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Upper estimates for the eigenfunctions of the Schrödinger operator

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For a series of questions concerning spectral theory of non-selfadjoint differential operators we need some estimates for the eigenfunctions.

In the present note we shall generalize the former results of IL'IN and Joó [3], [4], [5].

Let (a, b) be a finite interval and consider the formal differential operator

$$ly = -y'' + qy$$

with the complex potential $q \in L^1(a, b)$. A function u_i having absolutely continuous derivative on every closed subinterval of (a, b) is said to be an eigenfunction of order i of the operator l with the complex eigenvalue λ if there exist functions u_k $(k=1, 2, \ldots, i-1)$ with the same properties such that the equations

$$lu_k(x) = \lambda u_k(x) + u_{k-1}(x) \quad (k = 0, 1, 2, \ldots, i) \tag{1}$$

hold for almost all $x \in (a, b)$, with $u_{-1} \equiv 0$.

We prove the following

Theorem. *Every eigenfunction u_i of order i for the eigenvalue λ of the operator l has absolutely continuous derivatives on the closed interval $[a, b]$. Furthermore, setting for convenience $\lambda = \mu^2$ with $0 \leqq \arg \mu < \pi$, the following estimates hold:*

$$\|u_{k-1}\|_\infty \leqq C_k(1 + |\mu|)(1 + \operatorname{Im} \mu)\|u_k\|_\infty, \tag{2}$$

$$\|u_k\|_\infty \leqq C_k(1 + \operatorname{Im} \mu)^{\frac{1}{p}}\|u_k\|_p \qquad (1 \leqq p \leqq \infty), \tag{3}$$

$$\|u_k'\|_\infty \leqq C_k(1 + |\mu|)\|u_k\|_\infty \tag{4}$$

for $k = 0, 1, \ldots, i$; the constants $C_k = C_k(b-a, \|q\|_1)$ do not depend on λ.

Remark. The estimates (2), (3), (4) strengthen and generalize the corresponding results of IL'IN [3] for the case of the Schrödinger operator with $q \in C^1[a, b]$. Our theorem was formulated in [5] and its proof is based only on the use of mean-

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value formulas, an essentially new idea, which is necessary if the potential q is not smooth. For fixed k, a, b and q the order of the estimates (2), (3), (4) in λ cannot be improved. (This will be established in a forthcoming paper [6]). Indeed, for numerous applications this is the most important aspect.

For the proof of the Theorem we need the following extensions of Titchmarsh classical formulae [2, p. 26].

Lemma. *We have*

(5) $$u_k(x-t)+u_k(x+t) = 2u_k(x)\cos\mu t + \int_{x-t}^{x+t} [q(\xi)u_k(\xi)-u_{k-1}(\xi)]\frac{\sin\mu(t-|x-\xi|)}{\mu}\,d\xi \quad \text{if} \quad \mu\neq 0,$$

$$u_k(x-t)+u_k(x+t) = 2u_k(x) + \int_{x-t}^{x+t} [q(\xi)u_k(\xi)-u_{k-1}(\xi)](t-|x-\xi|)\,d\xi \quad \text{if} \quad \mu = 0;$$

(6) $$u_{k-1}(x)t\sin\mu t = \int_{x-t}^{x+t} u_{k-1}(\xi)\sin\mu(t-|x-\xi|)\,d\xi - \int_{x-t}^{x+t} [q(\xi)\,u_{k-1}(\xi)-u_{k-2}(\xi)] \int_{|x-\xi|}^{t} \frac{\sin\mu(\eta-|x-\xi|)}{\mu}\sin\mu(t-\eta)\,d\eta\,d\xi \quad \text{if} \quad \mu\neq 0,$$

$$u_{k-1}(x)\,t^2 = \int_{x-t}^{x+t} u_{k-1}(\xi)(t-|x-\xi|)\,d\xi - \int_{x-t}^{x+t} [q(\xi)\,u_{k-1}(\xi)-u_{k-2}(\xi)] \int_{|x-\xi|}^{t} (\eta-|x-\xi|)(t-\eta)\,d\eta\,d\xi \quad \text{if} \quad \mu = 0.$$

Proof. (Only for $\mu\neq 0$; the case $\mu=0$ is similar.) We can write by (1)

$$\int_{x-t}^{x+t} [q(\xi)\,u_k(\xi)-u_{k-1}(\xi)]\frac{\sin\mu(t-|x-\xi|)}{\mu}\,d\xi = \int_{x-t}^{x+t} [u''(\xi)+\mu^2 u_k(\xi)]\frac{\sin\mu(t-|x-\xi|)}{\mu}\,d\xi;$$

integrating by parts, we obtain (5).

On the other hand, in view of (5),

$$\int_{x-t}^{x+t} u_{k-1}(\xi)\sin\mu(t-|x-\xi|)\,d\xi = \int_0^t [u_{k-1}(x-\eta)+u_{k-1}(x+\eta)]\sin\mu(t-\eta)\,d\eta = \int_0^t 2u_{k-1}(x)\cos\mu\eta\sin\mu(t-\eta)\,d\eta + \int_0^t\int_{x-\xi}^{x+\xi} [q(\xi)u_{k-1}(\xi)-u_{k-2}(\xi)]\frac{\sin\mu(\eta-|x-\xi|)}{\mu}\,d\xi\sin\mu(t-\eta)\,d\eta;$$

applying the Fubini theorem, a short computation gives (6).

We shall also need the following elementary inequalities:

(7) $$|\sin z|,\ |\cos z| < 2;\ |\sin z| < 2|z| \quad \text{whenever} \quad |\mathrm{Im}\, z| \leqq 1;$$

(8) $$|\sin z| > \frac{1}{3}|z| \quad \text{if} \quad |z| \leqq 2;$$

(9) $$\sup_{1/2<\alpha<1} |\sin \alpha z| > \frac{1}{3} \quad \text{whenever} \quad |\mathrm{Im}\, z| \leqq 1 \quad \text{and} \quad |z| \geqq 2.$$

Proof of the Theorem. It is well known [1] that $u_k, u_k' \in L^\infty(a, b)$ and $u_k'' \in L^1(a, b)$. Next we show the auxiliary estimate

(10) $$\|u_k\|_\infty \leqq 10 \max_{[a+\delta,\, b-\delta]} |u_k| + 4\delta \min\left(2\delta, \frac{2}{|\mu|}\right)\|u_{k-1}\|_\infty \quad \text{for} \quad 0 \leqq \delta \leqq R,$$

where $R = \min\left\{\frac{b-a}{4}, \frac{1}{\mathrm{Im}\,\mu}, \frac{1}{4\|q\|_1}\right\}$. Indeed, for each $x \in \left[a, \frac{a+b}{2}\right]$ and $0 \leqq \delta \leqq R$ we obtain from (5) and (7)

(11) $$|u_k(x)| \leqq |u_k(x+2\delta)| + 4|u_k(x+\delta)| + 2\delta\|q\|_1\|u_k\|_\infty + 2\delta \min\left(2\delta, \frac{2}{|\mu|}\right)\|u_{k-1}\|_\infty.$$

An analogous estimate holds for $x \in \left[\frac{a+b}{2}, b\right]$, and hence

$$\|u_k\|_\infty \leqq 5 \max_{[a+\delta,\, b-\delta]} |u_k| + \frac{1}{2}\|u_k\|_\infty + 2\delta \min\left(2\delta, \frac{2}{|\mu|}\right)\|u_{k-1}\|_\infty.$$

Now we prove (2) by induction on k. The case $k=0$ is trivial (we set $C_0=1$). Suppose (2) holds with $k-1$ in place of k and consider the eigenfunction u_k. Comparing the expressions for the term $\int_{x-t}^{x+t} u_{k-1}(\xi) \sin \mu(t-|x-\xi|)\,d\xi$ in (5) and (6), respectively, and using (7) we obtain

$$|u_{k-1}(x)|\delta|\sin \delta\mu| \leqq (6+2\delta\|q\|_1)|\mu|\|u_k\|_\infty +$$
$$+2\delta^2 \min(2, 2\delta|\mu|)\|u_{k-1}\|_\infty + 2\delta^2 \min\left(\frac{4}{|\mu|}, 4\delta^2|\mu|\right)\|u_{k-2}\|_\infty$$

for all $x \in [a+b, b-\delta]$ and $0 \leqq \delta \leqq R$, thus (taking into account that $2\delta\|q\|_1 \leqq 1$)

$$\max_{[a+\delta,\, b-\delta]} |u_{k-1}|\, \delta|\sin \delta\mu| \leqq 7|\mu|\,\|u_k\|_\infty +$$
$$+2\delta \min(1, \delta|\mu|)\|u_{k-1}\|_\infty + 8\delta^2 \min\left(\frac{1}{|\mu|}, \delta^2|\mu|\right)\|u_{k-2}\|_\infty.$$

Applying (10) for u_{k-1} instead of u_k and expressing hence $\max_{[a+\delta, b-\delta]} |u_{k-1}|$ we get

$$\left\{\|u_{k-1}\|_\infty \cdot \frac{1}{10} - \frac{8}{10}\delta \min\left(\delta, \frac{1}{|\mu|}\right)\|u_{k-2}\|_\infty\right\} \delta\,|\sin \delta\mu| \leqq$$

$$\leqq 7\,|\mu|\,\|u_k\|_\infty + 4\delta^2\|q\|_1 \min(1, \delta|\mu|)\|u_{k-1}\|_\infty + 8\delta^2 \min\left(\frac{1}{|\mu|}, \delta^2|\mu|\right)\|u_{k-2}\|_\infty.$$

Using the induction hypothesis (i.e. $\|u_{k-2}\|_\infty \leqq C_{k-1}(1+|\mu|)(1+\operatorname{Im}\mu)\|u_{k-1}\|_\infty$)

$$\text{(12)} \qquad \frac{\delta^2}{7}\left\{\left|\frac{\sin\delta\mu}{\delta\mu}\right|\left[\frac{1}{10} - \delta\,\frac{C_{k-1}(1+|\mu|)(1+\operatorname{Im}\mu)}{|\mu|}\min(1, \delta|\mu|)\right] - \right.$$

$$\left. - \frac{4}{|\mu|}\|q\|_1 \min(1, \delta|\mu|) - \frac{8}{|\mu|}[\min(1, \delta|\mu|)]^2 \frac{C_{k-1}(1+|\mu|)(1+\operatorname{Im}\mu)}{|\mu|}\right\}\|u_{k-1}\|_\infty \leqq \|u_k\|_\infty.$$

Set $\delta_k = \min\left\{R, \left[960 C_{k-1}(1+\operatorname{Im}\mu)\left(1+\frac{b-a}{4}\right)\right]^{-1}, [480\|q\|_1]^{-1}\right\}$.

To examine (12), we distinguish two cases: a) $\delta_k|\mu| \leqq 2$, b) $\delta_k|\mu| > 2$.

Case a). In view of (8) and the fact $\delta(1+|\mu|) \leqq \delta + 1 \leqq 1 + \frac{b-a}{4}$, an application of (12) to $\delta = \frac{\delta_k}{2}$ yields

$$\frac{\delta^2}{7}\left\{\frac{1}{3}\left[\frac{1}{10} - \frac{1}{40}\right] - \frac{1}{120} - \frac{1}{120}\right\}\|u_{k-1}\|_\infty \leqq$$

$$\leqq \frac{\delta^2}{7}\left\{\frac{1}{3}\left[\frac{1}{10} - \delta^2 C_{k-1}(1+|\mu|)(1+\operatorname{Im}\mu)\right] - 4\delta\|q\|_1 - \right.$$

$$\left. - 8\delta^2 C_{k-1}(1+|\mu|)1 + \operatorname{Im}\mu)\right\}\|u_{k-1}\|_\infty \leqq \|u_k\|_\infty.$$

Thus, from the definition of δ we obtain

$$\text{(13)} \qquad \frac{\|u_{k-1}\|_\infty}{\|u_k\|_\infty} \leqq \frac{28 \cdot 120}{\delta_k^2}.$$

Case b). According to (9) we may choose $\alpha \in \left(\frac{1}{2}, 1\right)$ such that $|\sin \alpha\delta_k\mu| > \frac{1}{3}$. Thus by setting $\delta = \alpha\delta_k$ in (12) we have

$$\frac{\delta^2}{7}\left\{\frac{1/30}{\delta|\mu|} - \frac{\delta(1+|\mu|)(1+\operatorname{Im}\mu)}{|\mu|} C_{k-1}\frac{1/3}{\delta|\mu|} - \frac{4}{|\mu|}\|q\|_1 - \right.$$

$$\left. - \frac{8}{|\mu|}\frac{C_{k-1}(1+|\mu|)(1+\operatorname{Im}\mu)}{|\mu|}\right\}\|u_{k-1}\|_\infty \leqq \|u_k\|_\infty.$$

Observe that

$$\frac{1+|\mu|}{|\mu|}=1+\frac{1}{|\mu|}\leqq 1+\frac{\delta_k}{2}\leqq 1+\frac{b-a}{4}.$$

Therefore

$$\frac{\delta}{|\mu|}\frac{1/7}{120}\|u_{k-1}\|_\infty \leqq \frac{\delta}{7|\mu|}\left\{\frac{1}{30}-\frac{\delta}{3}(1+\operatorname{Im}\mu)\left(1+\frac{b-a}{4}\right)C_{k-1}-\right.$$

$$\left.-4\delta\|q\|_1-8\delta C_{k-1}\left(1+\frac{b-a}{4}\right)(1+\operatorname{Im}\mu)\right\}\|u_{k-1}\|_\infty\leqq\|u_k\|_\infty,$$

i.e.

$$\frac{\|u_{k-1}\|_\infty}{\|u_k\|_\infty}\leqq\frac{7\cdot 120|\mu|}{\delta}\leqq\frac{14\cdot 120|\mu|}{\delta_k}. \tag{14}$$

Summing up (13) and (14), and taking into account the definition of δ_k, estimate (2) follows with

$$C_k=28\cdot 120\left\{\left(\frac{4}{b-a}\right)^2+1+\left[960\left(1+\frac{b-a}{4}\right)C_{k-1}\right]^{-1}+[480\|q\|_1]^{-1}\right\}.$$

We prove (3) from (2). Integrating (11) by δ from 0 to δ_{k+1} we have for $x\in\left[a,\frac{a+b}{2}\right]$

$$\delta_{k+1}|u_k(x)|\leqq\int_0^{\delta_{k+1}}|u_k(x+2\delta)|\,d\delta+4\int_0^{\delta_{k+1}}|u_k(x+\delta)|\,d\delta+$$

$$+\delta_{k+1}^2\|q\|_1\|u_k\|_\infty+\min\left(\frac{4}{3}\delta_{k+1}^3,\frac{2\delta_{k+1}^2}{|\mu|}\right)\|u_{k-1}\|_\infty.$$

Applying Hölder's inequality and (2) it follows

$$\delta_{k+1}|u_k(x)|\leqq 5\delta_{k+1}^{1-1/p}\|u_k\|_p+\delta_{k+1}^2\|q\|_1\|u_k\|_\infty+$$

$$+\min\left(\frac{4}{3}\delta_{k+1}^3,\frac{2\delta_{k+1}^2}{|\mu|}\right)C_k(1+|\mu|)\,(1+\operatorname{Im}\mu)\|u_k\|_\infty,$$

whence (by considering the cases $|\mu|\leqq 1$ and $|\mu|>1$ separately)

$$|u_k(x)|\leqq 5\delta_{k+1}^{-1/p}\|u_k\|_p+\delta_{k+1}\|q\|_1\|u_k\|_\infty+4\delta_{k+1}C_k(1+\operatorname{Im}\mu)\|u_k\|_\infty.$$

An analogous inequality holds for $x\in\left[\frac{a+b}{2},b\right]$, and therefore

$$\|u_k\|_\infty\leqq 5\delta_{k+1}^{-1/p}\|u_k\|_p+\delta_{k+1}\|q\|_1\|u_k\|_\infty+4\delta_{k+1}C_k(1+\operatorname{Im}\mu)\|u_k\|_\infty,$$

i.e.

$$\|u_k\|_\infty\leqq 10\delta_{k+1}^{-1/p}\|u_k\|_p\leqq C_k(1+\operatorname{Im}\mu)^{1/p}\|u_k\|_p.$$

We turn to the proof of (4). In case of $x, x+t\in(a, b)$ we have

$$u_k(x+t) = u_k(x)\cos\mu t+u_k'(x)\frac{\sin\mu t}{\mu}+ \tag{15}$$

$$+\int_x^{x+t}[q(\xi)\,u_k(\xi)-u_{k-1}(\xi)]\,\frac{\sin\mu(x+t-\xi)}{\mu}\,d\xi \quad \text{if} \quad \mu\neq 0,$$

$$u_k(x+t) = u_k(x)+u_k'(x)\cdot t+$$

$$+\int_x^{x+t}[q(\xi)\,u_k(\xi)-u_{k-1}(\xi)](x+t-\xi)\,d\xi \quad \text{if} \quad \mu=0$$

((15) can be verified in a similar way as (5)). For each $x\in\left[a, \frac{a+b}{2}\right]$ and $t=\delta_{k+1}$ we obtain from (7) and (15)

$$|u_k'(x)|\left|\frac{\sin\mu t}{\mu}\right| \leqq (3+2\delta_{k+1}\|q\|_1)\|u_k\|_\infty+\delta_{k+1}\min\left(2\delta_{k+1}, \frac{2}{|\mu|}\right)\|u_{k-1}\|_\infty,$$

and therefore, applying (2) we get

$$|u_k'(x)|\left|\frac{\sin\mu t}{\mu}\right| \leqq (3+2\delta_{k+1}\|q\|_1)\|u_k\|_\infty+$$

$$+\delta_{k+1}\min\left(2\delta_{k+1}, \frac{2}{|\mu|}\right)C_k(1+|\mu|)(1+\operatorname{Im}\mu)\|u_k\|_\infty.$$

A similar estimate holds for $x\in\left[\frac{a+b}{2}, b\right]$. Hence by considering the cases $|\mu|\leqq 1$ and $|\mu|>1$ separately we conclude

$$\|u_k'\|_\infty\left|\frac{\sin\mu t}{\mu}\right| \leqq (3+2\delta_{k+1}\|q\|_1)\|u_k\|_\infty+4\delta_{k+1}C_k(1+\operatorname{Im}\mu)\|u_k\|_\infty.$$

If $\delta_{k+1}|\mu|\leqq 2$ then we get by (8)

$$\frac{1}{3}\|u_k'\| \leqq \delta_{k+1}^{-1}(3+2\delta_{k+1}\|q\|_1)\|u_k\|_\infty+4C_k(1+\operatorname{Im}\mu)\|u_k\|_\infty \leqq 5C_k\|u_k\|_\infty,$$

and if $\delta_{k+1}|\mu|>2$ then we have by (9) for $t=\alpha\delta_{k+1}$ instead of $t=\delta_{k+1}$ $\left(\alpha\in\left[\frac{1}{2}, 1\right]\right)$

$$\frac{1}{3}\frac{\|u_k'\|_\infty}{|\mu|} \leqq 5\|u_k\|_\infty+4\delta_{k+1}C_k(1+\operatorname{Im}\mu)\|u_k\|_\infty,$$

i.e.

$$\|u_k'\|_\infty \leqq 16(1+|\mu|)\|u_k\|_\infty.$$

The theorem is proved.

An important special case of (3) is

$$\|u_0\| \leqq 12\left\{\frac{1}{b-a}+\|q\|_1\right\}^{1/2}\|u_0\|_2 \qquad (\text{if } \lambda \geqq 0). \tag{16}$$

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