*Acta Sci. Math.*, 44 (1982), 133—135

# A note on hereditary radicals

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All rings in this paper are associative. Fundamental definitions and properties of radicals may be found in [4]. It is known ([3]) that to any radical $S$ there exist a unique maximal hereditary radical $h_S$ and a unique maximal left hereditary radical $lh_S$ contained in $S$. Of course $h_S \subseteq \bar{S} = \{A|$ any ideal of $A$ is in $S\}$ and $lh_S \subseteq \tilde{S} = \{A|$ any left ideal of $A$ is in $S\}$. It is easy to see that $\bar{S}$ and $\tilde{S}$ are radicals and $S$ is hereditary (left hereditary) if and only if $S = \bar{S}$ $(S = \tilde{S})$. The radicals $\bar{S}$ and $\tilde{S}$ were introduced in [2] to investigate hereditariness of strong and similar radicals. Obviously $h_S \subseteq \bar{\bar{S}} \subseteq \bar{S}$ and $lh_S \subseteq \tilde{\tilde{S}} \subseteq \tilde{S}$. In the note we prove that $h_S = \bar{\bar{S}}$ and $lh_S = \tilde{\tilde{S}}$, and that there exists a radical $S$ such that $h_S \neq \bar{S}$ and $lh_S \neq \tilde{S}$.

To denote that $I$ is an ideal (left ideal) of a ring $A$ we will write $I \lhd A$ $(I < A)$.

Lemma 1. *If $A$ is an $S$-radical ring and for some integer $n$, $A^{n+2} = 0$ then $A^{n+1} \in S$.*

Proof. It is easy to see that for any $a \in A^n$ the mapping $f_a : A \to A^{n+1}$ defined by $f_a(x) = ax$ is a ring homomorphism. But $f_a(A) = aA \lhd A^{n+1}$. Thus $aA \in S$ and $A^{n+1} = \sum_{a \in A^n} aA \in S$.

Proposition 1. *If $S$ is a radical such that any zero-$S$-ring is in $\bar{S}$ then $\bar{S} = h_S$.*

Proof. Let $J \lhd I \lhd A$. If $J^*$ is the ideal of $A$ generated by $J$ and $A \in \bar{S}$ then $J^*$ and $(J^*)^3$ are in $S$. Thus by Lemma 1 $(J^*)^2 \in S$. Now the assumption implies $J + (J^*)^2 \in S$. Since, by Andrunakievich lemma, $(J^*)^3 \subseteq J$, similarly, we obtain that $J \cap (J^*)^2 \in S$. This and the fact that $(J + (J^*)^2)/(J^*)^2 \approx J/((J^*)^2 \cap J)$ implies $J \in S$. Thus if $A \in \bar{S}$ then $A \in \bar{\bar{S}}$, so $\bar{\bar{S}} = \bar{S} = h_S$.

Of course for any radical $S$ the radical $\bar{S}$ satisfies the assumption of Proposition 1, so we have

Received July 22, 1980, and in revised form, May 5, 1981.

Corollary 1. *For any radical* $S$, $h_S=\bar{\bar{S}}$.

Remark. It is easy to check ([1]) that for any radical $S$, $h_S=aS=\{A|$ every accessible subring of $A$ is in $S\}$. Thus, by Corollary 1, $aS=\bar{\bar{S}}$ for any radical $S$.

Now we prove

Proposition 2. *For any radical* $S$, $lh_S=\tilde{\tilde{S}}$.

Proof. If $K<L<A$ then $LK<A$ and $LK\lhd K$. Thus if $A\in\tilde{\tilde{S}}$ then $I=LK\in\tilde{S}$. Now if $R<K$ then $R+I<L$ and, since $A\in\tilde{\tilde{S}}$, $R+I\in S$. Also $R\cap I\in S$ as $R\cap I<I$ and $I\in\tilde{S}$. These and the fact that $(R+I)/I\approx R/(R\cap I)$ imply $R\in S$. Hence if $A\in\tilde{\tilde{S}}$ then $L\in\tilde{\tilde{S}}$, so the radical $\tilde{\tilde{S}}$ is left hereditary. This and the fact that $lh_S\subseteq\tilde{\tilde{S}}$ ends the proof.

Example. Let $Q$ be the field of rational numbers, $Z$ the ring of integers and $R$ the ring of all $2\times 2$-matrices of the form $\begin{pmatrix}a, b\\0, 0\end{pmatrix}$ where $a, b\in Q$. Then $I=\left\{\begin{pmatrix}0, b\\0, 0\end{pmatrix}\middle| b\in Q\right\}$ is an ideal of $R$ and $J=\left\{\begin{pmatrix}0, z\\0, 0\end{pmatrix}\middle| z\in Z\right\}$ an ideal of $I$. Let $S$ be the lower radical determined by $\{R, I\}$. Since $R$ and $I$ are divisible rings, all $S$-rings are divisible. Thus $J\notin S$ and $R\notin\bar{S}$. Since $R\in\tilde{S}'$ and $\tilde{S}\subseteq\bar{S}$ therefore $\tilde{S}\neq\tilde{\tilde{S}}$ and $\bar{S}\neq\bar{\bar{S}}$.

The above example shows that generally $lh_S\neq\tilde{S}$. In the following proposition we will describe some radicals for which $lh_S=\tilde{S}$.

Proposition 3. *For a radical $S$ we have $lh_S=\tilde{S}$, provided* a) *$S$ contains all zero-rings, or* b) *$L<A$ and $A\in S$ imply $L=AL$.*

Proof. Let $A\in\tilde{S}$ and $K<L<A$. Since $LK<A$, we have $LK\in S$. But $LK\lhd K$ and $(K/LK)^2=0$, so if $S$ satisfies a) then $K\in S$. If $S$ satisfies b) then $K=LK\in S$ as $K<L$ and $L\in S$. Thus in both cases $K\in S$. In consequence $L\in\tilde{S}$ and $\tilde{S}$ is left hereditary. Hence $lh_S=\tilde{S}$.

Now we will show that there exist non-hereditary radicals satisfying condition b) of the proposition above. Let us define for any class $M$ of rings the class $M_1=\{A\in M|$ if $L<A$ then $AL=L\}$. We have

Proposition 4. *If a class $S$ is radical then so is $S_1$.*

Proof. Certainly the class $S_1$ is homomorphically closed and any ring which is the sum of a chain of $S_1$-ideals is in $S_1$. So it suffices to prove that if $I\lhd A$ and $I, A/I$ are in $S_1$ then $A$ is in $S_1$. Let $L<A$. Then $I(L\cap I)=L\cap I$. Also $(A/I)((L+I)/I)=(L+I)/I$, so $AL+I=L+I$. Thus if $l\in L$ then $l=m+i$ for some

$m\in AL$, $i\in I$. But since $AL\subseteqq L$, $i\in L\cap I$. Thus the equality $I(L\cap I)=L\cap I$ implies $i\in AL$ and $l\in AL$. Hence $L=AL$ and the result follows.

Corollary 2. *Let $S$ be the lower radical determined by a class $M$. If $M=M_1$ then $S=S_1$.*

Proof. Since $M\subseteqq S$ therefore $M=M_1\subseteqq S_1$. Now by Proposition 4, $S_1$ is a radical class containing $M$, so $S\subseteqq S_1$.

Let $M=\{R\}$, where $R$ is the ring of the Example. Then $M=M_1$ and by Corollary 2 the lower radical $S$ determined by $M$ satisfies condition b) of Proposition 3. It is easy to see that any non-zero $S$-ring contains a non-zero idempotent element. Thus $S$ is not hereditary as $R$ contains a non-zero nilpotent ideal.

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