Acta Sci. Math., 44 (1982), 137—149

Approximate decompositions of certain contractions

PEI YUAN WU

In this paper we obtain an approximate decomposition for contractions the outer factors of whose characteristic functions admit scalar multiples. We show that such a contraction is quasi-similar to the direct sum of its $C_{\cdot 1}$ and $C_{\cdot 0}$ parts. This class of operators includes, among other things, weak contractions and $C_{1\cdot}$ contractions with at least one defect index finite. In particular, our result generalizes the C_0-C_{11} decomposition for weak contractions. Applying this to $C_{1\cdot}$ contractions, we obtain that any $C_{1\cdot}$ contraction with at least one defect index finite is completely injection-similar to an isometry. As consequences, we are able to characterize, among $C_{1\cdot}$ contractions, those which are cyclic, have commutative commutants or satisfy the double commutant property.

In Section 1 below we first fix the notation and review some basic facts needed in the subsequent discussions. Then in Section 2 we prove the approximate decomposition and some of its consequences. In Section 3 we restrict ourselves to $C_{1\cdot}$ contractions.

1. Preliminaries. In this paper all the operators are acting on complex, separable Hilbert spaces. We will use extensively the contraction theory of SZ.-NAGY and FOIAŞ. The main reference is their book [8].

Let T be a contraction on the Hilbert space H. Denote by $\mathfrak{D}_T=\overline{\operatorname{ran}(I-T^*T)^{1/2}}$ and $\mathfrak{D}_{T*}=\overline{\operatorname{ran}(I-TT^*)^{1/2}}$ the *defect spaces* and $d_T=\operatorname{rank}(I-T^*T)^{1/2}$ and $d_{T*}=\operatorname{rank}(I-TT^*)^{1/2}$ the *defect indices* of T. T is *completely non-unitary (c.n.u.)* if there exists no non-trivial reducing subspace on which T is unitary. T is *of class* $C_{1\cdot}$ (resp. $C_{\cdot 1}$) if $T^nx\not\to 0$ (resp. $T^{*n}x\not\to 0$) for any $x\neq 0$; T is *of class* $C_{0\cdot}$ (resp. $C_{\cdot 0}$) if $T^nx\to 0$ (resp. $T^{*n}x\to 0$) for any x. $C_{\alpha\beta}=C_{\alpha\cdot}\cap C_{\cdot\beta}$ for $\alpha,\beta=0,1$. Let $T=\begin{bmatrix} T_1 & X \\ 0 & T_2\end{bmatrix}$ be the canonical triangulation of type $\begin{bmatrix} C_{\cdot 1} & * \\ 0 & C_{\cdot 0}\end{bmatrix}$ on $H=H_1\oplus H_2$. If T is c.n.u., then this triangulation corresponds to the canonical factorization $\Theta_T=$

Received February 25, 1981.

This research was partially supported by National Science Council of Taiwan, China.

$=\Theta_2\Theta_1$ of the characteristic function $\{\mathfrak{D}_T, \mathfrak{D}_{T*}, \Theta_T(\lambda)\}$ of T, where $\{\mathfrak{D}_T, \mathfrak{F}, \Theta_1(\lambda)\}$ and $\{\mathfrak{F}, \mathfrak{D}_{T*}, \Theta_2(\lambda)\}$ are the outer and inner factors of Θ_T, respectively. Moreover, the characteristic functions of T_1 and T_2 are the purely contractive parts of Θ_1 and Θ_2, respectively. For c.n.u. T, we will consider its *functional model*, that is, consider T being defined on the space $H=[H^2(\mathfrak{D}_{T*})\oplus\overline{\Delta_T L^2(\mathfrak{D}_T)}]\ominus\{\Theta_T w\oplus\Delta_T w: w\in H^2(\mathfrak{D}_T)\}$ by $T(f\oplus g)=P(e^{it}f\oplus e^{it}g)$, where $\Delta_T=(I-\Theta_T^*\Theta_T)^{1/2}$ and P denotes the (orthogonal) projection onto H. Then H_1 and H_2 can be represented as

$$H_1 = \{\Theta_2 u\oplus v: u\in H^2(\mathfrak{F}), v\in\overline{\Delta_T L^2(\mathfrak{D}_T)}\}\ominus\{\Theta_T w\oplus\Delta_T w: w\in H^2(\mathfrak{D}_T)\}$$

and

$$H_2 = [H^2(\mathfrak{D}_{T*})\ominus\Theta_2 H^2(\mathfrak{F})]\oplus\{0\}.$$

A contractive analytic function $\{\mathfrak{D}, \mathfrak{D}_*, \Theta(\lambda)\}$ is said to admit the *scalar multiple* $\delta(\lambda)$ if $\delta(\lambda)\not\equiv 0$ is a scalar-valued analytic function and there exists a contractive analytic function $\{\mathfrak{D}_*, \mathfrak{D}, \Omega(\lambda)\}$ such that $\Omega(\lambda)\Theta(\lambda)=\delta(\lambda)I_{\mathfrak{D}}$ and $\Theta(\lambda)\Omega(\lambda)=\delta(\lambda)I_{\mathfrak{D}_*}$ for all λ in $D=\{\lambda: |\lambda|<1\}$.

For an arbitrary operator T on H, let $\{T\}'$, $\{T\}''$ and Alg T denote its commutant, double commutant and the weakly closed algebra generated by T and I. Let Lat T, Lat $''T$ and Hyperlat T denote the lattices of invariant subspaces, bi-invariant subspaces and hyperinvariant subspaces of T, respectively. Let μ_T denote the *multiplicity* of T, that is, the least cardinal number of a subset K of H for which $H=\bigvee_{n\geqq 0} T^n K$. T is *cyclic* if $\mu_T=1$. For operators T_1 and T_2 on H_1 and H_2, respectively, $T_1\overset{i}{\prec}T_2$ (resp. $T_1\prec T_2$) denotes that there exists an injection $X: H_1\to H_2$ (resp. an injection $X: H_1\to H_2$ with dense range, called *quasi-affinity*) such that $T_2X=XT_1$. $T_1\overset{ci}{\prec}T_2$ denotes that there exists a family $\{X_\alpha\}$ of injections $X_\alpha: H_1\to H_2$ such that $H_2=\bigvee_\alpha X_\alpha H_1$ and $T_2X_\alpha=X_\alpha T_1$ for each α. T_1 and T_2 are *quasi-similar* ($T_1\sim T_2$) if $T_1\prec T_2$ and $T_2\prec T_1$; they are *injection-similar* ($T_1\overset{i}{\sim}T_2$) if $T_1\overset{i}{\prec}T_2$ and $T_2\overset{i}{\prec}T_1$; they are *completely injection-similar* ($T_1\overset{ci}{\sim}T_2$) if $T_1\overset{ci}{\prec}T_2$ and $T_2\overset{ci}{\prec}T_1$. Note that $T_1\prec T_2$ implies that $\mu_{T_1}\geqq\mu_{T_2}$.

2. Approximate decomposition. We start with the following major result.

Theorem 2.1. *Let T be a contraction on H and let $T=\begin{bmatrix}T_1 & X\\ 0 & T_2\end{bmatrix}$ on $H=H_1\oplus H_2$ be the canonical triangulation of type $\begin{bmatrix}C_{\cdot 1} & *\\ 0 & C_{\cdot 0}\end{bmatrix}$. If the characteristic function of T_1 admits a scalar multiple, then $T\sim T_1\oplus T_2$. Moreover, if T is c.n.u., then there exist quasi-affinities $Y: H\to H_1\oplus H_2$ and $Z: H_1\oplus H_2\to H$ which intertwine T and $T_1\oplus T_2$ and such that $YZ=\delta(T_1\oplus T_2)$ and $ZY=\delta(T)$ for some outer function δ.*

Proof. Let $T=U\oplus T'$ be decomposed as the direct sum of a unitary operator U and a c.n.u. contraction T'. Let $T'=\begin{bmatrix} T_1' & * \\ 0 & T_2' \end{bmatrix}$ be of type $\begin{bmatrix} C_{\cdot 1} & * \\ 0 & C_{\cdot 0} \end{bmatrix}$. Then

$$T=\begin{bmatrix} U & 0 & 0 \\ 0 & T_1' & * \\ 0 & 0 & T_2' \end{bmatrix},$$

where $\begin{bmatrix} U & 0 \\ 0 & T_1' \end{bmatrix}$ is of class $C_{\cdot 1}$ and T_2' is of class $C_{\cdot 0}$. Hence by the uniqueness of the canonical triangulation, we have $T_1=U\oplus T_1'$ and $T_2=T_2'$ (cf. [8], p. 73). Note that the characteristic functions of T_1 and T_1' coincide. Therefore the characteristic function of T_1' also admits a scalar multiple. If we can show that $T'\sim T_1'\oplus T_2'$, then $T==U\oplus T'\sim U\oplus T_1'\oplus T_2'=T_1\oplus T_2$. Hence without loss of generality, we may assume that T is c.n.u. As remarked before, we can consider the functional model of T. Let δ be an outer scalar multiple of Θ_1 (cf. [8], p. 217) and let $\{\mathfrak{F}, \mathfrak{D}_T, \Omega(\lambda)\}$ be a contractive analytic function such that $\Omega\Theta_1=\delta I_{\mathfrak{D}_T}$ and $\Theta_1\Omega=\delta I_{\mathfrak{F}}$. Define the operator S: $H_2\to H_1$ by $S(u\oplus 0)=P(0\oplus(-\Delta_T\Omega\Theta_2^* u))$ for $u\oplus 0\in H_2$. Note that $0\oplus(-\Delta_T\Omega\Theta_2^* u)$ is orthogonal to H_2 and therefore $P(0\oplus(-\Delta_T\Omega\Theta_2^* u))$ is indeed in H_1.

We first check that $T_1S-ST_2=\delta(T_1)X$. Note that for $u\oplus 0\in H_2$, we have

$$T_2(u\oplus 0)=(e^{it}u\oplus 0)-(\Theta_T w\oplus\Delta_T w)-(\Theta_2 u'\oplus v')=$$
$$=(e^{it}u-\Theta_T w-\Theta_2 u')\oplus(-\Delta_T w-v')=(e^{it}u-\Theta_T w-\Theta_2 u')\oplus 0$$

for some $w\in H^2(\mathfrak{D}_T)$ and $\Theta_2 u'\oplus v'\in H_1$, where the last equality follows from the fact that $T_2(u\oplus 0)\in H_2$. Moreover, $X(u\oplus 0)=\Theta_2 u'\oplus v'$. Hence

$$(T_1S-ST_2)(u\oplus 0)=$$
$$=T_1P(0\oplus(-\Delta_T\Omega\Theta_2^* u))-S((e^{it}u-\Theta_T w-\Theta_2 u')\oplus 0)=$$
$$=P(0\oplus(-e^{it}\Delta_T\Omega\Theta_2^* u))-P(0\oplus(-\Delta_T\Omega\Theta_2^*(e^{it}u-\Theta_T w-\Theta_2 u')))=$$
$$=P(0\oplus(-\Delta_T\Omega\Theta_2^*\Theta_T w-\Delta_T\Omega\Theta_2^*\Theta_2 u'))=P(0\oplus(-\Delta_T\delta w-\Delta_T\Omega u')).$$

On the other hand,

$$\delta(T_1)X(u\oplus 0)=\delta(T_1)(\Theta_2 u'\oplus v')=P(\delta\Theta_2 u'\oplus\delta v')=P(\Theta_T\Omega u'\oplus\delta v')=$$
$$=P(0\oplus(\delta v'-\Delta_T\Omega u')).$$

Since $-\Delta_T w-v'=0$, we obtain that $T_1S-ST_2=\delta(T_1)X$ as asserted.

Let $Y=\begin{bmatrix} \delta(T_1) & S \\ 0 & I \end{bmatrix}: H\to H_1\oplus H_2$ and $Z=\begin{bmatrix} I & V-S \\ 0 & \delta(T_2) \end{bmatrix}: H_1\oplus H_2\to H$, where V is the operator which appears in the triangulation of $\delta(T)$ with respect to $H_1\oplus H_2$:

$\delta(T)=\begin{bmatrix}\delta(T_1) & V\\ 0 & \delta(T_2)\end{bmatrix}$. We complete the proof in several steps. In each step the first statement is proved.

(i) $YT=(T_1\oplus T_2)Y$.

$$YT=\begin{bmatrix}\delta(T_1) & S\\ 0 & I\end{bmatrix}\begin{bmatrix}T_1 & X\\ 0 & T_2\end{bmatrix}=\begin{bmatrix}\delta(T_1)T_1 & \delta(T_1)X+ST_2\\ 0 & T_2\end{bmatrix}=$$
$$=\begin{bmatrix}T_1\delta(T_1) & T_1S\\ 0 & T_2\end{bmatrix}=\begin{bmatrix}T_1 & 0\\ 0 & T_2\end{bmatrix}\begin{bmatrix}\delta(T_1) & S\\ 0 & I\end{bmatrix}=(T_1\oplus T_2)Y.$$

(ii) $Z(T_1\oplus T_2)=TZ$. Since

$$\delta(T)T=\begin{bmatrix}\delta(T_1) & V\\ 0 & \delta(T_2)\end{bmatrix}\begin{bmatrix}T_1 & X\\ 0 & T_2\end{bmatrix}=\begin{bmatrix}\delta(T_1)T_1 & \delta(T_1)X+VT_2\\ 0 & \delta(T_2)T_2\end{bmatrix}=$$
$$=T\delta(T)=\begin{bmatrix}T_1 & X\\ 0 & T_2\end{bmatrix}\begin{bmatrix}\delta(T_1) & V\\ 0 & \delta(T_2)\end{bmatrix}=\begin{bmatrix}T_1\delta(T_1) & T_1V+X\delta(T_2)\\ 0 & T_2\delta(T_2)\end{bmatrix},$$

we have $\delta(T_1)X+VT_2=T_1V+X\delta(T_2)$. From $T_1S-ST_2=\delta(T_1)X$ we obtain that $T_1S-ST_2+VT_2=T_1V+X\delta(T_2)$. A simple computation using this relation shows that $Z(T_1\oplus T_2)=TZ$.

(iii) $ZY=\delta(T)$.

$$ZY=\begin{bmatrix}I & V-S\\ 0 & \delta(T_2)\end{bmatrix}\begin{bmatrix}\delta(T_1) & S\\ 0 & I\end{bmatrix}=\begin{bmatrix}\delta(T_1) & S+V-S\\ 0 & \delta(T_2)\end{bmatrix}=\delta(T).$$

(iv) $YZ=\delta(T_1\oplus T_2)$. Since

$$YZ=\begin{bmatrix}\delta(T_1) & S\\ 0 & I\end{bmatrix}\begin{bmatrix}I & V-S\\ 0 & \delta(T_2)\end{bmatrix}=\begin{bmatrix}\delta(T_1) & \delta(T_1)(V-S)+S\delta(T_2)\\ 0 & \delta(T_2)\end{bmatrix},$$

to complete the proof, it suffices to show that $\delta(T_1)(V-S)+S\delta(T_2)=0$. Note that $YT=(T_1\oplus T_2)Y$ implies that $Y\delta(T)=\delta(T_1\oplus T_2)Y$. But

$$Y\delta(T)=\begin{bmatrix}\delta(T_1) & S\\ 0 & I\end{bmatrix}\begin{bmatrix}\delta(T_1) & V\\ 0 & \delta(T_2)\end{bmatrix}=\begin{bmatrix}\delta(T_1)^2 & \delta(T_1)V+S\delta(T_2)\\ 0 & \delta(T_2)\end{bmatrix}$$

and

$$\delta(T_1\oplus T_2)Y=\begin{bmatrix}\delta(T_1) & 0\\ 0 & \delta(T_2)\end{bmatrix}\begin{bmatrix}\delta(T_1) & S\\ 0 & I\end{bmatrix}=\begin{bmatrix}\delta(T_1)^2 & \delta(T_1)S\\ 0 & \delta(T_2)\end{bmatrix}.$$

We conclude that $\delta(T_1)V+S\delta(T_2)=\delta(T_1)S$ as asserted.

(v) Y and Z are quasi-affinities. Since δ is outer, $\delta(T_1)$ and $\delta(T_2)$ are quasi-affinities (cf. [8], p. 118). It can be easily checked that Y and Z are also quasi-affinities.

It is interesting to contrast the preceding result with [14], Theorem 1, where the problem when T is similar to $T_1\oplus T_2$ was considered. Here we make a weaker assumption to obtain a (necessarily) weaker conclusion. Indeed, the intertwining operators Y and Z constructed here are closely related to the invertible intertwining operator appearing in the proof of [14], Theorem 1.

Corollary 2.2. *Let* $T=\begin{bmatrix} T_1 & X \\ 0 & T_2 \end{bmatrix}$ *be as in Theorem 2.1. Assume that T is c.n.u. Then* Lat $T \cong$ Lat $(T_1 \oplus T_2)$, Lat$''$ $T \cong$ Lat$''(T_1 \oplus T_2)$ *and* Hyperlat $T \cong$ Hyperlat $(T_1 \oplus T_2)$.

Proof. Let Y and Z be the operators constructed in the proof of Theorem 2.1. For $K \in \mathrm{Lat}\, T$ and $L \in \mathrm{Lat}\,(T_1 \oplus T_2)$, consider the mappings $K \to \overline{YK}$ and $L \to \overline{ZL}$. It is easily checked that they are inverses to each other and preserve the lattice operations. Hence $\mathrm{Lat}\, T \cong \mathrm{Lat}\,(T_1 \oplus T_2)$. To complete the proof, it suffices to show that (i) $K \in \mathrm{Lat}''\, T$ implies that $\overline{YK} \in \mathrm{Lat}''\,(T_1 \oplus T_2)$ and (ii) $K \in \mathrm{Hyperlat}\ T$ implies that $\overline{YK} \in \mathrm{Hyperlat}\,(T_1 \oplus T_2)$. Then by a symmetric argument we also obtain that $L \in \mathrm{Lat}''\,(T_1 \oplus T_2)$ and $L \in \mathrm{Hyperlat}\,(T_1 \oplus T_2)$ imply that $\overline{ZL} \in \mathrm{Lat}''\, T$ and $\overline{ZL} \in \mathrm{Hyperlat}\, T$, respectively.

To prove (i), let $S \in \{T_1 \oplus T_2\}''$. We first check that $ZSY \in \{T\}''$. Indeed, $YVZ \in \{T_1 \oplus T_2\}'$ for any $V \in \{T\}'$. Hence $ZSYVZ = ZYVZS = \delta(T)VZS = V\delta(T)ZS = VZ\delta(T_1 \oplus T_2)S = VZS\delta(T_1 \oplus T_2) = VZSYZ$. It follows that $ZSYV = VZSY$, and therefore $ZSY \in \{T\}''$ as asserted. Since $K \in \mathrm{Lat}''\, T$, we have $\overline{ZSYK} \subseteq K$. Hence $\overline{YZSYK} \subseteq \overline{YK}$. But $\overline{YZSYK} = \overline{\delta(T_1 \oplus T_2) SYK} = \overline{SY\delta(T)K} = \overline{SYK}$. We conclude that $\overline{SYK} \subseteq \overline{YK}$ which shows that $\overline{YK} \in \mathrm{Lat}''\,(T_1 \oplus T_2)$. An analogous but easier argument than above shows that (ii) is also true. This completes the proof.

Corollary 2.3. *Let* $T=\begin{bmatrix} T_1 & X \\ 0 & T_2 \end{bmatrix}$ *be as in Theorem 2.1. Then there exist bi-invariant subspaces K_1 and K_2 of T such that $K_1 \vee K_2 = H$, $K_1 \cap K_2 = \{0\}$, $T|K_1$ is of class C_{11} and $T|K_2$ is of class $C_{\cdot 0}$. Moreover, K_1 and K_2 can be chosen such that $K_1 = H_1$ and $T|K_2 \sim T_2$.*

Proof. As in the proof of Theorem 2.1, we may assume that T is c.n.u. Let Y and Z be the operators constructed there, and let $K_1 = \overline{Z(H_1 \oplus 0)}$ and $K_2 = \overline{Z(0 \oplus H_2)}$. Then $K_1, K_2 \in \mathrm{Lat}''T$, $K_1 \vee K_2 = H$ and $K_1 \cap K_2 = \{0\}$ by Corollary 2.2. From the definition of Z, it is easily seen that $K_1 = H_1$. On the other hand, since $Z|0 \oplus H_2$: $0 \oplus H_2 \to K_2$ and $Y|K_2 : K_2 \to 0 \oplus H_2$ are quasi-affinities which intertwine $0 \oplus T_2$ and $T|K_2$, we have $T|K_2 \sim T_2$. Moreover, it is easy to check that in this case $T|K_2$ must also be of class $C_{\cdot 0}$, completing the proof.

We remark that if $T=\begin{bmatrix} T_1' & X' \\ 0 & T_2' \end{bmatrix}$ is the type $\begin{bmatrix} C_{0\cdot} & * \\ 0 & C_{1\cdot} \end{bmatrix}$ canonical triangulation of the contraction T and if the characteristic function of T_2' admits a scalar multiple, then, by considering T^*, we obtain results analogous to Theorem 2.1 and Corollaries 2.2. and 2.3. Also note that weak contractions and $C_1.$ contractions with $d_T<\infty$ (cf. Lemma 3.2. below) are among the operators satisfying the assumption of Theorem 2.1. When applied to weak contractions, Theorem 2.1 yields the following result which has been obtained before in [15].

Corollary 2.4. *Let T be a c.n.u. weak contraction and let T_1 and T_1' be its C_{11} and C_0 parts. Then $T_1\sim T\oplus T_1'$.*

Proof. Let $T=\begin{bmatrix} T_1 & X \\ 0 & T_2 \end{bmatrix}$ and $T=\begin{bmatrix} T_1' & X \\ 0 & T_2' \end{bmatrix}$ be the triangulations of types $\begin{bmatrix} C_{\cdot 1} & * \\ 0 & C_{\cdot 0} \end{bmatrix}$ and $\begin{bmatrix} C_{0\cdot} & * \\ 0 & C_{1\cdot} \end{bmatrix}$, respectively. Since the characteristic functions of T_1 and T_2' admit scalar multiples (cf. [8], p. 325 and p. 217), by Theorem 2.1 and the remark above we have $T_1\oplus T_2\sim T\sim T_1'\oplus T_2'$. Note that T_1 and T_2' are of class C_{11} and T_2 and T_1' are of class C_0, it is routine to check that $T_1\sim T_2'$ and $T_2\sim T_1'$ (cf. proof of [15], Theorem 1). Hence $T\sim T_1\oplus T_1'$ as asserted.

Note that Corollary 2.2. generalizes the corresponding results for Lat$''T$ and Hyperlat T when T is a c.n.u. weak contraction with finite defect indices (cf. [18], Corollary 4.2. and [17], Theorem 3). Indeed, in this case Lat$''T\cong$Lat$''(T_1\oplus T_2)=$ Lat$''T_1\oplus$Lat$''T_2\cong$Lat$''T_1\oplus$Lat$''T_1'=$Lat$''(T_1\oplus T_1')$ and similarly for Hyperlat T, where T_1' denotes the C_0 part of T.

As for Corollary 2.3, it generalizes the C_0-C_{11} decomposition for c.n.u. weak contractions (cf. [8], pp. 331—332). To verify this, we have to show that, in the context of Corollary 2.3, if T is a c.n.u. weak contraction, then $T|K_2$ is the C_0 part of T. Since $T|K_2\sim T_2$ is of class C_0, we have $K_2\subseteq H_1'\equiv\{x\in H: T^n x\to 0 \text{ as } n\to\infty\}$. On the other hand, since $T_2\sim T|H_1'\equiv T_1'$ (cf. proof of Corollary 2.4), we have $T|K_2\sim\sim T_1'$. Note that $\sigma(T_1')\subseteq\sigma(T)$ (cf. [8], p. 332). Hence T_1' is a weak C_0 contraction. Let $W: H_1'\to K_2$ be a quasi-affinity intertwining T_1' and $T|K_2$ and let $V: K_2\to H_1'$ be the restriction of the identity operator. Then VW is an injection in $\{T_1'\}'$. We infer from [1], Corollary 2.8 that VW is a quasi-affinity. It follows that $K_2=H_1'$ whence $T|K_2$ is the C_0 part of T.

3. $C_1.$ contractions. In this section we restrict ourselves to $C_1.$ contractions with at least one defect index finite. We will show that they are completely injection-similar to isometries and characterize various algebras of operators associated with them. We start with the following lemma.

Lemma 3.1. *Let T be a c.n.u. $C_1.$ contraction with $d_T=d_{T*}<\infty$. Then T is of class C_{11}.*

Proof. Since T is of class $C_1.$, its characteristic function $\{\mathfrak{D}_T, \mathfrak{D}_{T*}, \Theta_T(\lambda)\}$ is a $*$-outer function. Hence $\Theta_T(\lambda)^*: \mathfrak{D}_{T*}\to\mathfrak{D}_T$ has dense range for all λ in D (cf.

[8], p. 191). We conclude from the assumption $d_T = d_{T*} < \infty$ that $\det \Theta_T \not\equiv 0$. By [8], Theorem VII. 6. 3 we infer that T is of class C_{11}.

Lemma 3.2. *Let T be a $C_{1\cdot}$ contraction with $d_T < \infty$ and let $T = \begin{bmatrix} T_1 & * \\ 0 & T_2 \end{bmatrix}$ be of type $\begin{bmatrix} C_{\cdot 1} & * \\ 0 & C_{\cdot 0} \end{bmatrix}$. Then T_1 and T_2 are of classes C_{11} and C_{10}, respectively.*

Proof. Obviously, T_1 is of class C_{11}. As in the proof of Theorem 2.1, we may assume that T is c.n.u.. Let $T_2 = \begin{bmatrix} T_3 & * \\ 0 & T_4 \end{bmatrix}$ be the triangulation of type $\begin{bmatrix} C_{0\cdot} & * \\ 0 & C_{1\cdot} \end{bmatrix}$. Note that T_3 is of class C_{00}. Indeed, since T_2 is of class $C_{\cdot 0}$, we have $T_2^{*n} = \begin{bmatrix} T_3^{*n} & 0 \\ * & T_4^{*n} \end{bmatrix} \to 0$ strongly. It follows that $T_3^{*n} \to 0$ strongly. Hence T_3 is of class $C_{\cdot 0}$ and thus of class C_{00}. We have

$$T = \begin{bmatrix} T_1 & * & * \\ 0 & T_3 & * \\ 0 & 0 & T_4 \end{bmatrix}.$$

Let $T' = \begin{bmatrix} T_1 & * \\ 0 & T_3 \end{bmatrix}$ with the corresponding regular factorization $\Theta_{T'} = \Theta_3 \Theta_1$, where $\{\mathfrak{D}_{T'}, \mathfrak{D}_{T'*}, \Theta_{T'}(\lambda)\}$ is factored as the product of $\{\mathfrak{D}_{T'}, \mathfrak{F}, \Theta_1(\lambda)\}$ and $\{\mathfrak{F}, \mathfrak{D}_{T'*}, \Theta_3(\lambda)\}$. Since T_1 and T_3 are of classes C_{11} and C_{00}, the purely contractive parts of Θ_1 and Θ_3 are outer and inner from both sides, respectively (cf. [8], p. 257). We deduce that $\dim \mathfrak{D}_{T'} = \dim \mathfrak{F}$ and $\dim \mathfrak{F} = \dim \mathfrak{D}_{T'*}$ (cf. [8], p. 192). It follows that $\dim \mathfrak{D}_{T'} = \dim \mathfrak{D}_{T'*}$, that is, $d_{T'} = d_{T'*}$. Note that T' is of class $C_{1\cdot}$ and $d_{T'} \leqq d_T < \infty$. Hence by Lemma 3.1, T' is of class C_{11}. This implies that T_3 is of class $C_{\cdot 1}$, contradicting the fact that T_3 is of class C_{00}. We conclude that T_2 itself must be of class $C_{1\cdot}$ and therefore of class C_{10}.

If T is a $C_{1\cdot}$ contraction with $d_T < \infty$, then as shown above T_1 is of class C_{11} and has finite defect indices. Hence its characteristic function admits a scalar multiple (cf. [8], p. 318) and therefore Theorem 2.1 is applicable. In particular, we have the following corollary.

Corollary 3.3. *Let T and S be $C_{1\cdot}$ contractions with finite defect indices and let $T = \begin{bmatrix} T_1 & * \\ 0 & T_2 \end{bmatrix}$ and $S = \begin{bmatrix} S_1 & * \\ 0 & S_2 \end{bmatrix}$ be the triangulations of type $\begin{bmatrix} C_{\cdot 1} & * \\ 0 & C_{\cdot 0} \end{bmatrix}$. Then $T \sim S$ if and only if $T_1 \sim S_1$ and $T_2 \sim S_2$.*

Proof. The conclusion follows easily from the preceding remark and [22], Theorem 6.

Lemma 3.4. *Let* $T=U_1\oplus\ldots\oplus U_p\oplus S_q$ *on* $H=L^2(E_1)\oplus\ldots\oplus L^2(E_p)\oplus H_q^2$, *where* $0\leqq p, q\leqq\infty$, E_j*'s are Borel subsets of the unit circle satisfying* $E_1\supseteq E_2\supseteq\ldots\supseteq E_p\neq\emptyset$, U_j *denotes the operator of multiplication by* e^{it} *on* $L^2(E_j)$, $j=1,\ldots,p$, *and* S_q *denotes the unilateral shift on* H_q^2. *Then* $\mu_T=p+q$.

Proof. Let $U=U_1\oplus\ldots\oplus U_p$. It is well known that $\mu_U=p$ and $\mu_{S_q}=q$. Hence $\mu_T\leqq\mu_U+\mu_{S_q}=p+q$. On the other hand, for almost all e^{it} in E_p, consider $H_t=\{h(e^{it})\colon h\in H\}$. Obviously, $H_t=\mathbf{C}^{p+q}$. We assume that $N\equiv\mu_T<\infty$ for otherwise the assertion is trivial. Let $K=\{h_1,\ldots,h_N\}$ be a set of vectors in H such that $H=\bigvee_{k=0}^{\infty}T^kK$. Then $H=\{p_1(T)h_1+\ldots+p_N(T)h_N\colon p_1,\ldots,p_N \text{ polynomials}\}^-$. We deduce that $H_t=\{p_1(e^{it})h_1(e^{it})+\ldots+p_N(e^{it})h_N(e^{it})\colon p_1,\ldots,p_N \text{ polynomials}\}^-$ for almost all e^{it} in E_p, that is, H_t is spanned by the set of N vectors $\{h_1(e^{it}),\ldots,h_N(e^{it})\}$. Hence we must have $p+q\leqq N$, and thus $\mu_T=N=p+q$.

Now we are ready to show the complete injection-similarity of $C_{1\cdot}$ contractions with isometries. The next theorem not only generalizes [20], Theorem 2.1 but the proof is much simpler.

Theorem 3.5. *Let T be a $C_{1\cdot}$ contraction with* $d_T<\infty$. *Then T is completely injection-similar to an isometry. If T is c.n.u., then* $U\oplus S_{m-n}\overset{\text{ci}}{\prec}T\prec U\oplus S_{m-n}$, *where* $m=d_{T^*}$, $n=d_T$, U *denotes the operator of multiplication by* e^{it} *on* $\overline{\Delta_T L_n^2}$ *and* S_{m-n} *denotes the unilateral shift on* H_{m-n}^2. *In particular,* $p+m-n\leqq\mu_T\leqq p+2(m-n)$, *where* $p=\mu_U$.

Proof. We may assume that T is c.n.u.. Let $T=\begin{bmatrix}T_1 & X\\ 0 & T_2\end{bmatrix}$ be the triangulation of type $\begin{bmatrix}C_{\cdot 1} & *\\ 0 & C_{\cdot 0}\end{bmatrix}$ with the corresponding factorization $\Theta_T=\Theta_2\Theta_1$. By the remark before Corollary 3.3, we have $T\sim T_1\oplus T_2$. Note that T_1, being of class C_{11}, is quasi-similar to U on $\overline{\Delta_1 L_n^2}=\overline{\Delta_T L_n^2}$, where $\Delta_1=(I-\Theta_1^*\Theta_1)^{1/2}$ (cf. [8], pp. 71—72). On the other hand, since the characteristic function of T_2 is the purely contractive part of Θ_2, we infer that $d_{T_2}=n-r$ and $d_{T_2^*}=m-r$ for some r with $0\leqq r\leqq n$. Hence for the C_{10} contraction T_2 we have $S_{m-n}\overset{\text{ci}}{\prec}T_2\prec S_{m-n}$ (cf. [7], Theorem 3). We conclude that $U\oplus S_{m-n}\overset{\text{ci}}{\prec}T\prec U\oplus S_{m-n}$. Finally we verify the assertion concerning μ_T. Note that $T\prec U\oplus S_{m-n}$ implies that $\mu_T\geqq\mu_{U\oplus S_{m-n}}=p+m-n$ by Lemma 3.4. On the other hand, we have $\mu_T=\mu_{T_1\oplus T_2}\leqq\mu_{T_1}+\mu_{T_2}\leqq p+2(m-n)$ (cf. [10], Theorem 2). This completes the proof.

Unfortunately, we are yet unable to show the uniqueness of the isometry completely injection-similar to T although its unitary part is indeed unique. This follows from the following lemma.

Lemma 3.6. *For $j=1,2$, let $V_j=U_j\oplus S_j$ be an isometry, where U_j is a unitary operator and S_j is a unilateral shift. If $V_1 \overset{i}{\sim} V_2$, then $U_1\cong U_2$.*

Proof. Assume that $V_j=U_j\oplus S_j$ is acting on $H_j=K_j\oplus L_j, j=1,2$. Let X: $H_1\to H_2$ and Y: $H_2\to H_1$ be the injections which intertwine V_1 and V_2. We claim that $XK_1\subseteq K_2$. Indeed, for any x in K_1 and $n\geqq 0, x=U_1^n y_n$ for some $y_n\in K_1$. Hence $Xx=XU_1^n y_n=XV_1^n y_n=V_2^n Xy_n\subseteq V_2^n H_2$ for any $n\geqq 0$. It follows that $Xx\in \bigcap_{n=0}^{\infty} V_2^n H_2=K_2$, as asserted. Similarly, we have $YK_2\subseteq K_1$. Thus $U_1 \overset{i}{\sim} U_2$. We conclude that U_1 and U_2 are unitarily equivalent to direct summands of each other (cf. [3], Lemma 4.1). By the third test problem in [5], this implies that $U_1\cong U_2$.

We conjecture that if $V_1 \overset{i}{\sim} V_2$ and $\mu_{U_1}<\infty$ then $V_1\cong V_2$.

The next two theorems characterize those $C_{1\cdot}$ contractions which are cyclic or have commutative commutants. Analogous results have been obtained before for $C_{\cdot 0}$ contractions (cf. [23], Theorems 1.3 and 1.5).

Theorem 3.7. *Let T be a c.n.u. $C_{1\cdot}$ contraction with $d_T<\infty$. Then the following statements are equivalent:*

(1) T is cyclic;

(2) T is of class C_{11} and $T\sim M_E$ or T is of class C_{10} and $T\sim S$, where M_E denotes the operator of multiplication by e^{it} on $L^2(E)$, E being a Borel subset of the unit circle, and S denotes the simple unilateral shift.

The proof is the same as the one for [20], Theorem 3.2.

Corollary 3.8. *Let T be a c.n.u. $C_{1\cdot}$ contraction with $d_T<\infty$. If T is cyclic, so is T^* but not conversely.*

Proof. If T is cyclic, then $T\sim M_E$ or $T\sim S$. Hence $T^*\sim M_E^*$ or $T^*\sim S^*$. In either case, T^* is cyclic. The converse example is given by $T=S\oplus S$ (cf. [4], Problem 126).

Theorem 3.9. *Let T be a c.n.u. $C_{1\cdot}$ contraction with $d_T<\infty$. Then the following statements are equivalent:*

(1) $\{T\}'=\{T\}''$;

(2) T is of class C_{11} and $T\sim M_E$ or T is of class C_{10} and $d_{T^*}-d_T=1$.

Proof. (2)⇒(1). If T is of class C_{11} and $T\sim M_E$, then obviously T is cyclic. Hence (1) follows from [9], Theorem 1. On the other hand, if T is of class C_{10} and $d_{T^*}-d_T=1$, then (1) follows from [23], Theorem 1.5.

(1)⇒(2). Let $T=\begin{bmatrix} T_1 & X \\ 0 & T_2 \end{bmatrix}$ on $H=H_1\oplus H_2$ be the triangulation of type

$\begin{bmatrix} C_{\cdot 1} & * \\ 0 & C_{\cdot 0} \end{bmatrix}$. As proved in Theorem 3.5, $T_1 \sim U$, the operator of multiplication by e^{it} on $\overline{\Delta_T L_n^2}$, and $T_2 \prec S_{m-n}$, where $m = d_{T^*}$ and $n = d_T$. We consider the following two cases:

(i) If $m=n$, then $T=T_1$ is of class C_{11} by Lemma 3.1. Note that there are quasi-affinities $Y: H \to \overline{\Delta_T L_n^2}$ and $Z: \overline{\Delta_T L_n^2} \to H$ which intertwine T and U and such that $YZ = \delta(U)$ and $ZY = \delta(T)$ for some outer function δ (cf. [21], Lemma 2.1). It is easily verified that $\{T\}' = \{T\}''$ implies that $\{U\}' = \{U\}''$. Therefore U is cyclic (cf. [6], § 3) and so $T \sim M_E$ for some Borel subset E.

(ii) If $m \neq n$, then there exist finitely many operators $Z_i: H^2_{m-n} \to \overline{\Delta_T L_n^2}$ which intertwine S_{m-n} and U and such that $\bigvee_i \operatorname{ran} Z_i = \overline{\Delta_T L_n^2}$ (cf. [2], pp. 299—300). Hence there exist $Y_i: H_2 \to H_1$ which intertwine T_2 and T_1 and such that $\bigvee_i \operatorname{ran} Y_i = H_1$. On the other hand, using Theorem 2.1 and the assumption $\{T\}' = \{T\}''$ we infer that $\{T_1 \oplus T_2\}' = \{T_1 \oplus T_2\}''$. Thus any operator $Y: H_2 \to H_1$ which intertwines T_2 and T_1 must be 0. We conclude from above that $H_1 = \{0\}$, that is, T is of class C_{10}. Moreover, $\{T\}' = \{T\}''$ implies that $m-n=1$ (cf. [23], Theorem 1.5).

Corollary 3.10. *Let T be a c.n.u. $C_{1\cdot}$ contraction with $d_T < \infty$. If T is cyclic, then $\{T\}' = \{T\}''$ but not conversely.*

Proof. The converse example is given in [10], pp. 321—322.

We remark that Corollaries 3.8 and 3.10 have been obtained before by Sz.-Nagy and Foiaş [9], Theorem 1 and [6].

In the final part of this paper, we determine when a $C_{1\cdot}$ contraction satisfies the double commutant property. Since a c.n.u. $C_{1\cdot}$ contraction T with $d_T < \infty$ is completely injection-similar to an isometry with an absolutely continuous unitary part, to motivate we first consider for such isometries. The next lemma partially generalizes [12], Theorem 3.3.

Lemma 3.11. *Let $V = U \oplus S$ be an isometry on $H = H_1 \oplus H_2$, where U is a unitary operator and S is a unilateral shift. Assume that U is absolutely continuous. Then the following statements are equivalent:*

(1) $S \neq 0$;

(2) *V is not unitary*;

(3) $\{V\}'' = \{\varphi(V): \varphi \in H^\infty\}$.

Proof. (1)$\Leftrightarrow$(2). Trivial.

(1)$\Rightarrow$(3). Let $T \in \{V\}''$. Then $T = T_1 \oplus T_2$ where $T_1 \in \{U\}''$ and $T_2 \in \{S\}''$. Since $S \neq 0$, there exists $\varphi \in H^\infty$ such that $T_2 = \varphi(S)$. As before, there are (possibly infinitely many) operators $Z_i: H_2 \to H_1$ which intertwine S and U and such that

$\bigvee_i \operatorname{ran} Z_i = H_1$ (cf. [2], pp. 299—300). Hence $\varphi(U)Z_i = Z_i\varphi(S) = Z_iT_2$ for all i. On the other hand, since $Y_i \equiv \begin{bmatrix} 0 & Z_i \\ 0 & 0 \end{bmatrix} \in \{V\}'$, we have $TY_i = Y_iT$. A simple computation shows that $T_1Z_i = Z_iT_2$. Thus $T_1Z_i = \varphi(U)Z_i$ for all i. We conclude that $T_1 = \varphi(U)$ and hence $T = \varphi(V)$.

(3)⇒(1). If $S=0$, then $V=U$ is a unitary operator. Hence $\{V\}'' = \{\psi(V)\colon \psi \in L^\infty\}$, which is certainly not equal to $\{\varphi(V)\colon \varphi \in H^\infty\}$.

Next we show that $C_1.$ contractions share similar properties. We need the following lemma.

Lemma 3.12. *Let T be a contraction on H and let $T = \begin{bmatrix} T_1 & X \\ 0 & T_2 \end{bmatrix}$ on $H = H_1 \oplus H_2$ be the triangulation of type $\begin{bmatrix} C_{\cdot 1} & * \\ 0 & C_{\cdot 0} \end{bmatrix}$. Then H_1 is hyperinvariant for T.*

Proof. Note that $H_2 = \{x \in H\colon T^{*n}x \to 0\}$ (cf. [8], p. 73). For $S \in \{T\}'$, we have $T^{*n}S^*x = S^*T^{*n}x \to 0$ as $n \to \infty$ for any $x \in H_2$. This shows that $S^*H_2 \subseteq H_2$. It follows that $SH_1 \subseteq H_1$, whence H_1 is hyperinvariant for T.

Theorem 3.13. *Let T be a c.n.u. $C_1.$ contraction with $d_T < \infty$. Let $m = d_{T^*}$ and $n = d_T$. Then the following statements are equivalent:*

(1) $m \neq n$;

(2) *T is not of class C_{11};*

(3) $\{T\}'' = \{\varphi(T)\colon \varphi \in H^\infty\}$.

Proof. (1)⇔(2). This follows from Lemma 3.1 and the fact that C_{11} contractions have equal defect indices.

(1)⇒(3). As in the proof of Theorem 3.9, if $m \neq n$ then there exist finitely many operators $Y_i\colon H_2 \to H_1$ which intertwine T_2 and T_1 and such that $\bigvee_i \operatorname{ran} Y_i = H_1$. Let $W \in \{T\}''$. By Lemma 3.12, $W = \begin{bmatrix} W_1 & * \\ 0 & W_2 \end{bmatrix}$ on $H = H_1 \oplus H_2$. Obviously, $W_2 \in \{T_2\}'$. We check that actually $W_2 \in \{T_2\}''$. Let $R \in \{T_2\}'$, and let Y and Z be the operators constructed in the proof of Theorem 2.1. It is easily seen that $Z(I \oplus R)Y \in \{T\}'$. Hence $Z(I \oplus R)YW = WZ(I \oplus R)Y$. A simple computation shows that $\delta(T_2)RW_2 = W_2\delta(T_2)R = \delta(T_2)W_2R$. Since $\delta(T_2)$ is an injection, we have $RW_2 = W_2R$ whence $W_2 \in \{T_2\}''$ as asserted. Thus there exists $\varphi \in H^\infty$ such that $W_2 = \varphi(T_2)$ (cf. [13], Theorem 1). We have $\varphi(T_1)Y_i = Y_i\varphi(T_2) = Y_iW_2$ for all i. On the other hand, since $X_i \equiv \begin{bmatrix} 0 & Y_i \\ 0 & 0 \end{bmatrix} \in \{T\}'$, we have $WX_i = X_iW$. It follows that $W_1Y_i = Y_iW_2$ whence $W_1Y_i = \varphi(T_1)Y_i$ for all i. We conclude that $W_1 = \varphi(T_1)$. Thus W is triangulated as $\begin{bmatrix} \varphi(T_1) & * \\ 0 & \varphi(T_2) \end{bmatrix}$. But we also have $\varphi(T) = \begin{bmatrix} \varphi(T_1) & * \\ 0 & \varphi(T_2) \end{bmatrix}$. Hence

$W-\varphi(T)=\begin{bmatrix}0 & Q\\ 0 & 0\end{bmatrix}\in\{T\}''$, say. To complete the proof, it suffices to show that $Q=0$. To this end, let $S\colon H_2\to H_1$ be the operator defined in the proof of Theorem 2.1 and let $A=\begin{bmatrix}\delta(T_1) & S\\ 0 & 0\end{bmatrix}$. It is clear that $A\in\{T\}'$. Hence $A(W-\varphi(T))=(W-\varphi(T))A$. A simple computation shows that $\delta(T_1)Q=0$. Since $\delta(T_1)$ is an injection, we conclude that $Q=0$, completing the proof.

(3)⇒(2). If T is of class C_{11}, then $\{T\}''$ has been given in [19], Lemma 2. We will show that it is not the same as $\{\varphi(T)\colon \varphi\in H^\infty\}$. Note that T is quasi-similar to the operator $U=U_1\oplus\ldots\oplus U_p$ on $K=L^2(E_1)\oplus\ldots\oplus L^2(E_p)$, where $0\leqq p\leqq n$, $E_j=\{e^{it}\colon \operatorname{rank}\Delta_T(e^{it})\geqq j\}$ are Borel subsets of the unit circle satisfying $E_1\supseteqq E_2\supseteqq\ldots\supseteqq E_p\neq\emptyset$ and U_j denotes the operator of multiplication by e^{it} on $L^2(E_j)$, $j=1,2,\ldots,p$ (cf. [16], Theorem 2). Let $\delta=\det\Theta_T$ and Ω be the algebraic adjoint of Θ_T. Since $\delta\not\equiv 0$, there exists some $\varepsilon>0$ such that $F=\{e^{it}\in E_1\colon |\delta(e^{it})|\geqq\varepsilon\}$ has positive Lebesgue measure. Let $G\subseteqq F$ be such that G and $F\setminus G$ both have positive Lebesgue measure. Let

$$V=P\begin{bmatrix}0 & 0\\ -\chi_G\frac{1}{\delta}\Delta_T\Omega & \chi_G\end{bmatrix}.$$

It is easily checked that $V\in\{T\}''$ (cf. [19], Lemma 2). If $V=\varphi(T)$ for some $\varphi\in H^\infty$, then $\chi_G=\varphi$ on $\overline{\Delta_T L_n^2}$. In particular, $\chi_G=\varphi$ a.e. on E_1. This is certainly impossible. We conclude that $\{T\}''\neq\{\varphi(T)\colon\varphi\in H^\infty\}$.

Corollary 3.14. *Let T be a c.n.u. $C_{1\cdot}$ contraction with $d_T<d_{T^*}\leqq\infty$. If T is cyclic, then $\{T\}'=\{\varphi(T)\colon\varphi\in H^\infty\}$.*

Proof. This follows from Corollary 3.10 and Theorem 3.13.

The preceding corollary has been obtained before in [11], Lemma 1.

Corollary 3.15. *Let T be a c.n.u. $C_{1\cdot}$ contraction with $d_T<\infty$. Then the following statements are equivalent:*

(1) $\{T\}''=\operatorname{Alg}T$;

(2) *either $d_T\neq d_{T^*}$ or $d_T=d_{T^*}$ and $\Theta_T(e^{it})$ is isometric for e^{it} in a set of positive Lebesgue measure.*

Proof. The assertion follows from Theorem 3.13 and [18], Theorem 3.8.

References

[1] H. BERCOVICI, C_0-Fredholm operators. I, *Acta Sci. Math.*, **41** (1979), 15—31.

[2] R. G. DOUGLAS, On the hyperinvariant subspaces for isometries, *Math. Z.*, **107** (1968), 297—300.

[3] R. G. DOUGLAS, On the operator equation $S^*XT=X$ and related topics, *Acta Sci. Math.*, **30** (1969), 19—32.

[4] P. R. HALMOS, *A Hilbert space problem book*, van Nostrand (Princeton, New Jersey, 1967).

[5] R. V. KADISON and I. M. SINGER, Three test problems in operator theory, *Pacific J. Math.*, **7** (1957), 1101—1106.

[6] B. SZ.-NAGY, Cyclic vectors and commutants, *Linear operator and approximation*, Birkhäuser (Basel—Stuttgart, 1972), 62—67.

[7] B. SZ.-NAGY, Diagonalization of matrices over H^∞, *Acta Sci. Math.*, **38** (1976), 223—238.

[8] B. SZ.-NAGY and C. FOIAŞ, *Harmonic analysis of operators on Hilbert space*, North Holland — Akadémiai Kiadó (Amsterdam—Budapest, 1970).

[9] B. SZ.-NAGY and C. FOIAŞ, Vecteurs cycliques et commutativité des commutants, *Acta Sci. Math.*, **32** (1971), 177—183.

[10] B. SZ.-NAGY and C. FOIAŞ, Jordan model for contractions of class $C_{\cdot 0}$, *Acta Sci. Math.*, **36** (1974), 305—322.

[11] B. SZ.-NAGY and C. FOIAŞ, Vecteurs cycliques et commutativité des commutants. II, *Acta Sci. Math.*, **39** (1977), 169—174.

[12] T. R. TURNER, Double commutants of isometries, *Tôhoku Math. J.*, **24** (1972), 547—549.

[13] M. UCHIYAMA, Double commutants of $C_{\cdot 0}$ contractions. II, *Proc. Amer. Math. Soc.*, **74** (1979), 271—277.

[14] P. Y. WU, On nonorthogonal decompositions of certain contractions, *Acta Sci. Math.*, **37** (1975), 301—306.

[15] P. Y. WU, Quasi-similarity of weak contractions, *Proc. Amer. Math. Soc.*, **69** (1978), 277—282.

[16] P. Y. WU, Jordan model for weak contractions, *Acta Sci. Math.*, **40** (1978), 189—196.

[17] P. Y. WU, Hyperinvariant subspaces of weak contractions, *Acta Sci. Math.*, **41** (1979), 259—266.

[18] P. Y. WU, Bi-invariant subspaces of weak contractions, *J. Operator Theory*, **1** (1979), 261—272.

[19] P. Y. WU, C_{11} contractions are reflexive, *Proc. Amer. Math. Soc.*, **77** (1979), 68—72.

[20] P. Y. WU, On contractions of class $C_{1\cdot}$, *Acta Sci. Math.*, **42** (1980), 205—210.

[21] P. Y. WU, On a conjecture of Sz.-Nagy and Foiaş, *Acta Sci. Math.*, **42** (1980), 331—338.

[22] P. Y. WU, On the quasi-similarity of hyponormal contractions, *Illinois J. Math.*, **25** (1981), 498—503.

[23] P. Y. WU, $C_{\cdot 0}$ contractions: cyclic vectors, commutants and Jordan models, *J. Operator Theory*, **5** (1981), 53—62.

DEPARTMENT OF APPLIED MATHEMATICS
NATIONAL CHIAO TUNG UNIVERSITY
HSINCHU, TAIWAN, CHINA