*Acta Sci. Math.*, 44 (1982), 157—163

# Injection-similar isometries

L. KÉRCHY

**1.** To construct canonical models for contractions of classes $C_{11}$ and $C_0$ on complex separable Hilbert spaces B. SZ.-NAGY and C. FOIAŞ generalized the notion of similarity (cf. [3, ch. II, sec. 3] and [4]). They called an operator $T_1 \in \mathscr{L}(\mathfrak{H}_1)$ a *quasi-affine transform* of the operator $T_2 \in \mathscr{L}(\mathfrak{H}_2)$, $T_1 \prec T_2$, if there exists a quasi-affinity (an injection with dense range) $X \in \mathscr{L}(\mathfrak{H}_1, \mathfrak{H}_2)$ which intertwines these operators, that is, $XT_1 = T_2X$. $T_1$ and $T_2$ are said to be *quasi-similar*, $T_1 \sim T_2$, if they are quasi-affine transforms of each other, $T_1 \prec T_2$ and $T_2 \prec T_1$. Finding Jordan-models for contractions of class $C_{\cdot 0}$ even quasi-similarity proved to be insufficient. Therefore SZ.-NAGY and FOIAŞ [5] introduced the notion of injection-similarity. Operators $T_1 \in \mathscr{L}(\mathfrak{H}_1)$ and $T_2 \in \mathscr{L}(\mathfrak{H}_2)$ are *injection-similar*, $T_1 \overset{i}{\sim} T_2$, if they can be injected into each other, $T_1 \overset{i}{\prec} T_2$ and $T_2 \overset{i}{\prec} T_1$, that is, there are injections $X \in \mathscr{L}(\mathfrak{H}_1, \mathfrak{H}_2)$ and $Y \in \mathscr{L}(\mathfrak{H}_2, \mathfrak{H}_1)$ such that $XT_1 = T_2X$ and $YT_2 = T_1Y$. $T_1$ and $T_2$ are *completely injection-similar*, $T_1 \overset{c.i}{\sim} T_2$, if they can be completely injected into each other, $T_1 \overset{c.i}{\prec} T_2$ and $T_2 \overset{c.i}{\prec} T_1$, that is, there exist families of intertwining injections $\{X_\alpha\}_\alpha \subseteq \mathscr{L}(\mathfrak{H}_1, \mathfrak{H}_2)$ and $\{Y_\beta\}_\beta \subseteq \mathscr{L}(\mathfrak{H}_2, \mathfrak{H}_1)$ such that $\bigvee_\alpha \operatorname{ran} X_\alpha = \mathfrak{H}_2$ and $\bigvee_\beta \operatorname{ran} Y_\beta = \mathfrak{H}_1$.

Recently P. Y. WU [1] has shown that every contraction $T$ of class $C_{1\cdot}$, with at least one defect index finite, $d_T < \infty$, is completely injection-similar to an isometry. More precisely he proved that

$$U \oplus S^{(\alpha)} \overset{c.i}{\prec} T \prec U \oplus S^{(\alpha)}.$$

Here $U$ is a unitary operator of the form $U = U_1 \oplus U_2$, where $U_1$ is the unitary part of the contraction $T$ (cf. [3, Th. I.3.2]), and $U_2$ denotes the operator of multiplication by $e^{it}$ on the space $(\Delta_T L^2(\mathfrak{D}_T))^-$ $(\Delta_T(e^{it}) = (I - \Theta_T(e^{it})^* \Theta_T(e^{it}))^{1/2}$, where $\Theta_T$ is the characteristic function of $T$). On the other hand $S^{(\alpha)}$ is the unilateral shift of multiplicity $\alpha = d_{T^*} - d_T$.

Received July 1, 1981.

As for uniqueness of this isometry, Wu has shown that the unitary parts of injection-similar isometries are unitarily equivalent. Moreover he made the conjecture that injection-similar isometries are really unitarily equivalent, at least in the case, when their unitary parts have finite multiplicities. (Hoover [7] proved that quasi-similarity even implies unitary equivalence between isometries.)

In the present paper we give a negative answer to this conjecture and describe the isometries being completely injection-similar to the contraction $T$ above. We follow the notation and terminology of [3]. For arbitrary operators $T_1 \in \mathcal{L}(\mathfrak{H}_1)$ and $T_2 \in \mathcal{L}(\mathfrak{H}_2)$, $\mathcal{I}(T_1, T_2)$ will denote the set of intertwining operators, that is, $\mathcal{I}(T_1, T_2) = \{X \in \mathcal{L}(\mathfrak{H}_1, \mathfrak{H}_2) \mid T_2 X = X T_1\}$.

**2.** We recall that every isometry $V$ has a unique decomposition $V = U \oplus S^{(\alpha)}$ such that $U$ is a unitary operator and $S^{(\alpha)}$ denotes the direct sum of $\alpha$ copies of the simple unilateral shift $S$. ($S^{(\alpha)}$ is a completely non-unitary (c. n. u.) isometry with multiplicity $\alpha$.) (Cf. [3, Th. I.1.1.]) The following proposition shows that Wu's conjecture has an affirmative answer, if $V$ is a c. n. u. isometry or $U$ is a singular unitary (s. u.) operator (the spectral measure of $U$ is singular with respect to Lebesgue measure).

Proposition 1. *Let $V_1$ and $V_2$ be injection-similar isometries, $V_1 \overset{i}{\sim} V_2$. Let us assume that $V_1$ is c. n. u. or its unitary part is a s. u. operator. Then these operators are unitarily equivalent, $V_1 \cong V_2$.*

Proof. Let $V_1$ and $V_2$ act on the Hilbert spaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$, respectively. Let us consider the canonical decompositions of these operators: $V_1 = U_1 \oplus S^{(\alpha)}$, $V_2 = U_2 \oplus S^{(\beta)}$ on the spaces $\mathfrak{H}_1 = \mathfrak{K}_1 \oplus \mathfrak{L}_1$ and $\mathfrak{H}_2 = \mathfrak{K}_2 \oplus \mathfrak{L}_2$. We know by [1, Lemma 3.6] that $U_1 \cong U_2$. If $V_1$ is c. n. u., then $\mathfrak{K}_1 = \{0\}$, and so we obtain that $S^{(\alpha)} = V_1 \overset{i}{\sim} V_2 = S^{(\beta)}$. Now [5, Th. 5/6] results that $S^{(\alpha)} \cong S^{(\beta)}$. Consequently in this case we have that $V_1 \cong V_2$.

Let us assume now that $\mathfrak{K}_1 \neq \{0\}$ and $U_1$ is a s. u. operator. Let us suppose further that for instance $\mathfrak{L}_1 \neq \{0\}$. (The case $\mathfrak{L}_1 = \mathfrak{L}_2 = \{0\}$ is trivial.) Let $X \in \mathcal{I}(V_1, V_2)$ be an injection, and consider the matrix $\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}$ of $X$ with respect to the decompositions above. It follows easily that $X_{12} \in \mathcal{I}(S^{(\alpha)}, U_2)$. Having denoted by $S_b^{(\alpha)}$ the minimal unitary dilation of $S^{(\alpha)}$, we define an operator $Y \in \mathcal{I}(S_b^{(\alpha)}, U_2)$ by the equation $Y(S_b^{(\alpha)})^{-n} f := U_2^{-n} X_{12} f$ $(f \in \mathfrak{L}_1, n \geqq 0)$ and by taking bounded closure. Since, being a bilateral shift, $S_b^{(\alpha)}$ is an absolutely continuous unitary (a. c. u.) operator, we infer by [8, Theorem 3] that $Y = 0$. Taking into account that $X_{12} = Y | \mathfrak{L}_1$, it follows that $X_{12} = 0$. We conclude that $X_{22} \in \mathcal{I}(S^{(\alpha)}, S^{(\beta)})$ is an injection. In particular we infer that $\mathfrak{L}_2 \neq \{0\}$, and so a similar argument shows that we have $S^{(\beta)} \overset{i}{\prec} S^{(\alpha)}$

also. Therefore $S^{(\alpha)} \overset{i}{\sim} S^{(\beta)}$, and [5, Th. 5/6] implies again $S^{(\alpha)} \cong S^{(\beta)}$. The proof is completed.

**3.** In this section we shall see that the setting is contrary to the one in section 2, if the isometry $V$ is not c. n. u. and its unitary part is not a s. u. operator. The following lemma plays an essential role in the sequel.

Lemma 2. *Let $E$ be a measurable set on the unit circle $C=\{z\in\mathbf{C}\,|\,|z|=1\}$, and let $M_E$ denote the operator of multiplication by $e^{it}$ on the space $L^2(E)$. (We consider the normalized Lebesgue measure $m$ on $C$.) If $m(E)>0$, then we have*

$$M_E \oplus S \prec M_E.$$

Proof. Let $\varphi_1 \in L^\infty(E)$ be a function such that $\varphi_1(e^{it}) \neq 0$ a. e. and $\int_E \log|\varphi_1(e^{it})|\,dm = -\infty$. On the other hand let $\varphi_2 \in L^\infty(E)$ be a function such that $|\varphi_2(e^{it})|=1$ a.e.. We consider $S$ as the operator of multiplication by $e^{it}$ on the Hardy space $H^2$. Now let us define the operator $X$ as follows: $X\colon L^2(E)\oplus H^2 \to L^2(E)$, $X\colon f\oplus g \mapsto \varphi_1 f + \varphi_2 (g|E)$. It is obvious that $X \in \mathscr{I}(M_E \oplus S, M_E)$ is a quasi-surjection.

Let us assume now that $X(f\oplus g)=0$. Let us suppose further that $g\neq 0$. Then we have $g(e^{it})\neq 0$ a. e., and so $f(e^{it})\neq 0$ a. e. on $E$. From the assumption it immediately follows that $|\varphi_1(e^{it})|\cdot|f(e^{it})| = |g(e^{it})|$ a. e. on $E$. But this implies

$$\log|\varphi_1(e^{it})| = \log|g(e^{it})| - \log|f(e^{it})| \geqq \log|g(e^{it})| + 1 - |f(e^{it})|,$$

and so we infer that

$$-\infty = \int_E \log|\varphi_1(e^{it})|\,dm \geqq \int_E \log|g(e^{it})|\,dm + m(E) - \int_E |f(e^{it})|\,dm > -\infty$$

(cf. [3, ch. III]). This being a contradiction we conclude that $g=0$ and this results $f=0$. Therefore $X$ is a quasi-affinity, and so $M_E \oplus S \prec M_E$.

Corollary 3. *Let $M_E$ be as before. Then for any $\alpha = 1, 2, \ldots, \infty$ we have*

$$M_E \oplus S^{(\alpha)} \prec M_E.$$

Proof. By induction we immediately infer that the statement holds for every natural number. Let us now assume that $\alpha=\infty$. Let $\{E_n\}_{n=1}^\infty$ be a sequence of pairwise disjoint measurable subsets of $E$ such that $\bigcup_{n=1}^\infty E_n = E$ and $m(E_n)>0$ for every $n$. Then we have $M_E \oplus S^{(\infty)} \cong \bigoplus_{n=1}^\infty (M_{E_n} \oplus S) \prec \bigoplus_{n=1}^\infty M_{E_n} \cong M_E$ by Lemma 2, and the proof is finished.

Corollary 4. *Let $V\in\mathcal{L}(\mathfrak{H})$ be a non-c. n. u. isometry, and let us assume that its unitary part $U\in\mathcal{L}(\mathfrak{K})$ $(\mathfrak{K}\neq\{0\})$ is not a s. u. operator. Then we have:*

(i) $V\overset{i}{\sim}U$, *more precisely* $U\overset{i}{\prec}V\prec U$;

(ii) *if even* $\mathfrak{H}\ominus\mathfrak{K}\neq\{0\}$ *holds, then* $V\overset{c.i}{\sim}U\oplus S$, *more precisely* $U\oplus S\overset{c.i}{\prec}V\prec\prec U\oplus S$.

Proof. After decomposing $U$ into the direct sum of its singular and its absolutely continuous parts, $U=U_s\oplus U_a$, and considering the functional model of $U_a$ (cf. [9]), we conclude these statements by Corollary 3.

On account of Corollary 4 we can state:

Proposition 5. *Let $V_1$ and $V_2$ be isometries, and let $U_1, U_2$ denote their unitary parts, respectively. Let us assume that $V_1$ is not c. n. u., and $U_1$ is not a s. u. operator. Then we have:*

(i) $V_1\overset{i}{\sim}V_2$ *if and only if* $U_1\cong U_2$;

(ii) $V_1\overset{c.i}{\sim}V_2$ *if and only if* $U_1\cong U_2$ *and* $V_1, V_2$ *are unitaries in the same time.*

Proof. These statements follow immediately by [1, Lemma 3.6] and the preceding corollary. We have only to note that for any operator $X\in\mathcal{I}(V_1, V_2)$ we have $(X\mathfrak{K}_1)^-\subseteq\mathfrak{K}_2$, where $\mathfrak{K}_i\in\operatorname{Lat}V_i$ is the subspace corresponding to $U_i$ $(i=1,2)$. (Cf. the proof of [1, Lemma 3.6].)

**4.** Now let $T$ be a contraction of class $C_{1\cdot}$, with at least one finite defect index, $d_T<\infty$. Consider the triangulation $\begin{bmatrix}T_1 & *\\ 0 & T_2\end{bmatrix}$ of the type $\begin{bmatrix}C_{\cdot 1} & *\\ 0 & C_{\cdot 0}\end{bmatrix}$ of $T$. We know from [1] that $T_1\in C_{11}, T_2\in C_{10}$ and $T\sim T_1\oplus T_2$ (cf. [1, Th. 2.1 and Lemma 3.2]). Now it follows easily by [3, Prop. II.3.5] and [6, Th. 3] that

$$U\oplus S^{(\alpha)}\overset{c.i}{\prec}T\prec U\oplus S^{(\alpha)},$$

where $U$ is a unitary operator and $S^{(\alpha)}$ is the unilateral shift of multiplicity $\alpha=d_{T^*}-d_T$. (Cf. [1, Th. 3.5].) Moreover we know by [1, Lemma 3.6] that the unitary part of every isometry, being injection-similar to $T$, is unitarily equivalent to $U$.

We shall say that $T$ is *mixed with absolutely continuous part (m. w. a. c. p.)*, if $T\notin C_{11}\cup C_{10}$ and $T_1$ is not a s. u. operator in the previous triangulation. Now we obtain immediately by Proposition 1:

Theorem 6. *If $T\in C_{1\cdot}, d_T<\infty$ and $T$ is not m. w. a. c. p., then $V=U\oplus S^{(\alpha)}$, $\alpha=d_{T^*}-d_T$, is the unique isometry which is completely injection-similar to $T$.*

On the other hand, in the contrary case we can state:

Theorem 7. *If* $T \in C_{1\cdot}, d_T < \infty$ *and* $T$ *is m. w. a. c. p., then*

$$U \oplus S^{(\alpha)} \overset{\text{c.i}}{\prec} T \prec U \oplus S^{(\alpha)}$$

*holds, if and only if* $1 \leqq \alpha \leqq d_{T*} - d_T$.

To prove this theorem we need:

Lemma 8. *If* $T$ *is a contraction of class* $C_{10}$ *and* $d_T < \infty$, *then* $\dim \ker T^* = d_{T*} - d_T$.

Proof. We can assume that $T$ is given by its functional model. That is, $T$ is the compression of the unilateral shift $U_+$ on the vector-valued Hardy space $H^2(\mathfrak{E}_*)$ to the subspace $\mathfrak{H} = H^2(\mathfrak{E}_*) \ominus \Theta_T H^2(\mathfrak{E})$ ($\in \operatorname{Lat} U_+^*$), where $\dim \mathfrak{E}_* = d_{T*}$, $\dim \mathfrak{E} = d_T$ and $\Theta_T$ denotes the characteristic function of $T$. $T$ being of class $C_{10}$, its characteristic function $\Theta_T$ is inner and $*$-outer (cf. [3, Prop. VI. 3.5]).

Since $T^* = U_+^* | \mathfrak{H}$, we infer that $\ker T^* = \mathfrak{H} \cap \ker U_+^* = \mathfrak{H} \cap \mathfrak{E}_*$. Let $v \in \mathfrak{E}_*$ be an arbitrary vector. We have that $v \in \mathfrak{H}$, if and only if $v$ is orthogonal to $\Theta_T H^2(\mathfrak{E})$. But this is the case, if and only if $v$ is orthogonal to $\Theta_T H^2(\mathfrak{E}) \ominus \lambda \Theta_T H^2(\mathfrak{E}) = \Theta_T (H^2(\mathfrak{E}) \ominus \lambda H^2(\mathfrak{E})) = \Theta_T \mathfrak{E}$. (We have used that $\Theta_T$ is an isometry.) Now, for any vector $w \in \mathfrak{E}$, we have $\langle v, \Theta_T w \rangle = \int_C \langle v, \Theta_T(e^{it}) w \rangle \, dm = \int_C \langle \Theta_T(e^{-it})^* v, w \rangle \, dm = \langle \Theta_T^{\sim} v, w \rangle = \langle P_{\mathfrak{E}} \Theta_T^{\sim} v, w \rangle$, where $P_{\mathfrak{E}}$ denotes the orthogonal projection of $H^2(\mathfrak{E})$ to the subspace $\mathfrak{E}$. Therefore, we conclude that $\ker T^* = \ker (P_{\mathfrak{E}} \Theta_T^{\sim} | \mathfrak{E}_*)$.

On the other hand, since $\Theta_T^{\sim}$ is an outer function, it follows that $H^2(\mathfrak{E}) = (\Theta_T^{\sim} H^2(\mathfrak{E}_*))^- = (\Theta_T^{\sim} \mathfrak{E}_*) \vee \lambda \Theta_T^{\sim} H^2(\mathfrak{E}_*) \subseteqq (\Theta_T^{\sim} \mathfrak{E}_*) \vee (\lambda H^2(\mathfrak{E})) = (P_{\mathfrak{E}} \Theta_T^{\sim} \mathfrak{E}_*)^- \oplus \lambda H^2(\mathfrak{E})$. Therefore the operator $P_{\mathfrak{E}} \Theta_T^{\sim} | \mathfrak{E}_* \in \mathscr{L}(\mathfrak{E}^*, \mathfrak{E})$ is quasi-surjective, and so, taking into account that $\dim \mathfrak{E} < \infty$, we infer that $\dim \ker (P_{\mathfrak{E}} \Theta_T^{\sim} | \mathfrak{E}_*) = \dim \mathfrak{E}_* - \dim \mathfrak{E} = d_{T*} - d_T$. The proof is completed.

Now we can prove Theorem 7.

Proof of Theorem 7. Let $T_1, T_2$ and $U$ be the operators as at the begining of this section. Since $T$ is m.w.a.c.p., it follows that the space of $U$ is not trivial (is not $\{0\}$), and that $U$ is not a s. u. operator. Applying Corollary 3 we can easily infer that $U \oplus S^{(\alpha)} \overset{\text{c.i}}{\prec} U \oplus S^{(d_{T*} - d_T)} \prec U \oplus S^{(\alpha)}$, for every $1 \leqq \alpha \leqq d_{T*} - d_T$. Therefore, it is enough to prove that $T \prec U \oplus S^{(\alpha)}$ implies $\alpha \leqq d_{T*} - d_T$.

So, let us assume that $T \prec U \oplus S^{(\alpha)}$. Then we have $U \oplus T_2 \prec T_1 \oplus T_2 \prec T \prec U \oplus S^{(\alpha)}$. Let $X \in \mathscr{I}(U \oplus T_2, U \oplus S^{(\alpha)})$ be a quasi-affinity. Since then $X^* \in \mathscr{I}(U^* \oplus S^{*(\alpha)}, U^* \oplus T_2^*)$ is also a quasi-affinity it follows that $X^* | \ker S^{*(\alpha)} : \ker S^{*(\alpha)} \to \ker T_2^*$ is an injection. Therefore we get that $\alpha = \dim \ker S^{*(\alpha)} \leqq \dim \ker T_2^*$. Taking into account that $d_{T*} - d_T = d_{T_2^*} - d_{T_2}$, we conclude by Lemma 8 that $\alpha \leqq d_{T*} - d_T$. The proof is finished.

Corollary 9. *If $T$ is a contraction as in Theorem 7, then for the multiplicity of $T^*$ we have:* $\mu_{T^*}=\mu_U$.

Proof. We infer by Theorem 7 and Lemma 2 that $T\prec U\oplus S\prec U$. It follows that $U^*\prec T^*$, and so $\mu_{T^*}\leqq\mu_{U^*}=\mu_U$. On the other hand $T^*\sim T_1^*\oplus T_2^*\sim U^*\oplus T_2^*$ implies $\mu_{T^*}\geqq\mu_{U^*}=\mu_U$.

5. Finally we show that if $T\in C_{1\cdot}$, $d_T<\infty$ and $T$ is m. w. a. c. p., then there always exists an isometry $V$ such that $V\prec T$. It can be easily seen that this is not the case, if $T$ is not m. w. a. c. p. (cf. [5, Th. 5 and Prop. 2]).

Theorem 10. *If $T\in C_{1\cdot}$, $d_T<\infty$, is a contraction m. w. a. c. p., then $U\oplus S^{(\alpha)}\prec\prec T$, where $\alpha=d_{T^*}$.*

Proof. Let $T_1, T_2$ and $U$ be the operators as in the begining of section 4. Since $T$ is m. w. a. c. p., it follows that these operators act on non-zero spaces, and that $U$ is not a s. u. operator. Therefore there exists a reducing subspace $\mathfrak{L}$ of $U$ such that $U|\mathfrak{L}\cong M_E$ for some measurable set $E$ $(m(E)>0)$. Taking into account that $T\sim T_1\oplus T_2\sim U\oplus T_2$, it is enough to prove that $M_E\oplus S^{(\alpha)}\prec M_E\oplus T_2$, where $\alpha=d_{T^*}$.

Let us consider the minimal isometric dilation $W\in\mathscr{L}(\mathfrak{K}_+)$ of the contraction $T_2\in\mathscr{L}(\mathfrak{H})$. Since $T_2\in C_{\cdot 0}$, it follows that $W$ is a unilateral shift of multiplicity $\alpha=d_{T^*}$ (cf. [3, Th. II.1.2 and II.2.1]). Therefore we infer by the proof of Corollary 3 that there exists an injection $Y\in\mathscr{I}(M_E\oplus W, M_E\oplus T_2)$ such that $(Y(L^2(E)\oplus\{0\}))^-= =(\operatorname{ran} Y)^-=L^2(E)\oplus\{0\}$. Let $P$ denote the orthogonal projection of the space $L^2(E)\oplus\mathfrak{K}_+$ onto its subspace $\{0\}\oplus\mathfrak{H}$. Then the operator $X=Y+P\in \in\mathscr{L}(L^2(E)\oplus\mathfrak{K}_+, L^2(E)\oplus\mathfrak{H})$ is obviously a quasi-affinity.

On the other hand, for any vector $f\oplus g\in L^2(E)\oplus\mathfrak{K}_+$ we have

$$(M_E\oplus T_2)X(f\oplus g)=(M_E\oplus T_2)Y(f\oplus g)+(M_E\oplus T_2)P(f\oplus g)=$$

$$=Y(M_E\oplus W)(f\oplus g)+(0\oplus T_2Pg)=Y(M_E\oplus W)(f\oplus g)+(0\oplus PWg)=$$

$$=X(M_E\oplus W)(f\oplus g).$$

Consequently we obtained that $M_E\oplus W\prec M_E\oplus T_2$, and so the proof is completed.

By Theorems 7 and 10 it follows immediately:

Corollary 11. *If $T\in C_{1\cdot}$, $d_T<\infty$, is a contraction m. w. a. c. p. and $d_{T^*}=\infty$, then we have*

$$T\sim U\oplus S^{(\infty)}.$$

If both defect indices of $T$ are finite, then it is in general not true that $T\sim \sim U\oplus S^{(\alpha)}$, where $\alpha=d_{T^*}-d_T$. Indeed, contractions $T$ with finite defect indices and quasi-similar to an isometry $V$, were characterized by P. Y. Wu [2]. We note that if $T \in C_{1\cdot}$, $d_T < \infty$ and $T$ is quasi-similar to an isometry $V$, then $V$ is necessarily unitarily equivalent to the operator $U \oplus S^{(\alpha)}$, where $\alpha = d_{T*} - d_T$. This follows easily by Theorems 6 and 7.

## References


[1] P. Y. Wu, Approximate decompositions of certain contractions, *Acta Sci. Math.*, **44** (1982), 137—149.

[2] P. Y. Wu, When is a contraction quasi-similar to an isometry?, *Acta Sci. Math.*, **44** (1982), 151—155.

[3] B. Sz.-Nagy and C. Foiaş, *Harmonic analysis of operators on Hilbert space*, North Holland — Akadémiai Kiadó (Amsterdam—Budapest, 1970).

[4] B. Sz.-Nagy and C. Foiaş, Modèles de Jordan pour une classe d'opérateurs de l'espace de Hilbert, *Acta Sci. Math.*, **31** (1970), 91—115.

[5] B. Sz.-Nagy and C. Foiaş, Jordan model for contractions of class $C_{\cdot 0}$, *Acta Sci. Math.*, **36** (1974), 305—322.

[6] B. Sz.-Nagy, Diagonalization of matrices over $H^\infty$, *Acta Sci. Math.*, **38** (1976), 223—238.

[7] T. B. Hoover, Quasi-similarity of operators, *Illinois J. Math.*, **16** (1972), 678—686.

[8] R. G. Douglas, On the operator equation $S^*XT = X$ and related topics, *Acta Sci. Math.*, **30** (1969), 19—32.

[9] N. Dunford and J. Schwartz, *Linear operators*, Part II (New York, 1963).



BOLYAI INSTITUTE
UNIVERSITY SZEGED
ARADI VÉRTANÚK TERE 1
H—6720 SZEGED, HUNGARY