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When is a contraction quasi-similar to an isometry?

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In this paper we answer the question in the title for contractions with finite defect indices. More precisely, we show that if T is a contraction with finite defect indices then T is quasi-similar to an isometry if and only if T is of class $C_{1.}$ and there exists a bounded analytic function Ω such that $\Omega\Theta_T=\delta I$ for some outer function δ, where Θ_T denotes the characteristic function of T. This condition is analogous to the one for a contraction similar to an isometry (cf. [3], Theorem 2.4.). We will also derive some related results.

In the following all the operators are acting on complex, separable Hilbert spaces. The main reference is the book of SZ.-NAGY and FOIAŞ [2]. Recall that for operators T_1 and T_2 on H_1 and H_2, respectively, $T_1 \prec T_2$ denotes that T_1 is a *quasi-affine transform* of T_2, that is, there exists a one-to-one operator $X: H_1 \to H_2$ with dense range (called *quasi-affinity*) such that $T_2X=XT_1$. T_1 and T_2 are *quasi-similar* ($T_1 \sim T_2$) if $T_1 \prec T_2$ and $T_2 \prec T_1$.

For a contraction T, let $d_T=\operatorname{rank}(I-T^*T)^{1/2}$ and $d_{T^*}=\operatorname{rank}(I-TT^*)^{1/2}$ denote its *defect indices* and let Θ_T denote its characteristic function. For any $n\geqq 1$, let S_n denote the unilateral shift on H_n^2. The next lemma characterizes those contractions which are quasi-similar to a unilateral shift.

Lemma 1. *Let T be a contraction with finite defect indices. Then the following statements are equivalent:*

(1) T is quasi-similar to a unilateral shift;

(2) T is of class C_{10} and there exists a bounded analytic function Ω such that $\Omega\Theta_T=\delta I$ for some outer function δ.

Proof. Let $n=d_T$ and $m=d_{T^*}$.

(1)$\Rightarrow$(2). That T is of class C_{10} follows from [8], Lemma 1. Consider the functional model of T, that is, consider T being acting on $\mathfrak{H}\equiv H_m^2\ominus\Theta_T H_n^2$ by $Tf=P(e^{it}f)$

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for $f\in\mathfrak{H}$, where P denotes the (orthogonal) projection onto $\mathfrak{H}$. Note that T must be quasi-similar to S_{m-n}. Indeed, this follows from the uniqueness of the Jordan model of T (cf. [4], Theorem 4). Let $Y\colon H^2_{m-n}\to\mathfrak{H}$ be the quasi-affinity intertwining S_{m-n} and T. Then Y is given by $Yg=P(\Phi g)$ for $g\in H^2_{m-n}$, where Φ is an $m\times(m-n)$ matrix valued bounded analytic function. Note that $\overline{\operatorname{ran} Y}=\mathfrak{H}$ if and only if $\Phi H^2_{m-n}+\Theta_T H^2_n$ is dense in H^2_m. Let Ψ denote the $m\times n$ matrix valued function $[\Phi, \Theta_T]$. Since $\Phi H^2_{m-n}+\Theta_T H^2_n=\Psi H^2_m$, we conclude from above that Ψ is an outer function. Let Ψ^A denote the algebraic adjoint of the matrix of Ψ. Say, $\psi^A=\begin{bmatrix}\Omega'\\ \Omega\end{bmatrix}$, where Ω' is $(m-n)\times m$ matrix valued and Ω is $n\times m$ matrix valued. Since $\Psi^A\Psi=\delta I$, where $\delta=\det\Psi$ is an outer function, we infer that $\Omega\Theta_T=\delta I$ as asserted.

(2)$\Rightarrow$(1). Consider the functional model of T and consider Ω as a multiplication operator from H^2_m to H^2_n. Let $\mathfrak{K}=\ker\Omega$. Define $X\colon\mathfrak{H}\to\mathfrak{K}$ by $Xf=\delta f-\Theta_T\Omega f$ for $f\in\mathfrak{H}$ and $Y\colon\mathfrak{K}\to\mathfrak{H}$ by $Yg=Pg$ for $g\in\mathfrak{K}$. Note that $\Omega Xf=\Omega\delta f-\Omega\Theta_T\Omega f=$ $=\Omega\delta f-\delta\Omega f=0$ for any $f\in\mathfrak{H}$. Hence X indeed maps $\mathfrak{H}$ to $\mathfrak{K}$. Let $S=S_m|\mathfrak{K}$. It is easily verified that X and Y intertwine T and S. Moreover, we have $XYg=XPg=$ $=X(g-\Theta_T w)=\delta(g-\Theta_T w)-\Theta_T\Omega(g-\Theta_T w)=\delta g-\Theta_T\Omega g=\delta g=\delta(S)g$ for any $g\in\mathfrak{K}$, where $w\in H^2_n$, and $YXf=Y(\delta f-\Theta_T\Omega f)=P(\delta f)-0=\delta(T)f$ for any $f\in\mathfrak{H}$. Since $\delta(S)$ and $\delta(T)$ are quasi-affinities, so are X and Y. This shows that T is quasi-similar to S, a unilateral shift, completing the proof.

We remark that the proof of (2)$\Rightarrow$(1) in the preceding lemma holds even without the finiteness assumption on the defect indices of T. Also note that Lemma 1 partially generalizes [4], Proposition 2 (for the case $d_T=1$ and $d_{T^*}=2$) and [6], Theorem 3.1 (for the case $d_{T^*}-d_T=1$). Next we consider contractions quasi-similar to isometries. We need the following lemma.

Lemma 2. *Let T be a contraction with finite defect indices. Then the following statements are equivalent:*

(1) *T is quasi-similar to an isometry;*

(2) *the completely non-unitary (c.n.u.) part of T is quasi-similar to an isometry.*

Proof. We have only to show (1)$\Rightarrow$(2). Assume that T is quasi-similar to the isometry V. By [8], Lemma 1, T is of class $C_{1\cdot}$. Let $V=U\oplus S$, where U is unitary and S is a unilateral shift, and let $T=T_1\oplus T_2$, where T_1 is unitary and T_2 is c.n.u. Let $T_2=\begin{bmatrix}T_3 & *\\ 0 & T_4\end{bmatrix}$ be the triangulation of type $\begin{bmatrix}C_{\cdot 1} & *\\ 0 & C_{\cdot 0}\end{bmatrix}$. Then T_3 is of class C_{11} and has finite defect indices. By [9], Theorem 2.1, $T_2\sim T_3\oplus T_4$. Hence $U\oplus S\sim T_1\oplus T_2$ $\sim T_1\oplus T_3\oplus T_4$. Note that U and $T_1\oplus T_3$ are of class C_{11}, S and T_4 are of class C_{10} (cf. [9], Lemma 3.2) and the defect indices of T_4 are finite. It follows from the proof of [8], Theorem 6 that $T_1\oplus T_3\sim U$ and $T_4\prec S$. Hence S must be the Jordan model of T_4 (cf. [8], Lemma 3), that is, $S=S_{m-n}$, where $m=d_{T_4^*}$ and $n=d_{T_4}$. Thus S has finite defect indices and we infer from [8], Theorem 6 again that $T_4 \sim S$. On the other hand, the C_{11} contraction T_3 is quasi-similar to a unitary operator (cf. [2], p. 72). We conclude from above that T_2 is quasi-similar to an isometry, completing the proof.

Theorem 3. *Let T be a contraction with finite defect indices and let* $T=\begin{bmatrix} T_1 & * \\ 0 & T_2 \end{bmatrix}$ *be the triangulation of type* $\begin{bmatrix} C_{\cdot 1} & * \\ 0 & C_{\cdot 0} \end{bmatrix}$. *Then the following statements are equivalent:*

(1) *T is quasi-similar to an isometry;*

(2) *T_1 is quasi-similar to a unitary operator and T_2 is quasi-similar to a unilateral shift;*

(3) *T is of class $C_{1\cdot}$ and there exists a bounded analytic function Ω such that $\Omega\Theta_T=\delta I$ for some outer function δ.*

Proof. By Lemma 2, it suffices to consider c.n.u. T.

(1)$\Rightarrow$(2) is proved in Lemma 2.

(2)$\Rightarrow$(3). By [8], Lemma 1, both T_1 and T_2 are of class $C_{1\cdot}$. A simple calculation shows that T must also be of class $C_{1\cdot}$. Let $\Theta_T=\Theta_2\Theta_1$ be the canonical factorization corresponding to the triangulation $T=\begin{bmatrix} T_1 & * \\ 0 & T_2 \end{bmatrix}$. Then the characteristic functions of T_1 and T_2 are the purely contractive parts of Θ_1 and Θ_2, respectively. Lemma 1 implies that there exists a bounded analytic function Ω_2 such that $\Omega_2\Theta_2=\delta_2 I$ for some outer function δ_2. On the other hand, T_1 is of class C_{11} implies that Θ_1 is outer (from both sides). Let Ω_1 be the algebraic adjoint of the matirx of Θ_1 and let $\Omega=\Omega_1\Omega_2$ and $\delta=\delta_2 \det\Theta_1$. Then $\Omega\Theta_T=\Omega_1\Omega_2\Theta_2\Theta_1=\Omega_1\delta_2\Theta_1=\delta I$, where δ is outer.

(3)$\Rightarrow$(1). As above, let $\Theta_T=\Theta_2\Theta_1$ be the factorization corresponding to $T=\begin{bmatrix} T_1 & * \\ 0 & T_2 \end{bmatrix}$. From $\Omega\Theta_T=\delta I$ we have $\Theta_1\Omega\Theta_T\Omega_1=\Theta_1\delta\,\Omega_1=\delta(\det\Theta_1)I$, where Ω_1 is the algebraic adjoint of Θ_1. It follows that $(\Theta_1\Omega)\Theta_2=\delta I$. Since T_2 is of class C_{10} (cf. [9], Lemma 3.2), we infer from Lemma 1 that T_2 is quasi-similar to a unilateral shift. On the other hand, T_1 is quasi-similar to a unitary operator and $T\sim T_1\oplus T_2$ (cf. [9], Theorem 2.1). We conclude that T is quasi-similar to an isometry as asserted.

Note that the isometry quasi-similar to T is unique up to unitary equivalence (cf. [1], Theorem 3.1). It also follows from the preceding proof that if T is c.n.u., then the isometry quasi-similar to T has an absolutely continuous unitary part. We may contrast Theorem 3 with the corresponding results for contractions similar to isometries: a contraction T is similar to an isometry if and only if there is a bounded analytic function Ω such that $\Omega\Theta_T=I$ (cf. [3], Theorem 2.4); a c.n.u. T is similar to an isometry if and only if T_1 is similar to a unitary operator and T_2 is similar to a unilateral shift (cf. [5], Theorem 2).

Corollary 4. *Let T be a c.n.u. contraction with finite defect indices and let $\mathfrak{H}_1$ be an invariant subspace for T.*

(1) *If T is quasi-similar to an isometry, so is $T|\mathfrak{H}_1$.*

(2) *If T is quasi-similar to a unilateral shift, so is $T|\mathfrak{H}_1$.*

Proof. (1) By [8], Lemma 1, T is of class $C_{1\cdot}$. Hence $T|\mathfrak{H}_1$ is also of class $C_{1\cdot}$. Let $\Theta_T=\Theta_2\Theta_1$ be the corresponding regular factorization and let Ω be such that $\Omega\Theta_T=\delta I$ for some outer δ. Then $(\Omega\Theta_2)\Theta_1=\delta I$ and by Theorem 3 we conclude that $T|\mathfrak{H}_1$ is quasi-similar to an isometry.

(2) By [8], Lemma 1, T is of class C_{10}. It is easy to check that $T|\mathfrak{H}_1$ is also of class C_{10}. Similar arguments as above finish the proof.

Corollary 5. *Let T be a c.n.u. contraction on $\mathfrak{H}$ with finite defect indices. If T is quasi-similar to an isometry V on $\mathfrak{K}$, then there exist quasi-affinities $X:\mathfrak{H}\to\mathfrak{K}$ and $Y:\mathfrak{K}\to\mathfrak{H}$ which intertwine T and V and such that $XY=\delta(V)$ and $YX=\delta(T)$ for some outer function δ.*

Proof. Let $T=\begin{bmatrix} T_1 & * \\ 0 & T_2 \end{bmatrix}$ be the triangulation of type $\begin{bmatrix} C_{\cdot 1} & * \\ 0 & C_{\cdot 0} \end{bmatrix}$. As before, since T_1 is of class C_{11} with finite defect indices, we have $T\sim T_1\oplus T_2$. Let $V=U\oplus S$ be the isometry quasi-similar to T, where U is unitary and S is a unilateral shift. As shown in the proof of Lemma 2, $T_1\sim U$ nad $T_2\sim S$. Note that all these three quasi-similarities can be implemented by quasi-affinities satisfying the corresponding properties in the conclusion of our assertion (cf. [9], Theorem 2.1, [7], Lemma 2.1 and proof of Lemma 1). Hence the same holds for the quasi-similarity of T and V.

For an operator T, let Lat T, Lat$''T$ and Hyperlat T denote, respectively, the lattices of invariant subspaces, bi-invariant subspaces and hyperinvariant subspaces of T. The next lemma will be needed in the proof of Theorem 7. It can be proved in the same fashion as [7], Lemma 2.3.

Lemma 6. *Let V be an isometry with an absolutely continuous unitary part and let $\mathfrak{N}\in\mathrm{Lat}\, V$. If δ is an outer function, then $\delta(V|\mathfrak{N})$ is a quasi-affinity on $\mathfrak{N}$.*

Theorem 7. *Let T be a c.n.u. contraction with finite defect indices. If T is quasi-similar to an isometry V, then* $\mathrm{Lat}\, T\cong\mathrm{Lat}\, V$, $\mathrm{Lat}''\, T\cong\mathrm{Lat}''\, V$ *and* $\mathrm{Hyperlat}\, T\cong \mathrm{Hyperlat}\, V$.

Proof. Note that T is of class $C_{1\cdot}$ by [8], Lemma 1. We may assume that T is not of class C_{11}, for otherwise the conclusion has already been proved in [7], Theorem 2.2.

Let X and Y be the quasi-affinities as in Corollary 5. For $\mathfrak{M}\in\mathrm{Lat}\, T$ and $\mathfrak{N}\in\mathrm{Lat}\, V$, consider the mappings $\mathfrak{M}\to\overline{X\mathfrak{M}}$ and $\mathfrak{N}\to\overline{Y\mathfrak{N}}$. Using Lemma 6, we can easily verify that they implement the lattice isomorphisms between Lat T and Lat V.

From [9], Theorem 3.13 and Lemma 3.11, we infer that Lat $T \cong$ Lat″ T and Lat $V \cong$ $\cong$ Lat″ V. Hence to complete the proof, it suffices to show that (i) $\mathfrak{M} \in$ Hyperlat T implies $\overline{X\mathfrak{M}} \in$ Hyperlat V and (ii) $\mathfrak{N} \in$ Hyperlat V implies $\overline{Y\mathfrak{N}} \in$ Hyperlat T. We only verify (i) and leave the verification of (ii) to the readers. Let $\mathfrak{M} \in$ Hyperlat T and $W \in \{V\}'$. Then $YWX \in \{T\}'$ and hence $\overline{YWX\mathfrak{M}} \subseteq \mathfrak{M}$. Applying X on both sides, we obtain $\overline{\delta(V)WX\mathfrak{M}} = \overline{XYWX\mathfrak{M}} \subseteq \overline{X\mathfrak{M}}$. Since $\delta(V)|\overline{WX\mathfrak{M}}$ is a quasi-affinity on $\overline{WX\mathfrak{M}}$ (by Lemma 6), we conclude that $\overline{WX\mathfrak{M}} \subseteq \overline{X\mathfrak{M}}$. This shows that $\overline{X\mathfrak{M}} \in$ Hyperlat V, completing the proof.

Corollary 8. *Let T be a c.n.u. contraction with finite defect indices. If T is quasi-similar to a unilateral shift, then* Lat $T=$ Lat″$T = \{\overline{\text{ran } W} : W \in \{T\}'\}$, *where* $\{T\}'$ *denotes the commutant of T.*

Proof. This follows easily from Theorem 7 and the fact that a unilateral shift S satisfies Lat $S=$ Lat″ $S = \{\overline{\text{ran } Z} : Z \in \{S\}'\}$.

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