case when $\varphi(t, m)=\lambda(m)$ is a slowly varying sequence, which is positive and nondecreasing, in particular, when $\varphi(t, m)=1$, the right-hand side of (2.2) is of the same order of magnitude as the right-hand side of (1.3) or (1.4). Thus, in this case the moment inequality (2.2) cannot be improved in the framework of our method.

Remark 2. Corollary 3 is proved in [3] by the socalled bisection technique with respect to the number m of the terms, which goes back to the proof of the well-known Rademacher—Menšov inequality (see, e.g. [4, p. 83]). The proof of Corollary 1 is based on the bisection technique with respect to the weight $f(b, m)$, which was firstly applied, it seems to us, by Erdős [1] concerning an upper estimation of the fourth moment of the partial sums of lacunary trigonometric series. Now, the proof of our Theorem presented in the next Section is based on an appropriate combination of these two bisection techniques. This combined technique was firstly used, as far as the author is aware, by Tandori [5] in order to obtain a special upper estimate for the second moment of the maximum of the partial sums of orthogonal series.

For a more detailed historical background of these moment inequalities see [3].

3. The proof of the theorem

Proof of (1.3). Setting

$$\Phi(t, 1) = \varphi(t, 1) \qquad (t \geqq 0)$$

and

$$\Phi(t, m) = \sum_{k=0}^{[\log m]-1} \varphi\left(\frac{t}{2^k}, \left[\frac{m}{2^{k+1}}\right]\right) \qquad (t \geqq 0,\ m \geqq 2),$$

it is clear that $\Phi(t, m)$ is also nondecreasing in both variables. This explicit expression for $\Phi(t, m)$ can be rewritten into the following recurrence one, which will be useful in the sequel:

$$\Phi(t, 1) = \Phi(t, 2) = \Phi(t, 3) = \varphi(t, 1) \quad (t \geqq 0) \tag{3.1}$$

and

$$\Phi(t, m) = \varphi\left(t, \left[\frac{m}{2}\right]\right) + \Phi\left(\frac{t}{2}, \left[\frac{m}{2}\right]\right) \qquad (t \geqq 0,\ m \geqq 4). \tag{3.2}$$

Now, statement (1.3) to be proved turns into

$$\int M^{\gamma}(b, m)\, d\mu \leqq 3^{\gamma-1} f(b, m)\, \Phi^{\gamma}(f(b, m), m). \tag{3.3}$$

The proof of (3.3) proceeds by induction on m. By (1.2) and (3.1), this is obvious for $m=1$ and for each b, even the factor $3^{\gamma-1}$ is superfluous on the right of (3.3) in this case.

In order to prove (3.3) for $m=2$ and 3 with arbitrary b, we use the trivial estimate

$$M(b, m) \leqq \sum_{k=b+1}^{b+m} |\xi_k|,$$

whence Minkowski's inequality and (1.2) provide that

$$\Big\{\int M^{\gamma}(b, m)\, d\mu\Big\}^{1/\gamma} \leqq \sum_{k=b+1}^{b+m} f^{1/\gamma}(k-1, 1)\, \varphi(f(k-1, 1), 1). \tag{3.4}$$

Taking into account the monotonicity of $\varphi(t, m)$ and making use of the elementary inequality

$$\sum_{k=b+1}^{b+m} t_k^{1/\gamma} \leqq m^{(\gamma-1)/\gamma} \Big(\sum_{k=b+1}^{b+m} t_k\Big)^{1/\gamma} \qquad (t_k \geqq 0,\ \gamma \geqq 1), \tag{3.5}$$

from (3.4) and (1.1) it follows that

$$\Big\{\int M^{\gamma}(b, m)\, d\mu\Big\}^{1/\gamma} \leqq \varphi(f(b, m), 1) \sum_{k=b+1}^{b+m} f^{1/\gamma}(k-1, 1) \leqq$$
$$\leqq m^{(\gamma-1)/\gamma} f^{1/\gamma}(b, m)\, \varphi(f(b, m), 1).$$

By (3.1) this is a sharpened form of (3.3) in case $m=2$, and (3.3) itself in case $m=3$.

Assume now as induction hypothesis that inequality (3.3) holds true for each nonnegative integer b and for each positive integer less than m, $m \geqq 4$, in the place of the second argument (we actually use that it is true for each positive integer not more than $[m/2]$). We will show that inequality (3.3) holds for m itself (and for arbitrary b).

We begin with an elementary observation. If $f(b, m)=0$ for some b and m, then, by (1.1), $f(b, k)=0$ and, by (1.2), $S(b, k)=0$ a.e. for each $k=1, 2, \ldots, m$, too. Consequently, $M(b, m)=0$ a.e. and thus (3.3) is obviously satisfied.

Henceforth we may and do assume that $f(b, m) \neq 0$. Then there exists an integer p, $1 \leqq p \leqq m$, such that

$$f(b, p-1) \leqq \frac{1}{2} f(b, m) < f(b, p), \tag{3.6}$$

where we agree to set $f(b, 0)=0$ on the left of (3.6) in case $p=1$. It is also convenient to set $S(b, 0)=M(b, 0)=0$. Now (1.1) and (3.6) imply

$$f(b+p, m-p) \leqq f(b, m)-f(b, p) < \frac{1}{2} f(b, m). \tag{3.7}$$

We distinguish three cases according as $p=1$, $2\leqq p\leqq m-1$, and $p=m$.
Case (i): $2\leqq p\leqq m-1$. Set

$$p_1 = \left[\frac{p-1}{2}\right] \quad \text{and} \quad q_1 = \begin{cases} p_1 & \text{if } p-1 \text{ is even,} \\ p_1+1 & \text{if } p-1 \text{ is odd;} \end{cases}$$

$$p_2 = \left[\frac{m-p}{2}\right] \quad \text{and} \quad q_2 = \begin{cases} p_2 & \text{if } m-p \text{ is even,} \\ p_2+1 & \text{if } m-p \text{ is odd.} \end{cases}$$

It is clear that $p_1+q_1=p-1$ and $p_2+q_2=m-p$.

We are going to establish appropriate upper bounds for $|S(b, k)|$ under various values of k between 1 and m. It is easy to check that

$$|S(b, k)| \leqq \begin{cases} M(b, p_1) & \text{for } 1 \leqq k \leqq p_1, \\ |S(b, q_1)|+M(b+q_1, p_1) & \text{for } q_1 \leqq k \leqq p-1, \\ |S(b, p)|+M(b+p, p_2) & \text{for } p \leqq k \leqq p+p_2, \\ |S(b, p+q_2)|+M(b+p+q_2, p_2) & \text{for } p+q_2 \leqq k \leqq m. \end{cases} \tag{3.8}$$

Hence we can derive a suitable upper estimate for $|S(b, k)|$ when k runs from 1 till m, which is independent of the value of k. Consequently, it will be an upper estimate for $M(b, m)$, as well:

$$\begin{aligned} M(b, m) \leqq\; & |S(b, q_1)|+|S(b+q_1, p-q_1)|+|S(b+p, q_2)|+ \\ & +\{M^\gamma(b, p_1)+M^\gamma(b+q_1, p_1)+M^\gamma(b+p, p_2)+M^\gamma(b+p+q_2, p_2)\}^{1/\gamma}. \end{aligned} \tag{3.9}$$

Applying Minkowski's inequality, we find that

$$\begin{aligned} \Big\{\int M^\gamma(b, m)\, d\mu\Big\}^{1/\gamma} \leqq\; & \Big\{\int |S(b, q_1)|^\gamma\, d\mu\Big\}^{1/\gamma}+\Big\{\int |S(b+q_1, p-q_1)|^\gamma\, d\mu\Big\}^{1/\gamma}+ \\ & +\Big\{\int |S(b+p, q_2)|^\gamma\, d\mu\Big\}^{1/\gamma}+\Big\{\int M^\gamma(b, p_1)\, d\mu+\int M^\gamma(b+q_1, p_1)\, d\mu+ \\ & +\int M^\gamma(b+p, p_2)\, d\mu+\int M^\gamma(b+p+q_2, p_2)\, d\mu\Big\}^{1/\gamma} = A+B, \end{aligned} \tag{3.10}$$

where A denotes the sum of the first three terms and B denotes the fourth term on the right-hand side of (3.10).

Due to (1.2) and the facts that

$$q_1, q_2 \leqq \left[\frac{m-2}{2}\right]+1 = \left[\frac{m}{2}\right] \quad \text{and} \quad p-q_1 = p_1+1 \leqq \left[\frac{m-2}{2}\right]+1 = \left[\frac{m}{2}\right],$$

we have that

$$A \leqq f^{1/\gamma}(b, q_1)\,\varphi\big(f(b, q_1), q_1\big)+f^{1/\gamma}(b+q_1, p-q_1)\,\varphi\big(f(b+q_1, p-q_1), p-q_1\big)+$$
$$+f^{1/\gamma}(b+p, q_2)\,\varphi\big(f(b+p, q_2), q_2\big) \leqq$$
$$\leqq \varphi\left(f(b, m), \left[\frac{m}{2}\right]\right)\{f^{1/\gamma}(b, q_1)+f^{1/\gamma}(b+q_1, p-q_1)+f^{1/\gamma}(b+p, q_2)\}.$$

Using the elementary inequality (3.5) for $m=3$, by (1.1) we obtain

$$A \leqq 3^{(\gamma-1)/\gamma} f^{1/\gamma}(b, m)\,\varphi\left(f(b, m), \left[\frac{m}{2}\right]\right). \tag{3.11}$$

On the other hand, by the induction hypothesis,

$$B^\gamma \leqq 3^{\gamma-1}\big\{f(b, p_1)\,\Phi^\gamma\big(f(b, p_1), p_1\big)+ \tag{3.12}$$
$$+f(b+q_1, p_1)\,\Phi^\gamma\big(f(b+q_1, p_1), p_1\big)+f(b+p, p_2)\,\Phi^\gamma\big(f(b+p, p_2), p_2\big)+$$
$$+f(b+p+q_2, p_2)\,\Phi^\gamma\big(f(b+p+q_2, p_2), p_2\big)\big\} = 3^{\gamma-1}(B_1+B_2+B_3+B_4).$$

First consider B_1. Taking (3.6) into account, and that $p_1 \leqq p-1$ and $p_1 \leqq [m/2]$, it follows that

$$B_1 \leqq f(b, p_1)\,\Phi^\gamma\big(f(b, p-1), p_1\big) \leqq f(b, p_1)\,\Phi^\gamma\left(\frac{f(b, m)}{2}, \left[\frac{m}{2}\right]\right).$$

Similarly, by (3.6) and (3.7) we have in turn

$$B_2 \leqq f(b+q_1, p_1)\,\Phi^\gamma\left(\frac{f(b, m)}{2}, \left[\frac{m}{2}\right]\right),$$

$$B_3 \leqq f(b+p, p_2)\,\Phi^\gamma\left(\frac{f(b, m)}{2}, \left[\frac{m}{2}\right]\right),$$

$$B_4 \leqq f(b+p+q_2, p_2)\,\Phi^\gamma\left(\frac{f(b, m)}{2}, \left[\frac{m}{2}\right]\right).$$

To sum up, (3.12) and the estimates for B_i just obtained yield

$$B^\gamma \leqq 3^{\gamma-1}\Phi^\gamma\left(\frac{f(b, m)}{2}, \left[\frac{m}{2}\right]\right)\{f(b, p_1)+f(b+q_1, p_1)+f(b+p, p_2)+ \tag{3.13}$$
$$+f(b+p+q_2, p_2)\} \leqq 3^{\gamma-1} f(b, m)\,\Phi^\gamma\left(\frac{f(b, m)}{2}, \left[\frac{m}{2}\right]\right),$$

the last inequality following by (1.1).

Finally, putting (3.10), (3.11), and (3.13) together, we arrive at the inequality

$$\left\{\int M^\gamma(b, m)\,d\mu\right\}^{1/\gamma} \leqq 3^{(\gamma-1)/\gamma} f^{1/\gamma}(b, m)\left\{\varphi\left(f(b, m), \left[\frac{m}{2}\right]\right)+\Phi\left(\frac{f(b, m)}{2}, \left[\frac{m}{2}\right]\right)\right\},$$

which is equivalent to (3.3) owing to (3.2).

Case (ii): $p=1$. Now $f(b,1)>\frac{1}{2}f(b,m)$ and thus $f(b+1,m-1)<\frac{1}{2}f(b,m)$. Setting

$$p_2=\left[\frac{m-1}{2}\right] \quad \text{and} \quad q_2=\begin{cases} p_2 & \text{if} \quad m-1 \quad \text{is even,} \\ p_2+1 & \text{if} \quad m-1 \quad \text{is odd;} \end{cases}$$

we have, $q_2=[m/2]$. Now instead of (3.9) we can estimate in a simpler way:

$$(3.14) \quad M(b,m) \leqq |S(b,1)|+|S(b+1,q_2)|+\{M^\gamma(b+1,p_2)+M^\gamma(b+q_2+1,p_2)\}^{1/\gamma}.$$

The further reasonings are very similar, but somewhat shorter, to those in Case (i). We do not enter into details.

Case (iii): $p=m$. Now $f(b,m-1)\leqq\frac{1}{2}f(b,m)$ and

$$(3.15) \quad M(b,m) \leqq |S(b,q_2)|+|S(b+m-1,1)|+\{M^\gamma(b,p_2)+M^\gamma(b+q_2,p_2)\}^{1/\gamma},$$

where p_2 and q_2 are the same as in Case (ii).

Thus inequality (1.3) has been completely proved.

Proof of (1.4). Setting

$$\tilde{\Phi}(t,m)=\sum_{k=0}^{[\log m]}\varphi\left(\frac{t}{2^k},\left[\frac{m}{2^k}\right]\right) \quad (t\geqq 0,\ m\geqq 1),$$

we have, instead of (3.1) and (3.2), the following recurrence relations:

$$\tilde{\Phi}(t,1)=\varphi(t,1) \quad (t\geqq 0) \quad \text{and} \quad \tilde{\Phi}(t,m)=\varphi(t,m)+\tilde{\Phi}\left(\frac{t}{2},\left[\frac{m}{2}\right]\right) \quad (t\geqq 0,\ m\geqq 2).$$

Statement (1.4) turns into

$$(3.16) \quad \int M^\gamma(b,m)\,d\mu \leqq \frac{5}{2}f(b,m)\,\Phi^\gamma(f(b,m),m).$$

This is obvious for $m=1$ even without the factor 5/2 on the right-hand side since $M(b,1)=S(b,1)$ for each b. In order to prove it for $m=2$ and for arbitrary b, we again use the trivial estimate

$$M(b,2)\leqq|\xi_{b+1}|+|\xi_{b+2}|,$$

whence Minkowski's inequality and (1.2) provide that

$$(3.17) \quad \left\{\int M^\gamma(b,2)\,d\mu\right\}^{1/\gamma}\leqq f^{1/\gamma}(b,1)\,\varphi(f(b,1),1)+f^{1/\gamma}(b+1,1)\,\varphi(f(b+1,1),1).$$

Making use of (1.1), we can conclude that either

$$f(b,1)\leqq\frac{1}{2}f(b,2) \quad \text{or} \quad f(b+1,1)\leqq\frac{1}{2}f(b,2).$$

Taking this and the monotonicity of $\varphi(t, m)$ into account, from (3.17) it follows that

$$\left\{\int M^\gamma(b, 2)\, d\mu\right\}^{1/\gamma} \leqq f^{1/\gamma}(b, 2)\left\{\varphi\big(f(b, 2), 1\big)+\varphi\left(\frac{f(b, 2)}{2}, 1\right)\right\} =$$
$$= f^{1/\gamma}(b, 2)\, \tilde{\Phi}\big(f(b, 2), 1\big),$$

which is a sharpened form of (3.16) for $m=2$.

The induction step is quite similar to that in the proof of (1.3), with the exception that this time one can start, instead of (3.9), from the following inequality, too:

$$M(b, m) \leqq \{|S(b, q_1)|^\gamma+|S(b, p)|^\gamma+|S(b, p+q_2)|^\gamma\}^{1/\gamma}+ \tag{3.18}$$
$$+\{M^\gamma(b, p_1)+M^\gamma(b+q_1, p_1)+M^\gamma(b+p, p_2)+M^\gamma(b+p+q_2, p_2)\}^{1/\gamma}$$

(and analogous inequalities also instead of (3.14) and (3.15)). If one begins the calculations with (3.18), then one can avoid using inequality (3.5), as a result of which one gets the smaller factor 5/2. Indeed, now

$$\left\{\int M^\gamma(b, m)\, d\mu\right\}^{1/\gamma} \leqq \tilde{A}+B,$$

where

$$\tilde{A} = \left\{\int |S(b, q_1)|^\gamma\, d\mu+\int |S(b, p)|^\gamma\, d\mu+\int |S(b, p+q_2)|^\gamma\, d\mu\right\}^{1/\gamma}$$

and B is the same as in (3.10). Due to (1.2), the monotonicity of $\varphi(t, m)$ and (3.6), one can easily deduce:

$$\tilde{A} \leqq \{f(b, q_1)\, \varphi^\gamma\big(f(b, q_1), q_1\big)+f(b, p)\, \varphi^\gamma\big(f(b, p), p\big)+$$
$$+f(b, p+q_2)\, \varphi\big(f(b, p+q_2), p+q_2\big)\}^{1/\gamma} \leqq \varphi\big(f(b, m), m\big)\{f(b, q_1)+f(b, p)+$$
$$+f(b, p+q_2)\}^{1/\gamma} \leqq \left(\frac{5}{2}\right)^{1/\gamma} f^{1/\gamma}(b, m)\, \varphi\big(f(b, m), m\big).$$

The further reasoning runs along the same line as in the proof of (1.3).

Thus our Theorem has been completely proved.

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