On the a.e. convergence of multiple orthogonal series. I (Square and spherical partial sums)

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1. Notations. Let Z^d be the set of d-tuples $k=(k_1, ..., k_d)$ with nonnegative integral coordinates. Let $\varphi=\{\varphi_k(x): k\in Z^d\}$ be an orthonormal system (in abbreviation: ONS) on the unit cube $x=(x_1, ..., x_d)\in I^d$, where $I=[0, 1]$. Consider the d-multiple orthogonal series

(1) $$\sum_{k\in Z^d} a_k\varphi_k(x) = \sum_{k_1=0}^{\infty}\cdots\sum_{k_d=0}^{\infty} a_{k_1,...,k_d}\varphi_{k_1,...,k_d}(x_1, ..., x_d),$$

where $a=\{a_k: k\in Z^d\}$ is a system of coefficients, for which

(2) $$\sum_{k\in Z^d} a_k^2 < \infty.$$

Fixing a sequence $Q=\{Q_r: r=0, 1, ...\}$ of finite sets in Z^d with properties

$$Q_0\subset Q_1\subset Q_2\subset ... \quad \text{and} \quad \bigcup_{r=0}^{\infty} Q_r = Z^d,$$

our main goal is to study the convergence behaviour of the sums

(3) $$s_r(x) = \sum_{k\in Q_r} a_k\varphi_k(x) \quad (r = 0, 1, ...),$$

which can be regarded as a certain kind of partial sums of series (1). The case

$$Q_r^1 = \{k\in Z^d: \max_{1\le j\le d} k_j \le r\}$$

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provides the square partial sums $s_r^1(x)$, while

$$Q_r^2 = \left\{k \in Z^d: |k| = \left(\sum_{j=1}^d k_j^2\right)^{1/2} \leqq r\right\}$$

provides the spherical partial sums $s_r^2(x)$ of (1).

2. A.e. convergence of $\{s_r(x): r=0, 1, \ldots\}$. Denote by $M(d, Q)$ the class of those systems $a=\{a_k: k \in Z^d\}$ of coefficients for which the sequence $\{s_r(x)\}$ defined by (3) converges a.e. for every ONS $\varphi=\{\varphi_k(x): k \in Z^d\}$ on I^d. The set of measure zero of the divergence points may vary with each φ.

One can easily see that if $a \in M(d, Q)$, then (2) is necessarily satisfied. This follows from the obvious fact that the d-multiple Rademacher system

$$\{r_k(x)\} = \left\{\prod_{j=1}^d r_{k_j}(x_j): k=(k_1, \ldots, k_d) \in Z^d \text{ and } x=(x_1, \ldots, x_d) \in I^d\right\}$$

consists of stochastically independent functions and thus, for every choice of the sequence $Q=\{Q_r: r=0, 1, \ldots\}$ of finite sets in Z^d, the sequence $\{s_r(x)\}$ defined by (3) for $\varphi=\{r_k(x)\}$ converges a.e. or diverges a.e. according as (2) is satisfied or not.

For a given system $a=\{a_k: k \in Z^d\}$ of coefficients we set

$$\mathscr{I}(a; d, Q, \varrho) = \sup \int_{I^d} \left(\max_{0 \leqq r \leqq \varrho} |s_r(x)|\right)^2 dx,$$

where the supremum is taken over all ONS $\varphi=\{\varphi_k(x): k \in Z^d\}$ on I^d and $dx = dx_1 \ldots dx_d$, further,

$$\|a; d, Q\| = \lim_{\varrho \to \infty} \mathscr{I}^{1/2}(a; d, Q, \varrho) \leqq \infty.$$

This limit exists since $\mathscr{I}(a; d, Q, \varrho)$ is nondecreasing in ϱ.

Theorem 1. (i) $a \in M(d, Q)$ *if and only if* $\|a; d, Q\| < \infty$;
(ii) $M(d, Q)$ *endowed with the norm* $\|\cdot; d, Q\|$ *is a separable Banach space.*

This theorem is essentially a reformulation of an earlier result of Tandori [11].

To this effect, let $\psi=\{\psi_{k_1}(x_1): k=0, 1, \ldots\}$ be a single ONS on I. Consider the ordinary orthogonal series

$$\sum_{k_1=0}^{\infty} c_{k_1} \psi_{k_1}(x_1), \tag{4}$$

where $c=\{c_{k_1}: k_1=0, 1, \ldots\}$ is a sequence of coefficients for which

$$\sum_{k_1=0}^{\infty} c_{k_1}^2 < \infty. \tag{5}$$

Fixing a sequence $\nu=\{\nu_r: r=0, 1, \ldots\}$ of integers with the property $0\leqq\nu_0<\nu_1<<\nu_2<\ldots$, denote by $M(\nu)$ the class of those sequences $c=\{c_{k_1}\}$ for which the ν_rth partial sums of series (4) converge a.e. for every ONS $\psi=\{\psi_{k_1}(x_1)\}$ on I.

For a given sequence $c=\{c_{k_1}\}$ of coefficients we set

$$\mathcal{I}(c;\, \nu, \varrho) = \sup \int_I \left(\max_{0\leqq r\leqq\varrho} \left| \sum_{k_1=0}^{\nu_r} c_{k_1}\psi_{k_1}(x_1) \right| \right)^2 dx_1, \tag{6}$$

where the supremum is taken over all ONS $\psi=\{\psi_{k_1}(x_1)\}$ on I, and

$$\|c;\, \nu\| = \lim_{\varrho\to\infty} \mathcal{I}^{1/2}(c;\, \nu, \varrho) \leqq \infty.$$

It is not hard to see that

$$\mathcal{I}(c;\, \nu, \varrho) = \sup \int_I \left(\max_{0\leqq r\leqq\varrho} \left| \sum_{m_1=0}^{r} C_{m_1}\Psi_{m_1}(x_1) \right| \right)^2 dx_1,$$

where

$$C_m = \left(\sum_{k_1=\nu_{m-1}+1}^{\nu_m} c_{k_1}^2 \right)^{1/2} \quad (m = 0, 1, \ldots;\ \nu_{-1} = -1)$$

and the supremum is taken over all ONS $\{\Psi_{m_1}(x_1)\}$ on I.

After these preliminaries the above-mentioned theorem of Tandori reads as follows.

Theorem A [11, Satz II]. (i) *$c\in M(\nu)$ if and only if $\|c;\nu\|<\infty$;*
(ii) *$M(\nu)$ endowed with the norm $\|\cdot\,;\nu\|$ is a separable Banach space.*

Now, it is a trivial observation that Theorem A remains valid if instead of the single ONS $\psi=\{\psi_{k_1}(x_1): k_1=0, 1, \ldots\}$ on I we consider the d-multiple ONS $\varphi==\{\varphi_k(x): k\in Z^d\}$ on I^d and take the integrals over I^d instead of I in (6). In fact, the sufficiency part in (i) is true over any measure space X (instead of $X=I$ or I^d), while the necessity part in (i) can be shown by the following simple observation: let $\nu_r=|Q_r|$, the number of the lattice points of Z^d contained in the set Q_r, and let $\varphi_k(x_1, \ldots, x_d)=\psi_{m_1}(x_1)$, where the mapping $k=k(m_1)$ is one-to-one for each pair $\nu_{r-1}<m_1\leqq\nu_r$ and $k\in Q_r\setminus Q_{r-1}$ ($r=0, 1, \ldots$; $\nu_{-1}=-1$ and $Q_{-1}=\emptyset$). Consequently, Theorem 1 is really a reformulation of Theorem A.

In the light of what has been said above, the result of [11, Satz III] can be reformulated as follows.

Theorem 2. *If two systems $a=\{a_k: k\in Z^d\}$ and $b=\{b_k: k\in Z^d\}$ of coefficients are such that*

$$B_r = \Big\{ \sum_{k\in Q_r\setminus Q_{r-1}} b_k^2 \Big\}^{1/2} \leqq \Big\{ \sum_{k\in Q_r\setminus Q_{r-1}} a_k^2 \Big\}^{1/2} = A_r \quad (r = 0, 1, \ldots),$$

then

$$\|b;\, d, Q\| \leqq \|a;\, d, Q\|;$$

consequently, if $a \in M(d, Q)$ *then* $b \in M(d, Q)$.

It is of interest to give an upper estimate for the norm $\|\cdot\,; d, Q\|$ which turns out to be exact in certain cases.

Theorem 3. *In each case we have*

$$\|a;\, d, Q\| \leqq C_1 \left\{ \sum_{r=0}^{\infty} \Big(\sum_{k \in Q_r \setminus Q_{r-1}} a_k^2 \Big) \log^2 (r+2) \right\}^{1/2}, \tag{7}$$

and in the special case when

$$A_r = \Big\{ \sum_{k \in Q_r \setminus Q_{r-1}} a_k^2 \Big\}^{1/2} \geqq \Big\{ \sum_{k \in Q_{r+1} \setminus Q_r} a_k^2 \Big\}^{1/2} = A_{r+1} \quad (r = 0, 1, \ldots)$$

an inequality opposite to (7) *holds also true:*

$$\|a;\, d, Q\| \geqq C_2 \left\{ \sum_{r=0}^{\infty} \Big(\sum_{k \in Q_r \setminus Q_{r-1}} a_k^2 \Big) \log^2 (r+2) \right\}^{1/2}.$$

Here C_1 *and* C_2 *are positive constants depending only on* d.

To prove Theorem 3 one has to start with the results of [7, Theorems 1 and 2] and to argue in a similar manner as it is done during the proof of [11, Satz VII].

We note that in the cases of the square and the spherical partial sums the right-hand sides in inequality (7) coincide, up to a constant:

$$\|a;\, d, Q^i\| \leqq C_1 \Big\{ \sum_{k \in Z^d} a_k^2 \log^2 (|k|+2) \Big\}^{1/2} \quad (i = 1, 2).$$

In spite of this fact, the norms $\|a; d, Q^1\|$ and $\|a; d, Q^2\|$ are not equivalent to each other in case $d \geqq 2$.

Theorem 4. *If* $d \geqq 2$, *then there exists a system* $a = \{a_k\colon k \in Z^d\}$ *of coefficients for which*

$$\|a;\, d, Q^1\| < \infty \quad \text{and} \quad \|a;\, d, Q^2\| = \infty,$$

and vice versa, there exists a system $a = \{a_k\colon k \in Z^d\}$ *of coefficients for which*

$$\|a;\, d, Q^1\| = \infty \quad \text{and} \quad \|a;\, d, Q^2\| < \infty.$$

This is an easy consequence of Theorem 1 and [7, Theorem 3].

We note that the result stated in [7, Theorem 3] can be strengthened in the following way:

Let T be a regular method of summation (see. e.g., [14, p. 74]). *Then there exists a double orthogonal series* (1) *such that* (2) *is satisfied, its square partial sums converge* *a.e., but its spherical partial sums are not summable by the method T a.e. on I^2; and vice versa.*

In the proof of the latter assertion one has to use a result of [4, p. 183]:

For every regular method T of summation there exists a strictly increasing sequence $\{\mu_r: r=0, 1, \dots\}$ of positive integers such that the a.e. T-summability of series (4) *under condition* (5) *involves the a.e. convergence of the μ_rth partial sums of* (4).

Keeping in mind the proof of [7, Theorem 3] one's task is essentially reduced to the construction of a single orthogonal series (4) with condition (5), the μ_rth partial sums of which diverge a.e., while the μ_{2r}th partial sums of which converge a.e. on I. This construction can be certainly done if the ratio μ_{r+1}/μ_r is large enough $(r=0, 1, \dots)$, and the last condition may be assumed without loss of generality.

3. A.e. $(C, \delta>0)$-summability of the spherical partial sums. Up to this point we studied the convergence properties of series (1) in the setting when $a=\{a_k: k\in Z^d\}$ is a fixed system of coefficients, while $\varphi=\{\varphi_k(x): k\in Z^d\}$ runs over all the ONS on I^d. From now on we consider an individual ONS $\varphi=\{\varphi_k\}$ on I^d with some nice properties and let $a=\{a_k\}$ run over all the systems of coefficients satisfying condition (2).

To this aim, we assume that $\varphi=\{\varphi_k(x): k\in Z^d\}$ is a product ONS on I^d in the sense that there exists a single ONS $\psi=\{\psi_{k_1}(x_1): k_1=0, 1, \dots\}$ on I such that

$$\varphi_k(x) = \prod_{j=1}^{d} \psi_{k_j}(x_j), \quad k=(k_1, \dots, k_d) \text{ and } x=(x_1, \dots, x_d); \tag{8}$$

furthermore, we assume that the system $\psi=\{\psi_{k_1}(x_1)\}$ is such that for every sequence $c=\{c_{k_1}: k_1=0, 1, \dots\}$ of coefficients we have

$$\int_I \left(\max_{0\le r\le \varrho} \left|\sum_{k_1=0}^{r} c_{k_1}\psi_{k_1}(x_1)\right|\right)^2 dx_1 \leqq C \sum_{k_1=0}^{\varrho} c_{k_1}^2 \quad (\varrho=0, 1, \dots), \tag{9}$$

where C is a positive constant. Inequality (9) implies, among others, that series (4) converges a.e. under condition (5). The fact that inequality (9) is satisfied for the ordinary trigonometric system $\psi=\{1, \cos 2\pi k_1x_1, \sin 2\pi k_1 x_1: k_1=1, 2, \dots\}$ is due to Hunt [3], while for the Walsh system $\psi=\{w_{k_1}(x_1): k_1=0, 1, \dots\}$ is due to Sjölin [8].

It is not hard to conclude from (9) the following upper estimate for the maximum of the square partial sums $s_r^1(x)$ of series (1):

$$\int_{I^d} (\max_{0\le r\le\varrho} |s_r^1(x)|)^2\, dx \leqq 2^d C^d \sum_{k\in Q_\varrho} a_k^2 \quad (\varrho=0, 1, \dots).$$

This means that the square partial sums $s_r^1(x)$ converge a.e. on I^d provided (2) is satisfied. (For more details, see [12] and [6].)

The question of a.e. convergence of the spherical partial sums $s_r^2(x)$ of series (1) under condition (2) seems to us to be an open problem for $d \geqq 2$. As to the multiple trigonometric system, we cite here two papers by Russian mathematicians. On the one hand, TEVZADZE [13] published in 1973 that he managed to prove that the spherical partial sums of the double Fourier expansion of a function $f(x_1, x_2)$ from $L^p(I^2)$ with $p>1$ converge a.e. on I^2, but the proof turned out to be false even in case $p=2$. On the other hand, BUADZE [2] announced in 1976 the existence of a continuous function $f(x_1, x_2)$ on I^2 such that the spherical partial sums of the double Fourier expansion of $f(x_1, x_2)$ diverge everywhere, but the construction has not yet appeared.

We are unable to decide this question. However, we can prove the a.e. $(C, \delta>0)$-summability of the spherical partial sums $s_r^2(x)$ of series (1) under the only conditions that $\varphi=\{\varphi_k(x)\}$ is an ONS with properties (8) and (9), and $a=\{a_k\}$ is a system of coefficients satisfying (2). To this end, we recall that the (C, δ)-means $\sigma_\varrho^\delta(x)$ in question are defined as follows:

$$\sigma_\varrho^\delta(x) = \frac{1}{A_\varrho^\delta} \sum_{r=0}^{\varrho} A_{\varrho-r}^{\delta-1} s_r^2(x) =$$

$$= \frac{1}{A_\varrho^\delta} \sum_{r=0}^{\varrho} A_{\varrho-r}^{\delta} \Big(\sum_{r-1<|k|\leqq r} a_k \varphi_k(x) \Big),$$

where

$$A_\varrho^\delta = \binom{\varrho+\delta}{\varrho} \quad (\varrho = 0, 1, \ldots; \ \delta > 0).$$

For a positive integer δ one can consider the following modified (C, δ)-means, too:

$$\tilde{\sigma}_\varrho^\delta(x) = \frac{1}{A_\varrho^\delta} \sum_{|k|\leqq\varrho} A_{\varrho-|k|}^\delta a_k \varphi_k(x),$$

in particular, for $\delta=1$,

$$\tilde{\sigma}_\varrho^1(x) = \sum_{|k|\leqq\varrho} \Big(1 - \frac{|k|}{\varrho+1}\Big) a_k \varphi_k(x).$$

Unfortunately, we can prove the statement that

$$\sigma_\varrho^\delta(x) - \tilde{\sigma}_\varrho^\delta(x) \to 0 \quad \text{as } \varrho \to \infty \quad \text{a.e. on } I$$

only in case $\delta=1$. In fact, writing

$$\sigma_\varrho^1(x) - \tilde{\sigma}_\varrho^1(x) = \frac{1}{\varrho+1} \sum_{r=0}^{\varrho} \Big(\sum_{r-1<|k|\leqq r} (r-|k|) a_k \varphi_k(x) \Big),$$

by virtue of the Kronecker lemma (see, e.g. [1, p. 72]) it is enough to show that the single orthogonal series

$$\sum_{r=0}^{\infty} \frac{1}{r+1} \Big(\sum_{r-1<|k|\leqq r} (r-|k|) a_k \varphi_k(x) \Big)$$

converges a.e. on I^d. But by the well-known Rademacher—Menšov theorem this is the case provided (2) is satisfied.

After these preliminaries we state the following

Theorem 5. *Assume that* $\varphi=\{\varphi_k(x)\}$ *is a product* ONS *on* I^d *given by* (8) *and satisfying condition* (9), $a=\{a_k\}$ *is a system of coefficients satisfying* (2), *and* δ *is a positive number. Then the spherical partial sums* $s_r^2(x)$ *of series* (1) *are* (C, δ)*-summable a.e. on* I^d.

Taking into account of what has been said above on the trigonometric and Walsh systems, hence it follows immediately the following

Corollary. *If* $\varphi=\{\varphi_k(x)\}$ *is the d-multiple trigonometric or Walsh system, then the spherical partial sums* $s_r^2(x)$ *of series* (1) *are* $(C, \delta>0)$*-summable a.e. on* I^d *provided* (2) *is satisfied.*

Remarks. (a) In the case when φ is the d-multiple trigonometric system, Stein [9] proved that the Bochner—Riesz means $\tilde{\sigma}_\varrho^\delta(x)$ of series (1) defined by

$$\tilde{\sigma}_\varrho^\delta(x) = \sum_{|k|<\varrho} \left(1-\frac{|k|^2}{\varrho^2}\right)^\delta a_k \varphi_k(x) \quad (\varrho, \delta > 0)$$

converge to $f(x)$ a.e. on I^d provided series (1) is the d-multiple Fourier expansion of a function $f(x) \in L^p(I^d)$, where

$$\delta > \frac{d-1}{2}\left(\frac{2}{p}-1\right) \quad \text{and} \quad 1 < p \leqq 2.$$

In particular, under condition (2) the means $\tilde{\sigma}_\varrho^\delta(x)$ converge a.e. on I^d again for every $\delta>0$.

(b) As to the multiple Haar system, Kemhadze [5] proved that the spherical partial sums of the expansion of a function $f(x)$ with respect to the d-multiple Haar system converge a.e. on I^d provided $f(x) \in L(\log^+ L)^{d-1}(I^d)$.

Proof of Theorem 5. Our starting point is that under the conditions of the theorem the square partial sums $s_r^1(x)$ of series (1) converge a.e. on I^d. We assume that $d \geqq 2$, since in case $d=1$ we have $s_r^2(x) \equiv s_r^1(x)$ $(r=0, 1, \ldots)$.

We will show that the subsequence $\{s_{d^m}^2(x) : m=0, 1, \ldots\}$ of the spherical partial sums of (1) also converges a.e. on I^d. This is an immediate consequence of Beppo Levi's theorem since

$$\sum_{m=0}^{\infty} \int_{I^d} (s_{d^m}^1(x) - s_{d^m}^2(x))^2 \, dx = \sum_{m=0}^{\infty} \Big(\sum_{k \in Q_{d^m}^1 \setminus Q_{d^m}^2} a_k^2 \Big) \leqq \sum_{k \in Z^d} a_k^2 < \infty.$$

Here we took into account that $\{Q_{d^m}^1 \setminus Q_{d^m}^2 : m=0, 1, \ldots\}$ is a disjoint sequence of sets. In fact, if $k\in Q^1_{d^m}\setminus Q^2_{d^m}$ for a certain $m\geqq 1$, then $\max_{1\leqq j\leqq d} k_j=d^m$ and hence

$$|k| \leqq d^{1/2} \max_{1\leqq j\leqq d} k_j \leqq d^{m+1/2},$$

i.e., $k\notin Q^1_{d^n}\setminus Q^2_{d^n}$ for $n\geqq m+1$. On the other hand,

$$\max_{1\leqq j\leqq d} k_j \geqq d^{-1/2}|k| > d^{m-1/2},$$

whence $k\notin Q^1_{d^n}\setminus Q^2_{d^n}$ follows for $n\leqq m-1$. We note that we should have taken the "thicker" subsequence $\{s^2_{[d^{m/2}]}(x)\}$ too, where $[\cdot]$ means the integral part.

In order to make the proof complete, we apply a result of TANDORI [10] in a somewhat more general setting as stated originally and add some supplements. To this effect, let $\nu=\{\nu_r: r=0, 1, \ldots\}$ be, as earlier, a sequence of integers, $0\leqq\nu_0<<\nu_1<\nu_2<\ldots$, and consider the ν_rth partial sums

$$\tilde{s}_{\nu_r}(x_1) = \sum_{k_1=1}^{\nu_r} c_{k_1}\psi_{k_1}(x_1)$$

of the orthogonal series (4) under condition (5). Now we form the $(C, \delta>0)$-means $\sigma^\delta_\varrho(\nu; x_1)$ of the subsequence $\{\tilde{s}_{\nu_r}(x_1)\}$:

$$\sigma^\delta_\varrho(\nu; x_1) = \frac{1}{A^\delta_\varrho}\sum_{r=0}^{\varrho} A^{\delta-1}_{\varrho-r}\tilde{s}_{\nu_r}(x_1) = \tag{10}$$

$$= \frac{1}{A^\delta_\varrho}\sum_{r=0}^{\varrho} A^\delta_{\varrho-r}\Bigg(\sum_{k_1=\nu_{r-1}+1}^{\nu_r} c_{k_1}\psi_{k_1}(x_1)\Bigg)$$

$$(\varrho = 0, 1, \ldots;\ \nu_{-1} = -1).$$

Then the above-mentioned theorem of Tandori can be stated in a more general form as follows.

Theorem B ([10, Hilfssatz I]). *Let $\nu=\{\nu_r\}$ be a strictly increasing sequence of nonnegative integers, and let $\delta>0$ and $q>1$. Then, under condition* (5), *we have*

(i) $\tilde{s}_{\nu_{[q^m]}}(x_1)-\sigma^1_{[q^m]}(\nu; x_1) \to 0$ *as* $m\to\infty$, *and*

(ii) $\max_{[q^m]<r<[q^{m+1}]}\big(\sigma^1_r(\nu; x_1)-\sigma^1_{[q^m]}(\nu; x_1)\big) \to 0$ *as* $m\to\infty$

a.e. on I.

This theorem is proved in [10] for the special case $q=2$, but the proof can be executed, without essential changes, for general $q>1$, too.

Now, using the reasonings made in [4, pp. 186—187] for the special case $\nu_r\equiv r$, one can supplement (i)—(ii) as follows.

Theorem C. *Let* $\nu=\{\nu_r\}$ *be a strictly increasing sequence of nonnegative integers and let* $\delta>1/2$. *Then, under condition* (5), *we have*

(iii) $$\frac{1}{\varrho+1}\sum_{r=0}^{\varrho}(\sigma_r^{\delta-1}(\nu;x_1)-\sigma_r^{\delta}(\nu;x_1))^2\to 0 \quad \text{as } \varrho\to\infty$$

a.e. on I. Consequently, if

$$\sigma_r^{\delta}(\nu;x_1)\to f(x_1) \quad \text{as } r\to\infty$$

a.e. on I, then

$$\frac{1}{\varrho+1}\sum_{r=0}^{\varrho}(\sigma_r^{\delta-1}(\nu;x_1)-f(x_1))^2\to 0 \quad \text{as } \varrho\to\infty$$

a.e. on I.

Finally, we insert an elementary lemma which can be found e.g. in [4, p. 189]:

(iv) *If* $\delta>-1/2$ *and*

$$\frac{1}{\varrho+1}\sum_{r=0}^{\varrho}(\sigma_r^{\delta}-s)^2\to 0 \quad \text{as } \varrho\to\infty,$$

where the σ_r^{δ} *are the* (C,δ)*-means of a numerical series, then, for every* $\varepsilon>0$, *we have*

$$\sigma_r^{\delta+1/2+\varepsilon}\to s \quad \text{as } r\to\infty.$$

Combining (i)—(iv) in such a manner as it is done in [4, pp. 189—190] for the case $\nu_r\equiv r$, one can conclude the following statement:

Under condition (5), *the a.e. convergence of the subsequence* $\{\tilde{s}_{\nu_{[q^m]}}(x_1)\colon m=0,1,\ldots\}$ *of the partial sums of the orthogonal series* (4) *is equivalent to the a.e. convergence of the means* $\{\sigma_\varrho^{\delta}(\nu;x_1)\colon \varrho=0,1,\ldots\}$ *defined by* (10), *where* $\delta>0$ *and* $q>1$ *are fixed numbers.*

On closing, one more remark: the latter statement clearly holds true if the interval I of orthogonality is replaced by any measure space X, in particular, by $X=I^d$.

This completes the proof of Theorem 5.

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